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Research Article

Rational Biparameter Homotopy Perturbation Method and Laplace-Padé Coupled Version

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The fact that most of the physical phenomena are modelled by nonlinear differential equations underlines the importance of having reliable methods for solving them. This work presents the rational biparameter homotopy perturbation method (RBHPM) as a novel tool with the potential to find approximate solutions for nonlinear differential equations. The method generates the solutions in the form of a quotient of two power series of different homotopy parameters. Besides, in order to improve accuracy, we propose the Laplace-Padé rational biparameter homotopy perturbation method (LPRBHPM), when the solution is expressed as the quotient of two truncated power series. The usage of the method is illustrated with two case studies. On one side, a Ricatti nonlinear differential equation is solved and a comparison with the homotopy perturbation method (HPM) is presented. On the other side, a nonforced Van der Pol Oscillator is analysed and we compare results obtained with RBHPM, LPRBHPM, and HPM in order to conclude that the LPRBHPM and RBHPM methods generate the most accurate approximated solutions.

1. Introduction

Solving nonlinear differential equations is an important issue in sciences because many physical phenomena are modelled using such classes of equations. One of the most powerful methods to approximately solve nonlinear differential equations is the homotopy perturbation method (HPM) [1–45]. The HPM method is based on the use of a power series, which transforms the original nonlinear differential equation into a series of linear differential equations. In this work, we propose a generalization of the aforementioned concept by using

a quotient of two power series of different homotopy parameters, which will be denominated as the rational biparameter homotopy perturbation method (RBHPM). In the same fashion, like HPM, the use of this quotient transforms the nonlinear differential equation into a series of linear differential equations. The generated solutions are expressed as the quotient of two truncated power series and they constitute the approximate solutions. Besides, we propose an after-treatment to the approximate solutions with the Laplace-Padé (LP) transform [46] in order to improve the accuracy of the solutions. This coupled method will be denominated as the LPRBHPM. In addition, the method is applied to two case studies, a Ricatti nonlinear differential equation [47] and a Van der Pol Oscillator [8, 48], without external forcing.

This paper is organized as follows. In Section 2, the basic idea of the RBHPM method is given. Section 3 presents a convergence analysis for the proposed method. In Section 4, the basic concept of Padé approximants is explained. In Section 5, the coupling of the RBHPM method with the Laplace-Padé transform is recast. In Section 6, the first case study, a Riccati nonlinear differential equation, is solved by using the proposed method and HPM. In Section 7, the second case study, a nonlinear oscillator problem, is treated. Section 8 is devoted to discuss the resulting solutions and some numeric analysis issues. Finally, a brief conclusion is given in Section 9.

2. Basic Concept of RBHPM

The RBHPM and HPM methods share common foundations. For both methods, it can be considered that a nonlinear differential equation can be expressed as

$$A(u) - f(r) = 0, \quad \text{where } r \in \Omega \tag{2.1}$$

with the boundary condition given by

$$B\left(u, \frac{\partial u}{\partial \eta}\right) = 0, \quad \text{where } r \in \Gamma, \tag{2.2}$$

where A is a general differential operator, $f(r)$ is a known analytic function, B is a boundary operator, Γ is the boundary of the domain Ω, and $\partial u/\partial \eta$ denotes differentiation along the normal drawn outwards from Ω [49]. In most cases, the A operator can be split into two operators, namely L and N, which represent the linear and the nonlinear operators, respectively. Hence, (2.1) can be rewritten as

$$L(u) + N(u) - f(r) = 0. \tag{2.3}$$

Now, a homotopy formulation can be given as

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p(L(v) + N(v) - f(r)) = 0, \quad p \in [0,1], \tag{2.4}$$

where u_0 is the trial function (initial approximation) for (2.3) that satisfies the boundary conditions, and p is known as the perturbation homotopy parameter.

The equation above exhibits specific behaviors at the limit values $p = 0$ and $p = 1$, as given in

$$H(v,0) = L(v) - L(u_0) = 0,$$
$$H(v,1) = L(v) + N(v) - f(r) = 0. \tag{2.5}$$

For the HPM method [11–14], we assume, without loss of generality, that the solution for (2.4) can be expressed as a power series of p

$$v = p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots. \tag{2.6}$$

In the limit, when $p \to 1$, the approximate solution for (2.1) is give as

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots, \tag{2.7}$$

where $v_0, v_1, v_2, \ldots$ are unknown functions to be determined by the HPM method. The series in (2.7) is convergent in most cases [1, 2, 11, 14, 36].

For the RBHPM method, the homotopy in (2.4) can be rewritten as

$$H(v,p) = (1-Q)[L(v) - L(u_0)] + Q\big(L(v) + N(v) - f(r)\big) = 0, \quad p \in [0,1],\ q \in [0,1], \tag{2.8}$$

$$Q = ap + (1-a)q, \quad 0 < a < 1, \tag{2.9}$$

where p and q are the homotopy parameters, and a is a weight factor that affects them in complementary proportions.

Now, we assume that the solution for (2.8) can be written as the quotient of the power series of both homotopy parameters:

$$v = \frac{p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots}{q^0 w_0 + q^1 w_1 + q^2 w_2 + \cdots}, \tag{2.10}$$

where $v_0, v_1, v_2, \ldots$ and $w_1, w_2, \ldots$ are unknown functions to be determined by the RBHPM method. In addition, w_0 is an arbitrary trial function, which is chosen in order to improve the RBHPM convergence in the same way as the trial function u_0 does for the HPM method [50].

