Research Article

Optimal Bounds for Seiffert Mean in terms of One-Parameter Means

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The authors present the greatest value r_1 and the least value r_2 such that the double inequality $J_{r_1}(a, b) < T(a, b) < J_{r_2}(a, b)$ holds for all $a, b > 0$ with $a \neq b$, where $T(a, b)$ and $J_p(a, b)$ denote the Seiffert and p th one-parameter means of two positive numbers a and b , respectively.

1. Introduction

For $p \in \mathbb{R}$ the p th one-parameter mean $J_p(a, b)$ and the Seiffert mean $T(a, b)$ of two positive real numbers a and b are defined by

$$J_p(a, b) = \begin{cases} \frac{p(a^{p+1} - b^{p+1})}{(p+1)(a^p - b^p)}, & a \neq b, p \neq 0, -1, \\ \frac{a - b}{\log a - \log b}, & a \neq b, p = 0, \\ \frac{ab(\log a - \log b)}{a - b}, & a \neq b, p = -1, \\ a, & a = b, \end{cases} \quad (1.1)$$

$$T(a, b) = \begin{cases} \frac{a - b}{2 \arctan((a - b)/(a + b))}, & a \neq b, \\ a, & a = b, \end{cases} \quad (1.2)$$

respectively. Recently, both mean values have been the subject of intensive research. In particular, many remarkable inequalities and properties for J_p and T can be found in the literature [1–14].

It is well known that the one-parameter mean $J_p(a, b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Many mean values are the special case of the one-parameter mean, for example:

$$\begin{aligned} J_1(a, b) &= \frac{(a+b)}{2}, & \text{the arithmetic mean,} \\ J_{1/2}(a, b) &= \frac{(a + \sqrt{ab} + b)}{3}, & \text{the Heronian mean,} \\ J_{-1/2}(a, b) &= \sqrt{ab}, & \text{the geometric mean,} \\ J_{-2}(a, b) &= \frac{2ab}{(a+b)}, & \text{the harmonic mean.} \end{aligned} \quad (1.3)$$

Seiffert [4] proved that the double inequality

$$M_1(a, b) < T(a, b) < M_2(a, b) \quad (1.4)$$

holds for all $a, b > 0$ with $a \neq b$, where $M_r(a, b) = [(a^r + b^r)/2]^{1/r}$ ($r \neq 0$) and $M_0(a, b) = \sqrt{ab}$ is the r th power mean of a and b .

In [15–17], the authors presented the best possible bounds for the Seiffert mean in terms of the Lehmer, power-type Heron, and one-parameter Gini means as follows:

$$\begin{aligned} L_0(a, b) &< T(a, b) < L_{1/3}(a, b), \\ H_{\log 3 / \log(\pi/2)}(a, b) &< T(a, b) < H_{5/2}(a, b), \\ S_1(a, b) &< T(a, b) < S_{5/3}(a, b). \end{aligned} \quad (1.5)$$

for all $a, b > 0$ with $a \neq b$, where $L_r(a, b) = (a^{r+1} + b^{r+1}) / (a^r + b^r)$, $H_k(a, b) = [(a^k + (ab)^{k/2} + b^k)/3]^{1/k}$ ($k \neq 0$) and $H_0(a, b) = \sqrt{ab}$, and $S_\alpha(a, b) = [(a^{\alpha-1} + b^{\alpha-1}) / (a + b)]^{1/(\alpha-2)}$ ($\alpha \neq 2$) and $S_2(a, b) = (a^a b^b)^{1/(a+b)}$ denote the Lehmer, power-type Heron, and one-parameter Gini means of a and b , respectively.

The purpose of this paper is to answer the question: what are the greatest value r_1 and the least value r_2 such that the double inequality

$$J_{r_1}(a, b) < T(a, b) < J_{r_2}(a, b) \quad (1.6)$$

holds for all $a, b > 0$ with $a \neq b$?

2. Lemma

In order to establish our main result we need the following lemma.

Lemma 2.1. If $p = 2/(\pi - 2) = 1.75 \dots$, $t \geq 1$ and $g(t) = -(p-1)t^{2p+2} + (p+1)t^{2p} + p(p+1)t^{p+3} - 2(p+1)^2t^{p+2} + 2p(p+3)t^{p+1} - 2(p+1)^2t^p + p(p+1)t^{p-1} + (p+1)t^2 - (p-1)$, then there exists $\lambda \in (1, \infty)$ such that $g(t) > 0$ for $t \in (1, \lambda)$ and $g(t) < 0$ for $t \in (\lambda, \infty)$.