On one side, the order of the approximation for the HPM method is determined by the highest power of p considered in the formulation. On the other side, the order for the RBHPM method is given as $[i,k]$, where i and k are the highest power of p and q employed in the numerator and denominator of (2.10), respectively.

The limit of (2.10), when $p \to 1$ and $q \to 1$, provides an approximate solution for (2.3) given as

$$u = \lim_{\substack{p \to 1 \\ q \to 1}} v = \frac{v_0 + v_1 + v_2 + \cdots}{w_0 + w_1 + w_2 + \cdots}. \tag{2.11}$$

The limit above exists in the case that both limits

$$\lim_{p\to 1}\left(\sum_{i=0}^{\infty} v_i\right),$$
$$\lim_{q\to 1}\left(\sum_{i=0}^{\infty} w_i\right), \quad \text{where } \sum_{i=0}^{\infty} w_i \neq 0 \tag{2.12}$$

exist.

3. Convergence of RBHPM Method

In order to analyse the convergence of the RBHPM method, (2.8) is rewritten as

$$L(v) = L(u_0) + Q\left[f(r) - N(v) - L(u_0)\right] = 0. \tag{3.1}$$

After applying the inverse operator, L^{-1}, on both sides of (3.1), we obtain

$$v = u_0 + Q\left[L^{-1}f(r) - L^{-1}N(v) - u_0\right]. \tag{3.2}$$

By assuming that (see (2.10))

$$v = \frac{\sum_{i=0}^{\infty} p^i v_i}{\sum_{i=0}^{\infty} q^i w_i}, \tag{3.3}$$

and after substituting (3.3) in the right-hand side of (3.2) in the following form:

$$v = u_0 + Q\left[L^{-1}f(r) - \left(L^{-1}N\right)\left[\frac{\sum_{i=0}^{\infty} p^i v_i}{\sum_{i=0}^{\infty} q^i w_i}\right] - u_0\right]. \tag{3.4}$$

The exact solution of (2.3) is obtained in the limit when $p \to 1$ and $q \to 1$ in (3.4), which results in

$$\begin{aligned} u &= \lim_{\substack{p\to 1\\ q\to 1}}\left(QL^{-1}f(r) - Q\left(L^{-1}N\right)\left[\frac{\sum_{i=0}^{\infty} p^i v_i}{\sum_{i=0}^{\infty} q^i w_i}\right] + u_0 - Qu_0\right), \\ &= L^{-1}f(r) - \left[\sum_{i=0}^{\infty}\left(L^{-1}N\right)\left(\frac{v_i}{\beta}\right)\right], \quad \beta = \sum_{i=0}^{\infty} w_i, \end{aligned} \tag{3.5}$$

where

$$\lim_{\substack{p\to 1\\ q\to 1}} Q = \lim_{\substack{p\to 1\\ q\to 1}} (ap + (1-a)q) = 1. \tag{3.6}$$

In order to study the convergence of the RBHPM method, we use the Banach Theorem as reported in [1, 2, 7, 36]. This theorem relates the solution of (2.3) to the fixed point problem of the nonlinear operator N.

Theorem 3.1 (Sufficient Condition for Convergence). *Suppose that X and Y are Banach spaces and $N : X \to Y$ is a contractive nonlinear mapping, that is,*

$$\forall w, w^* \in X; \quad \|N(w) - N(w^*)\| \leq \gamma \|w - w^*\|; \quad 0 < \gamma < 1. \tag{3.7}$$

Then, according to Banach Fixed Point Theorem, N has a unique fixed point u, that is, $N(u) = u$. Assume that the sequence generated by the RBHPM method can be written as

$$W_n = N(W_{n-1}), \quad W_{n-1} = \sum_{i=0}^{n-1} \left(\frac{v_i}{\beta} \right), \quad n = 1, 2, 3 \ldots, \tag{3.8}$$

and suppose that $W_0 = v_0/\beta \in B_r(u)$, where $B_r(u) = \{w^* \in X | \|w^* - u\| < r\}$, then, under these conditions:

(i) $W_n \in B_r(u)$,

(ii) $\lim_{n\to\infty} W_n = u$.

Proof. (i) By inductive approach, for $n = 1$ we have

$$\|W_1 - u\| = \|N(W_0) - N(u)\| \leq \gamma \|w_0 - u\|. \tag{3.9}$$

Assuming that $\|W_{n-1} - u\| \leq \gamma^{n-1} \|w_0 - u\|$, as induction hypothesis, then

$$\|W_n - u\| = \|N(W_{n-1}) - N(u)\| \leq \gamma \|W_{n-1} - u\| \leq \gamma^n \|w_0 - u\|. \tag{3.10}$$

Using (i), we have

$$\|W_n - u\| \leq \gamma^n \|w_0 - u\| \leq \gamma^n r < r \Longrightarrow W_n \in B_r(u). \tag{3.11}$$

(ii) Because of $\|W_n - u\| \leq \gamma^n \|w_0 - u\|$ and $\lim_{n\to\infty} \gamma^n = 0$, $\lim_{n\to\infty} \|W_n - u\| = 0$, that is,

$$\lim_{n\to\infty} W_n = u. \tag{3.12}$$

□

4. Padé Approximants

A rational approximation to $f(x)$ on $[a,b]$ is the quotient of two polynomials $P_N(x)$ and $Q_M(x)$ of degrees N and M, respectively. We use the notation $R_{N,M}(x)$ to denote this quotient. The $R_{N,M}(x)$ Padé approximations [51, 52] to a function $f(x)$ are given by

$$R_{N,M}(x) = \frac{P_N(x)}{Q_M(x)}, \quad \text{for } a \le x \le b. \tag{4.1}$$

The method of Padé requires $f(x)$ and its derivative to be continuous at $x = 0$. The polynomials used in (4.1) are expressed as

$$\begin{aligned} P_N(x) &= p_0 + p_1 x + p_2 x^2 + \cdots + p_N(x), \\ Q_M(x) &= q_0 + q_1 x + q_2 x^2 + \ \cdots + q_M(x). \end{aligned} \tag{4.2}$$

The polynomials in (4.2) are constructed so that $f(x)$ and $R_{N,M}(x)$ agree at $x = 0$ as well as their derivatives up to $N + M$ agree at $x = 0$. A special case occurs for $Q_0(x) = 1$, wherein the approximation in (4.1) becomes the Maclaurin expansion for $f(x)$. For a fixed value of $N + M$ the error is smallest when $P_N(x)$ and $Q_M(x)$ have the same degree or when $P_N(x)$ has degree one higher than $Q_M(x)$.

Notice that the constant coefficient of Q_M is $q_0 - 1$. This is permissible, because it can be noted that 0 and $R_{N,M}(x)$ are not changed when both $P_N(x)$ and $Q_M(x)$ are divided by the same constant. Hence the rational function $R_{N,M}(x)$ has $N + M + 1$ unknown coefficients. Assume that $f(x)$ is analytic and has the Maclaurin expansion

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots . \tag{4.3}$$

And from the difference $f(x)Q_M(x) - P_N(x) = Z(x)$

$$\left[\sum_{i=0}^{\infty} a_i x^i\right]\left[\sum_{i=0}^{M} q_i x^i\right] - \left[\sum_{i=0}^{N} p_i x^i\right] = \left[\sum_{i=N+M+1}^{\infty} c_i x^i\right]. \tag{4.4}$$

The lower index $j = N + M + 1$ in the summation on the right side of (4.4) is chosen because the first $N + M$ derivatives of $f(X)$ and $R_{N,M}(x)$ should agree at $x = 0$.

When the left side of (4.4) is multiplied out and the coefficients of the powers of x^i are set equal to zero for $k = 0, 1, 2, \ldots, N + M$, the result is a system of $N + M + 1$ linear equations:

$$\begin{gathered} a_0 - p_0 = 0, \\ q_1 a_0 + a_1 - p_1 = 0, \\ q_2 a_0 + q_1 a_1 + a_2 - p_2 = 0, \end{gathered}$$

$$q_3 a_0 + q_2 a_1 + q_1 a_2 + a_2 - p_3 = 0,$$
$$q_M a_{N-M} \big| q_{M-1} a_{N-M-1} \big| a_N - p_N = 0, \tag{4.5}$$

$$q_M a_{N-M+1} + q_{M-1} a_{N-M+2} + \cdots + q_1 a_N + a_{N+2} = 0,$$
$$q_M a_{N-M+2} + q_{M-1} a_{N-M+3} + \cdots + q_1 a_{N+1} + a_{N+3} = 0, \tag{4.6}$$
$$\vdots$$
$$q_M a_N + q_{M-1} a_{N+1} + \cdots + q_1 a_{N+M+1} + a_{N+M} = 0.$$

Notice that the sum of the subscripts of the terms of each product is the same in each equation, and it increases consecutively from 0 to $N + M$. The M equations in (4.6) involve only the unknowns $q_1, q_2, \ldots, q_M$ and must be firstly solved. Then the equations in (4.5) are used successively to find $p_1, p_2, \ldots, p_N$ [51].

5. Laplace-Padé Transform and RBHPM Method Coupling

The coupling of Laplace transform and Padé approximant [46] is used in order to recover part of the lost information due to the truncated power series [51, 53–60]. The process can be recast as follows.

(1) First, Laplace transformation is applied to power series.

(2) Next, s is substituted by $1/x$ in the resulting equation.

(3) After that, we convert the transformed series into a meromorphic function by forming its Padé approximant of order $[N/M]$. N and M are arbitrarily chosen, but they should be of smaller value than the order of the power series. In this step, the Padé approximant extends the domain of the truncated series solution to obtain better accuracy and convergence.

(4) Then, x is substituted by $1/s$.

(5) Finally, by using the inverse Laplace s transformation, we obtain the modified approximate solution.

In order to improve the approximate rational solutions generated by the RBHPM method, we propose to apply the Laplace-Padé method, separately, to the denominator and the numerator in (2.11), only when they are expressed as power series. We will denominate to this process as the Laplace-Padé rational biparameter homotopy perturbation method (LPRBHPM).