Proof. Let $g_1(t) = g'(t)/t$, $g_2(t) = t^{4-p}g_1'(t)$ and $g_3(t) = t^{4-p}g_2^{(5)}(t)/[4p^2(p-1)^2(p+1)^2]$. Then simple computations lead to

$$g(1) = 0, \quad (2.1)$$

$$\lim_{t \rightarrow +\infty} g(t) = -\infty, \quad (2.2)$$

$$\begin{aligned} g_1(t) = & -2(p-1)(p+1)t^{2p} + 2p(p+1)t^{2p-2} + p(p+1)(p+3) \\ & \times t^{p+1} - 2(p+1)^2(p+2)t^p + 2p(p+1)(p+3)t^{p-1} \\ & - 2p(p+1)^2t^{p-2} + p(p+1)(p-1)t^{p-3} + 2(p+1), \end{aligned} \quad (2.3)$$

$$g_1(1) = 0, \quad (2.4)$$

$$\lim_{t \rightarrow +\infty} g_1(t) = -\infty, \quad (2.5)$$

$$\begin{aligned} g_2(t) = & -4p(p-1)(p+1)t^{p+3} + 4p(p+1)(p-1)t^{p+1} + p(p+1)^2 \\ & \times (p+3)t^4 - 2p(p+1)^2(p+2)t^3 + 2p(p-1)(p+1)(p+3) \\ & \times t^2 - 2p(p+1)^2(p-2)t + p(p+1)(p-1)(p-3), \end{aligned} \quad (2.6)$$

$$g_2(1) = 0, \quad (2.7)$$

$$\lim_{t \rightarrow +\infty} g_2(t) = -\infty, \quad (2.8)$$

$$\begin{aligned} g_2'(t) = & -4p(p-1)(p+1)(p+3)t^{p+2} + 4p(p+1)^2(p-1)t^p \\ & + 4p(p+1)^2(p+3)t^3 - 6p(p+1)^2(p+2)t^2 \\ & + 4p(p-1)(p+1)(p+3)t - 2p(p+1)^2(p-2), \end{aligned} \quad (2.9)$$

$$g_2'(1) = 0, \quad (2.10)$$

$$\lim_{t \rightarrow +\infty} g_2'(t) = -\infty, \quad (2.11)$$

$$\begin{aligned} g_2''(t) = & -4p(p-1)(p+1)(p+2)(p+3)t^{p+1} + 4p^2(p+1)^2 \\ & \times (p-1)t^{p-1} + 12p(p+1)^2(p+3)t^2 - 12p(p+1)^2 \\ & \times (p+2)t + 4p(p-1)(p+1)(p+3), \end{aligned} \quad (2.12)$$

$$g_2''(1) = 12p(2-p)(p+1)^2 > 0, \quad (2.13)$$

$$\lim_{t \rightarrow +\infty} g_2''(t) = -\infty, \quad (2.14)$$

$$g_2'''(t) = -4p(p-1)(p+1)^2(p+2)(p+3)t^p + 4p^2(p+1)^2 \times (p-1)^2 t^{p-2} + 24p(p+1)^2(p+3)t - 12p(p+1)^2(p+2), \quad (2.15)$$

$$g_2'''(1) = 12p(2-p)(2p+3)(p+1)^2 > 0, \quad (2.16)$$

$$\lim_{t \rightarrow +\infty} g_2'''(t) = -\infty, \quad (2.17)$$

$$g_2^{(4)}(t) = -4p^2(p-1)(p+1)^2(p+2)(p+3)t^{p-1} + 4p^2(p+1)^2 \times (p-1)^2(p-2)t^{p-3} + 24p(p+1)^2(p+3), \quad (2.18)$$

$$g_2^{(4)}(1) = 8p(p+1)^2(-4p^3 + 2p^2 + 5p + 9) > 8p(p+1)^2[-4 \times 1.8^3 + 2 \times 1.75^2 + 5 \times 1.75 + 9] \quad (2.19)$$

$$= 4.376p(p+1)^2 > 0,$$

$$\lim_{t \rightarrow +\infty} g_2^{(4)}(t) = -\infty, \quad (2.20)$$

$$g_3(t) = -(p+2)(p+3)t^2 + (p-2)(p-3) \leq -(p+2)(p+3) + (p-2)(p-3) = -10p < 0 \quad (2.21)$$

for $t \in [1, \infty)$.