6. Case Study 1: Riccati Nonlinear Differential Equation

Consider the Riccati nonlinear differential equation [7, 47]

$$y'(x) - y(x)^2 + 1 = 0, \qquad y(0) = 0, \tag{6.1}$$

having the exact solution given as

$$y(x) = -\tanh(x). \tag{6.2}$$

6.1. Solution Calculated by RBHPM

We establish the following homotopy equation:

$$\left(1 - \left(ap + (1-a)q\right)\right)\left(L(v) - L(u_0)\right) + \left(ap + (1-a)q\right)\left(v' - v^2 + 1\right) = 0, \tag{6.3}$$

where we have chosen $a = 0.25$. Besides, the linear operator L is given as

$$L(v) = v' + v + 1, \tag{6.4}$$

and the trial function is

$$u_0 = -1 + \exp(-x). \tag{6.5}$$

We suppose that the solution of (6.3) is of order $[2,2]$, which is expressed as follows

$$v = \frac{v_0 + v_1 p + v_2 p^2}{w_0 + w_1 q + w_2 q^2}, \tag{6.6}$$

where the trial w-function is chosen as $w_0 = 1$.

After substituting (6.6) into (6.3), regrouping, and equating the terms having the following powers: p^0q^0, p^1, p^2, q^1, and q^2, it can be solved for v_0, v_1, v_2, w_1, and w_2. In order to fulfil the initial conditions of (6.1), it follows that $v_0(0) = 0, v_1(0) = 0, v_2(0) = 0, w_1(0) = 0$, and $w_2(0) = 0$.

The results are recast in the following system of differential equations:

$$\begin{aligned}
p^0q^0 &: w_0v_0' + v_0w_0 - v_0w_0' + w_0^2 = 0, \qquad v_0(0) = 0,\\
p^1 &: w_0v_1' + v_1w_0 - v_1w_0' - av_0w_0 - av_0^2 = 0, \qquad v_1(0) = 0,\\
p^2 &: w_0v_2' - 2av_0v_1 + v_2w_0 - v_2w_0' - av_1w_0 = 0, \qquad v_2(0) = 0,\\
q^1 &: -v_0w_1' - v_0w_0 - v_0^2 + 2w_0w_1 + v_0'w_1 + av_0^2 + av_0w_0 + v_0w_1 = 0, \qquad w_1(0) = 0,\\
q^2 &: -v_0w_2' + 2w_0w_2 + av_0w_1 + w_1^2 + v_0w_2 + v_0'w_2 - v_0w_1 = 0, \qquad w_2(0) = 0.
\end{aligned} \tag{6.7}$$

Solving (6.7) yields

$$\begin{aligned}
v_0 &= -1 + \exp(-x),\\
v_1 &= a\left(-x - \exp(-x) + 1\right)\exp(-x),
\end{aligned}$$

$$
\begin{aligned}
v_2 &= -a^2\left(-\frac{1}{2}x^2 + \exp(-x) + x - 2x\exp(-x) - \exp(-2x)\right)\exp(-x),\\
w_1 &= \frac{(1-a)\left(-x-\exp(-x)\right) - a + 1}{\exp(x) - 1},\\
w_2 &= \begin{cases} 0, & x = 0,\\ \dfrac{\left((x-2)\exp(x) + x + 2\right)(-1+a)^2 x}{2\left(\exp(x)-1\right)^2}, & x \neq 0.\end{cases}
\end{aligned}
\tag{6.8}
$$

By substituting (6.8) into (6.6), and calculating the limits when $p \to 1$ and $q \to 1$, we obtain

$$
y(x) = \lim_{\substack{p\to 1\\ q\to 1}} v = \frac{v_0 + v_1 + v_2}{w_0 + w_1 + w_2}. \tag{6.9}
$$

6.2. Solution by HPM

We establish the following homotopy equation:

$$
(1-p)(L(v) - L(u_0)) + p\left(v' - v^2 + 1\right) = 0, \tag{6.10}
$$

where the linear operator L and the trial function u_0 are (6.4) and (6.5), respectively.

Substituting (2.6) into (6.10), regrouping, and equating the terms with identical powers of p, it can be solved for v_0, v_1, v_2, and so on (in order to fulfil initial conditions from $v(0) = y(0) = 0$, it follows that $v_0(0) = 0, v_1(0) = 0, v_2(0) = 0$ and so on).

The result is recast in the following system of differential equations:

$$
\begin{aligned}
&p^0 : v_0' + v_0 + 1 = 0, \qquad v_0(0) = 0,\\
&p^1 : v_1' + v_1 - v_0^2 - v_0 = 0, \qquad v_1(0) = 0,\\
&p^2 : v_2' + v_2 - 2v_0v_1 - v_1 = 0, \qquad v_2(0) = 0,\\
&p^3 : v_3' + v_3 - v_1^2 - v_2 - 2v_0v_2 = 0, \qquad v_3(0) = 0,\\
&p^4 : v_4' + v_4 - 2v_1v_2 - v_3 - 2v_0v_3 = 0, \qquad v_4(0) = 0,\\
&\qquad\vdots
\end{aligned}
\tag{6.11}
$$

By solving (6.11), we obtain

$$
\begin{aligned}
v_0 &= -1 + \exp(-x),\\
v_1 &= -\left(x + \exp(-x) - 1\right)\exp(-x),\\
v_2 &= \frac{1}{2}\left((4x-2)\exp(-x) + 2\exp(-2x) + x^2 - 2x\right)\exp(-x),
\end{aligned}
$$

$$
\begin{aligned}
v_3 &= -\frac{1}{6}\Big(-12x\exp(-x) + x^3 - 3x^2 - 6\exp(-2x) + 12x^2\exp(-x) \\
&\qquad + 18x\exp(-2x) + 6\exp(-3x)\Big)\exp(-x), \\
v_4 &= \frac{1}{24}\Big(-4x^3 - 48x^2\exp(-x) - 72x\exp(-2x) - 24\exp(-3x) \\
&\qquad + x^4 + 96x\exp(-3x) + 108x^2\exp(-2x) + 32x^3\exp(-x) + 24\exp(-4x)\Big)\exp(-x), \\
&\qquad \vdots
\end{aligned}
\tag{6.12}
$$

By using (6.12) and (2.6) and calculating the limit when $p \to 1$, we obtain the second and four order approximations

$$y(x) = \lim_{p\to 1}\left(\sum_{i=0}^{2} v_i p^i\right) = v_0 + v_1 + v_2, \tag{6.13}$$

$$y(x) = \lim_{p\to 1}\left(\sum_{i=0}^{4} v_i p^i\right) = v_0 + v_1 + v_2 + v_3 + v_4, \tag{6.14}$$

respectively.

7. Case Study 2: Van Der Pol Oscillator

Consider the Van der Pol Oscillator problem [8, 48] without external forcing

$$u'' + u' + u + u^2u' = 0, \qquad u(0) = 0, \qquad u'(0) = 1, \tag{7.1}$$

7.1. Solution by the RBHPM Method

We establish the following homotopy equation:

$$\big(1 - (ap + (1-a)q)\big)(L(v) - L(u_0)) + \big(ap + (1-a)q\big)\left(v'' + v' + v + v^2v'\right) = 0, \tag{7.2}$$

where the linear operator L is

$$L(v) = v'', \tag{7.3}$$

and the trial function is

$$u_0 = t. \tag{7.4}$$

We assume that the solution for (7.2) is order [2, 2], which is expressed as follows:

$$v = \frac{v_0 + v_1 p + v_2 p^2}{w_0 + w_1 q + w_2 q^2}, \tag{7.5}$$

where the trial w-function is chosen as $w_0 = 1$.

Substituting (7.5) into (7.2), regrouping, and equating the terms having the following powers: p^0q^0, p^1, p^2, q^1, and q^2, it can be solved for v_0, v_1, v_2, w_1, and w_2. In order to fulfil the initial conditions of (7.1), it follows that $v_0(0) = 0$, $v_0'(0) = 1$ and the rest are $v_i(0) = 0$, $v_i'(0) = 0$, $w_i(0) = 0$ and $w_i'(0) = 0$, $(i = 1, 2, 3, \ldots)$. Then, considering that $w_0 = 1$, we establish the following system:

$$\begin{aligned}
&p^0 : v_0'' = 0, \qquad v_0(0) = 0, \qquad v_0'(0) = 1,\\
&p^1 : v_1'' + av_0^2v_0' + av_0 + av_0' = 0, \qquad v_1(0) = 0, \qquad v_1'(0) = 0,\\
&p^2 : v_2'' + av_0^2v_1' + av_1' + 2av_0v_1v_0' + av_1 = 0, \qquad v_2(0) = 0, \qquad v_2'(0) = 0,\\
&q^1 : -v_0w_1'' + 3v_0''w_1 + v_0' + v_0 - 2v_0'w_1' - av_0^2v_0'\\
&\quad - av_0 - av_0' + v_0^2v_0' = 0, \qquad w_1(0) = 0, \qquad w_1'(0) = 0,\\
&q^2 : -v_0w_2'' + v_0^2v_0'w_1 + av_0^3w_1' - v_0^3w_1' - av_0^2v_0'w_1 + av_0w_1' - 3av_0w_1 - 4v_0'w_1'w_1\\
&\quad - 2v_0w_1''w_1 - 3av_0'w_1 - 2v_0'w_2' + 2v_0\left(w_1'\right)^2 + 3v_0w_1 + 3v_0'w_1\\
&\quad + 3v_0''w_1^2 - v_0w_1' + 3v_0''w_2 = 0, \qquad w_2(0) = 0, \qquad w_2'(0) = 0.
\end{aligned} \tag{7.6}$$

By solving (7.6), we obtain

$$\begin{aligned}
v_0 &= t,\\
v_1 &= -\frac{1}{12}at^2\left(t^2 + 2t + 6\right),\\
v_2 &= \frac{1}{2520}a^2t^3\left(30t^4 + 77t^3 + 315t^2 + 210t + 420\right),\\
w_1 &= \frac{1}{12}(1-a)t^3 + \frac{1}{6}(1-a)t^2 - \frac{1}{2}at + k_1,\\
w_2 &= -\frac{5}{1008}(-1+a)^2t^6 - \frac{1}{360}(-1+a)^2t^5\\
&\quad + \frac{1}{5040}\left(-28a + 98 - 70a^2\right)t^4 + \frac{1}{12}(-1+a)\left(a - \frac{1}{6} + k_1\right)t^3\\
&\quad + \frac{1}{5040}\left(420a^2 + (-840 - 840k_1)a + 840k_1\right)t^2 - \frac{1}{2}k_1(a-3)t + k_2,
\end{aligned} \tag{7.7}$$

where k_1 and k_2 are integration constants.

Substituting (7.7) into (7.5), and calculating the limits when $p \to 1$ and $q \to 1$, we obtain

$$u(t) = \lim_{\substack{p\to 1\\ q\to 1}} v = \frac{v_0 + v_1 + v_2}{w_0 + w_1 + w_2}. \tag{7.8}$$

We select the parameters as $a = 27/100$, $k_1 = 22/53$, and $k_2 = -17/39$ by using the numerical procedure reported in [35–37].