From the inequality (2.21) we clearly see that $g_2^{(4)}(t)$ is strictly decreasing in $[1, \infty)$. Then (2.19) and (2.20) lead to the conclusion that there exists $\lambda_1 > 1$ such that $g_2^{(4)}(t) > 0$ for $t \in [1, \lambda_1)$ and $g_2^{(4)}(t) < 0$ for $t \in (\lambda_1, \infty)$. Hence, $g_2'''(t)$ is strictly increasing in $[1, \lambda_1]$ and strictly decreasing in $[\lambda_1, \infty)$.

It follows from (2.16) and (2.17) together with the monotonicity of $g_2'''(t)$ that there exists $\lambda_2 > 1$ such that $g_2'''(t) > 0$ for $t \in [1, \lambda_2)$ and $g_2'''(t) < 0$ for $t \in (\lambda_2, \infty)$. Therefore, $g_2''(t)$ is strictly increasing in $[1, \lambda_2]$ and strictly decreasing in $[\lambda_2, \infty)$.

From (2.13) and (2.14) together with the monotonicity of $g_2''(t)$ we know that there exists $\lambda_3 > 1$ such that $g_2''(t) > 0$ for $t \in [1, \lambda_3)$ and $g_2''(t) < 0$ for $t \in (\lambda_3, \infty)$. So, $g_2'(t)$ is strictly increasing in $[1, \lambda_3]$ and strictly decreasing in $[\lambda_3, \infty)$.

Equations (2.10) and (2.11) together with the monotonicity of $g_2'(t)$ imply that there exists $\lambda_4 > 1$ such that $g_2'(t) > 0$ for $t \in (1, \lambda_4)$ and $g_2'(t) < 0$ for $t \in (\lambda_4, \infty)$. Hence, $g_2(t)$ is strictly increasing in $[1, \lambda_4]$ and strictly decreasing in $[\lambda_4, \infty)$.

It follows from (2.7) and (2.8) together with the monotonicity of $g_2(t)$ that there exists $\lambda_5 > 1$ such that $g_2(t) > 0$ for $t \in (1, \lambda_5)$ and $g_2(t) < 0$ for $t \in (\lambda_5, \infty)$. Therefore, $g_1(t)$ is strictly increasing in $[1, \lambda_5]$ and strictly decreasing in $[\lambda_5, \infty)$.

From (2.4) and (2.5) together with the monotonicity of $g_1(t)$ we clearly see that there exists $\lambda_6 > 1$ such that $g_1(t) > 0$ for $t \in (1, \lambda_6)$ and $g_1(t) < 0$ for $t \in (\lambda_6, \infty)$. So, $g(t)$ is strictly increasing in $[1, \lambda_6]$ and strictly decreasing in $[\lambda_6, \infty)$.

Therefore, Lemma 2.1 follows from (2.1) and (2.2) together with the monotonicity of $g(t)$. $\square$

3. Main Result

Theorem 3.1. *The double inequality*

$$J_{2/(\pi-2)}(a, b) < T(a, b) < J_2(a, b) \quad (3.1)$$

holds for all $a, b > 0$ with $a \neq b$, and $J_{2/(\pi-2)}(a, b)$ and $J_2(a, b)$ are the best possible lower and upper one-parameter mean bounds for the Seiffert mean $T(a, b)$, respectively.

Proof. Without loss of generality, we assume that $a > b$. Let $t = a/b > 1$. Then from (1.1) and (1.2) we have

$$J_2(a, b) - T(a, b) = \frac{b(t^2 + t + 1)}{6(t + 1) \arctan((t - 1)/(t + 1))} \left[4 \arctan \frac{t - 1}{t + 1} - \frac{3(t^2 - 1)}{t^2 + t + 1} \right]. \quad (3.2)$$

Let

$$f(t) = 4 \arctan \frac{t - 1}{t + 1} - \frac{3(t^2 - 1)}{t^2 + t + 1}. \quad (3.3)$$

Then simple computations lead to

$$\begin{aligned} f(1) &= 0, \\ f'(t) &= \frac{(t - 1)^4}{(t^2 + 1)(t^2 + t + 1)^2} > 0, \end{aligned} \quad (3.4)$$

for $t > 1$.

Therefore, $T(a, b) < J_2(a, b)$ for all $a, b > 0$ with $a \neq b$ follows from (3.2)–(3.4).