In order to guarantee the validity of the approximate solution (7.8) for large t, the quotient of series solutions is transformed by the Laplace-Padé after-treatment. The procedure is applied separately to numerator and denominator of the expression in (7.8). First, Laplace transformation is applied to numerator of (7.8) and then $1/t$ is written in place of s in the equation. Then, the Padé approximant [3/3] is applied and $1/s$ is written in place of t. Finally, by using the inverse Laplace s transformation, we obtain the modified approximate solution for numerator

$$\begin{aligned} u_n(t) = &-0.0566603453624\exp(-1.71426194764t) \\ &+0.0566603453622\exp(-0.0402536974457t)\cos(0.736244931229t) \\ &+1.22941439189\exp(-0.0402536974457t)\sin(0.736244931229t). \end{aligned} \tag{7.9}$$

Now, we repeat the same process to the denominator by changing the order of the Padé approximant to [3/4], to obtain

$$\begin{aligned} u_d(t) = &\,0.003937675067\exp(-0.535832417112t)\cos(2.71442186582t) \\ &+0.003048633725\exp(-0.535832417112t)\sin(2.71442186582t) \\ &+0.228229226307\exp(-0.192147218554t)+0.747030002351\exp(0.628210432031t). \end{aligned} \tag{7.10}$$

Then, the approximate solution calculated by LPRBHPM is

$$u(t) = \frac{u_n(t)}{u_d(t)}. \tag{7.11}$$

7.2. Solution by HPM

We establish the following homotopy equation:

$$(1-p)(L(v)-L(u_0)) + p\left(v'' + v' + v + v^2 v'\right) = 0, \tag{7.12}$$

where the linear operator L and the trial function u_0 are (7.3) and (7.4), respectively.

Substituting (2.6) into (7.12), regrouping, and equating the terms with identical powers of p. In order to fulfil initial conditions of (7.1), it follows that $v_0(0) = 0, v_0'(0) = 1$,

and the rest are $v_i(0) = 0$ and $v_i'(0) = 0$ $(i = 1, 2, 3, \ldots)$. Then, we establish the following system:

$$\begin{aligned}
&p^0 : v_0'' = 0, \qquad v_0(0) = 0, \qquad v_0'(0) = 1,\\
&p^1 : v_1'' + v_0 + v_0^2 v_0' + v_0' = 0, \qquad v_1(0) = 0, \qquad v_1'(0) = 0,\\
&p^2 : v_2'' + v_0^2 v_1' + v_1 + 2v_0 v_1 v_0' + v_1' = 0, \qquad v_2(0) = 0, \qquad v_2'(0) = 0,\\
&p^3 : v_3'' + v_0^2 v_2' + 2v_0 v_1 v_1' + v_1^2 v_0' + v_2' + v_2 + 2v_0 v_2 v_0' = 0, \qquad v_3(0) = 0, \qquad v_3'(0) = 0,\\
&p^4 : v_4'' + 2v_1 v_2 v_0' + v_1^2 v_1' + 2v_0 v_3 v_0' + v_3' + v_0^2 v_3'\\
&\qquad + 2v_0 v_1 v_2' + 2v_0 v_2 v_1' + v_3 = 0, \qquad v_4(0) = 0, \qquad v_4'(0) = 0.
\end{aligned} \tag{7.13}$$

By solving (7.13), we obtain

$$\begin{aligned}
v_0 &= t,\\
v_1 &= -\frac{1}{12}t^4 - \frac{1}{6}t^3 - \frac{1}{2}t^2,\\
v_2 &= \frac{1}{84}t^7 + \frac{11}{360}t^6 + \frac{1}{8}t^5 + \frac{1}{12}t^4 + \frac{1}{6}t^3,\\
v_3 &= -\frac{19}{10080}t^{10} - \frac{67}{10080}t^9 - \frac{53}{1680}t^8 - \frac{31}{720}t^7 - \frac{67}{720}t^6 - \frac{1}{40}t^5 - \frac{1}{24}t^4,\\
v_4 &= \frac{737}{2358720}t^{13} + \frac{1889}{1330560}t^{12} + \frac{8347}{1108800}t^{11} + \frac{14039}{907200}t^{10}\\
&\quad + \frac{13921}{362880}t^9 + \frac{11}{336}t^8 + \frac{25}{504}t^7 + \frac{1}{180}t^6 + \frac{1}{120}t^5.
\end{aligned} \tag{7.14}$$

By substituting the solutions from (7.14) into (2.6) and calculating the limit when $p \to 1$, we can obtain the second order approximation

$$u(t) = \lim_{p \to 1}\left(\sum_{i=0}^{2} v_i p^i\right) = t - \frac{1}{2}t^2 + \frac{1}{84}t^7 + \frac{11}{360}t^6 + \frac{1}{8}t^5. \tag{7.15}$$

In the same way, we obtain the four order approximation

$$\begin{aligned}
u(t) = \lim_{p \to 1}\left(\sum_{i=0}^{4} v_i p^i\right) &= t - \frac{1}{2}t^2 + \frac{31}{1680}t^7 - \frac{41}{720}t^6 + \frac{13}{120}t^5 + \frac{12329}{907200}t^{10}\\
&\quad + \frac{11509}{362880}t^9 + \frac{1}{840}t^8 - \frac{1}{24}t^4 + \frac{737}{2358720}t^{13}\\
&\quad + \frac{1889}{1330560}t^{12} + \frac{8347}{1108800}t^{11}.
\end{aligned} \tag{7.16}$$

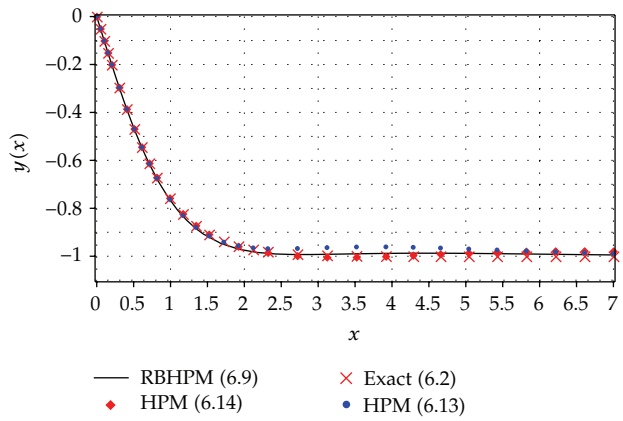

Figure 1: Exact solution (6.2) (diagonal cross) for (6.1) and its approximate solutions (6.9) (solid line), (6.13) (solid circles), and (6.14) (solid diamonds).

Table 1: Comparison between exact solution (6.2) for (6.1) to the results of approximations (6.9), (6.13), and (6.14).

Order x	Exact (6.2)	Homotopy (6.3) [2, 2] RBHPM (6.9)	Homotopy (6.10) 2 HPM (6.13)	Homotopy (6.10) 4 HPM (6.14)
0.00	0.000000000000	0.000000000000	0.000000000000	0.000000000000
0.50	−0.462117157260	−0.462720433177	−0.462402288846	−0.462119211588
1.50	−0.905148253645	−0.918326771288	−0.911212665075	−0.90475203875
2.00	−0.964027580076	−0.975747936207	−0.960889970046	−0.965901380964
2.50	−0.986614298151	−0.991622046961	−0.968972449805	−0.992586779171
3.50	−0.998177897611	−0.989972317008	−0.961440997373	−1.00443324553
4.00	−0.999329299739	−0.98793701472	−0.961349802243	−1.0000486249
4.50	−0.999753210848	−0.987368801973	−0.96441914623	−0.993370392728
5.00	−0.999909204263	−0.988053048524	−0.969315733164	−0.986865576296
5.50	−0.999966597156	−0.98949557545	−0.974818141376	−0.981957532724
6.00	−0.999987711651	−0.991249069742	−0.980108525233	−0.979148093068
6.50	−0.999995479351	−0.993004075085	−0.984752811149	−0.97832221949
7.00	−0.999998336944	−0.994584980578	−0.988591496328	−0.979053617564
	A.A.E.	8.15 (−3)	1.84 (−2)	8.93 (−3)

8. Numerical Simulation and Discussion

On one hand, Figure 1 and Table 1 show a comparison between the exact solution (6.2) for the Riccati nonlinear differential equation (6.1) and the analytic approximations (6.9), (6.13), and (6.14). The RBHPM method of order [2, 2] gives the smallest average absolute error (A.A.E.) of all solutions, followed by the solutions obtained with HPM of order 4 and 2. It is remarkable to observe that the RBHPM method generates an accurate rational approximation that successfully replicates the asymptotic behaviour of (6.2).

On the other hand, Figure 2 and Table 2 show a comparison between the Fehlberg fourth-fifth order Runge-Kutta method with degree four interpolant (RKF45) [61, 62] solution (built-in function of Maple software) for the Van der Pol Oscillator (7.1)

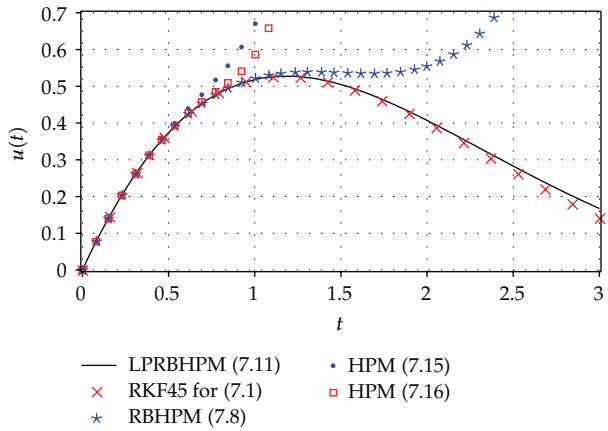

Figure 2: RKF45 solution for Van Der Pol Oscillator problem (7.1) (diagonal cross) and its approximate solutions obtained by RBHPM (7.8) (asterisk), LPRBHPM (7.11) (solid line), HPM (7.15) (solid circles), and HPM (7.16) (empty squares).

and the analytic approximations (7.8), (7.11), (7.15), and (7.16). The LPRBHPM method of order [2, 2] (see (7.11)) yields the smallest A.A.E. of all solutions, followed by (7.8) obtained by the RBHPM method of order [2, 2], and finally by approximations calculated by HPM of order 4 and 2. In this case study, both LPRBHPM and RBHPM generate more accurate solutions than HPM.