Next, we prove that

$$T(a, b) > J_{2/(\pi-2)}(a, b) \quad (3.5)$$

for all $a, b > 0$ with $a \neq b$.

Let $p = 2/(\pi - 2) = 1.75 \dots$. Then (1.1) and (1.2) lead to

$$\begin{aligned} T(a, b) - J_p(a, b) &= \frac{bp(t^{p+1} - 1)}{2(p + 1)(t^p - 1) \arctan((t - 1)/(t + 1))} \left[\frac{(p + 1)(t - 1)(t^p - 1)}{p(t^{p+1} - 1)} - 2 \arctan \frac{t - 1}{t + 1} \right]. \end{aligned} \quad (3.6)$$

Let

$$G(t) = \frac{(p+1)(t-1)(t^p-1)}{p(t^{p+1}-1)} - 2 \arctan \frac{t-1}{t+1}. \quad (3.7)$$

Then simple computations lead to

$$\lim_{t \rightarrow 1} G(t) = \lim_{t \rightarrow +\infty} G(t) = 0, \quad (3.8)$$

$$G'(t) = \frac{g(t)}{p(t^{p+1}-1)^2(t^2+1)}, \quad (3.9)$$

where $g(t)$ is defined as in Lemma 2.1.

From Lemma 2.1 and (3.9) we know that there exists $\lambda > 1$ such that $G(t)$ is strictly increasing in $[1, \lambda]$ and strictly decreasing in $[\lambda, \infty)$. Then (3.8) leads to that

$$G(t) > 0, \quad (3.10)$$

for $t > 1$.

Therefore, the inequality (3.5) follows from (3.6), (3.7), and (3.10).

Finally, we prove that $J_{2/(\pi-2)}(a, b)$ and $J_2(a, b)$ are the best possible lower and upper one-parameter mean bounds for the Seiffert mean $T(a, b)$, respectively.

Let $p = 2/(\pi - 2)$, $0 < \varepsilon < 2$ and $x > 0$. Then from (1.1) and (1.2) one has

$$\lim_{x \rightarrow +\infty} \frac{J_{p+\varepsilon}(x, 1)}{T(x, 1)} = \frac{p+\varepsilon}{p+\varepsilon+1} \times \frac{\pi}{2} > \frac{p}{p+1} \times \frac{\pi}{2} = 1, \quad (3.11)$$

$$T(1+x, 1) - J_{2-\varepsilon}(1+x, 1) = \frac{h(x)}{2(3-\varepsilon) \left[(1+x)^{2-\varepsilon} - 1 \right] \arctan(x/(2+x))}, \quad (3.12)$$

where

$$h(x) = (3-\varepsilon)x \left[(1+x)^{2-\varepsilon} - 1 \right] - 2(2-\varepsilon) \left[(1+x)^{3-\varepsilon} - 1 \right] \arctan \frac{x}{2+x}. \quad (3.13)$$

Letting $x \rightarrow 0$ and making use of Taylor expansion we get

$$\begin{aligned} h(x) &= (3-\varepsilon)x \left[(2-\varepsilon)x + \frac{(2-\varepsilon)(1-\varepsilon)}{2}x^2 - \frac{\varepsilon(1-\varepsilon)(2-\varepsilon)}{6}x^3 + o(x^3) \right] \\ &\quad - 2(2-\varepsilon) \left[(3-\varepsilon)x + \frac{(3-\varepsilon)(2-\varepsilon)}{2}x^2 + \frac{(1-\varepsilon)(2-\varepsilon)(3-\varepsilon)}{6}x^3 + o(x^3) \right] \\ &\quad \times \left[\frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{12} + o(x^3) \right] = \frac{1}{12}\varepsilon(2-\varepsilon)(3-\varepsilon)x^4 + o(x^4). \end{aligned} \quad (3.14)$$

The inequality (3.11) implies that for any $0 < \varepsilon < 2$, there exists $X = X(\varepsilon) > 1$, such that $T(x, 1) < J_{2/(\pi-2)+\varepsilon}(x, 1)$ for $x \in (X, +\infty)$.

Equations (3.12)–(3.14) imply that for any $0 < \varepsilon < 2$, there exists $\delta = \delta(\varepsilon) > 0$ such that $T(1+x, 1) > J_{2-\varepsilon}(1+x, 1)$ for $x \in (0, \delta)$. $\square$

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