The coupling of Laplace and Padé with RBHPM was a key factor for improving the accuracy in the Van der Pol problem. Both numerator and denominator of the rational expression (7.8) were considered as truncated power series. Laplace-Padé after-treatment [46] was successfully applied, and it yields better accuracy and it allows for larger ranges of the domain. Additionally, the trial functions w_0 and u_0 play an important role in the behaviour of the RBHPM and LPRBHPM methods; therefore, future research must be done in order to understand what kinds of trial functions produce better results; in this context, the operators L and N will be a key aspect to consider. Furthermore, it is important to point out that the RBHPM and LPRBHPM methods do not resort to linearization, a perturbation parameter, or assumptions of weak nonlinearity, and it clearly results that the generated solution has a general character and it is more realistic compared to the method of simplifying the physical problems.

Finally, the homotopy in (2.4) can be replaced by homotopy schemes such as those reported in the literature [63–93]. This future line of research can lead us to improve the performance of RBHPM or LPRBHPM methods.

9. Conclusions

This paper presented the RBHPM and LPRBHPM methods as a novel tool with high potential to solve nonlinear differential equations. Furthermore, a comparison between the results of applying the proposed methods and HPM was given. For the Riccati nonlinear asymptotic problem, a comparison between the RBHPM and HPM methods was presented, showing how the RBHPM method generates highly accurate approximate solutions, similar to the results obtained when using the HPM method. Additionally, a Van der Pol Oscillator

Table 2: Comparison numerical solution (RKF45) for (7.1) and its first order approximate solutions given by RBHPM (7.8), LPRBHPM (7.11), HPM (7.15), and HPM (7.16).

Order		[2, 2]	[2, 2]	2	4
t	RKF45 for (7.1)	RBHPM (7.8)	LPRBHPM (7.11)	HPM (7.15)	HPM (7.16)
0.00	0.000000000000	0.000000000000	−2.04249011 (−13)	0.000000000000	0.000000000000
0.10	0.094996749389	0.0963305722762	0.0963305680033	0.095001281746	0.0949968616125
0.20	0.179964338949	0.181413957989	0.181413692649	0.180042107937	0.179964612584
0.30	0.254886117960	0.255755918036	0.255752978888	0.255328628571	0.254889069792
0.40	0.319824133934	0.319864038194	0.319847953401	0.321424660317	0.319850515572
0.50	0.374962243912	0.374261968751	0.374202151797	0.379476686508	0.3751195795
0.60	0.420620714410	0.419505461842	0.419331298484	0.431478857143	0.421336349604
0.70	0.457249830610	0.456198011306	0.455769940137	0.480583988889	0.459927483684
0.80	0.485408608033	0.485004250862	0.484075378004	0.531466565079	0.49405125938
0.90	0.505735525317	0.506659764874	0.504828319804	0.590743735714	0.53057647436
1.00	0.518917385708	0.521976574206	0.518630236646	0.66746031746	0.583931084348
1.10	0.525660073662	0.531844221386	0.526097880422	0.773643793651	0.683143422756
1.20	0.526664476070	0.537227026265	0.527855737463	0.924935314286	0.884097382101
1.30	0.522608206722	0.539158647966	0.524527343795	1.14130269603	1.2900013214
1.40	0.514133103782	0.538735520529	0.516726377800	1.44784142222	2.08441286596
1.50	0.501837942107	0.537111006964	0.505048313285	1.87566964286	3.58297069111
1.60	0.486275223293	0.535492254682	0.490063208382	2.4629231746	6.31238329453
1.70	0.467950980699	0.535141785803	0.472309972142	3.25585650079	11.128360908
1.80	0.447326548414	0.537385900716	0.452292231487	4.31005577143	19.3882241197
1.90	0.424821677896	0.54363212026	0.430475743238	5.69176980317	33.1990855168
2.00	0.400817960941	0.555398274173	0.407287171967	7.47936507937	55.769016169
2.10	0.375662771298	0.574356631824	0.383113985298	9.76491075	91.8967515218
2.20	0.349672525187	0.602397906631	0.358305196653	12.6558996317	148.645576503
2.30	0.323136129463	0.641722425466	0.333172699524	16.2771112079	236.259417439
2.40	0.296317411714	0.694969880802	0.307992974314	20.7726226286	369.394267712
2.50	0.269457558120	0.765406030734	0.28300899685	26.3079737103	568.756347393
2.60	0.242776634418	0.857196643743	0.258432227399	33.0724919365	863.260364544
2.70	0.216475028642	0.975820118585	0.234444603854	41.2817834571	1292.84749061
2.80	0.190734203918	1.12870912808	0.211200498849	51.1803960889	1912.13383476
2.90	0.165717437177	1.32628670185	0.188828626412	63.0446603151	2795.09702573
3.00	0.141570064350	1.58371504978	0.167433899647	77.1857142857	4041.05178634
	A.A.E.	0.231	0.00641	12.1	401

problem was tackled with the proposed methods and compared to solutions obtained by HPM. It resulted that the RBHPM and LPRBHPM methods produced the most accurate approximated solutions. Because RBHPM, LPRBHPM, and HPM are closely related, it is possible that differential equations solved by HPM can be solved by using RBHPM or LPRBHPM. Besides, further research should be done to apply the proposed methods to the calculation of approximate solutions of nonlinear partial differential equations, nonlinear fractional equations, and boundary value problems, among others. An important remark is that the RBHPM and LPRBHPM methods do not resort to any kind of linearization procedure or perturbation parameter. Thereupon, these methods promise to become important

mathematical tools, useful for scientist and engineers working in the area of mathematical modelling and computer simulation.

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