Coupled Fixed Point Theorems for a Pair of Weakly Compatible Maps along with CLRg Property in Fuzzy Metric Spaces

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The aim of this paper is to extend the notions of E.A. property and CLRg property for coupled mappings and use these notions to generalize the recent results of Xin-Qi Hu (2011). The main result is supported by a suitable example.

1. Introduction and Preliminaries

The concept of fuzzy set was introduced by Zadeh [1] and after his work there has been a great endeavor to obtain fuzzy analogues of classical theories. This problem has been searched by many authors from different points of view. In 1994, George and Veeramani [2] introduced and studied the notion of fuzzy metric space and defined a Hausdorff topology on this fuzzy metric space.

Bhaskar and Lakshmikantham [3] introduced the notion of coupled fixed points and proved some coupled fixed point results in partially ordered metric spaces. The work [3] was illustrated by proving the existence and uniqueness of the solution for a periodic boundary value problem. These results were further extended and generalized by Lakshmikantham and Cirić [4] to coupled coincidence and coupled common fixed point results for nonlinear contractions in partially ordered metric spaces.

Sedghi et al. [5] proved some coupled fixed point theorems under contractive conditions in fuzzy metric spaces. The results proved by Fang [6] for compatible and weakly
compatible mappings under \(\phi\)-contractive conditions in Menger spaces that provide a tool to Hu [7] for proving fixed points results for coupled mappings and these results are the genuine generalization of the result of [5].

Aamri and Moutawakil [8] introduced the concept of E.A. property in a metric space. Recently, Sintunavarat and Kuman [9] introduced a new concept of (CLRg). The importance of CLRg property ensures that one does not require the closeness of range subspaces.

In this paper, we give the concept of E.A. property and (CLRg) property for coupled mappings and prove a result which provides a generalization of the result of [7].

2. Preliminaries

Before we give our main result, we need the following preliminaries.

Definition 2.1 (see [1]). A fuzzy set \(A\) in \(X\) is a function with domain \(X\) and values in \([0, 1]\).

Definition 2.2 (see [10]). A binary operation \(* : [0, 1] \times [0, 1] \to [0, 1]\) is continuous \(t\)-norm, if \(([0, 1], *)\) is a topological abelian monoid with unit 1 such that \(a * b \leq c * d\) whenever \(a \leq c\) and \(b \leq d\) for all \(a, b, c, d \in [0, 1]\).

Some examples are below:

(i) \(* (a, b) = ab,\)

(ii) \(* (a, b) = \min (a, b).\)

Definition 2.3 (see [11]). Let \(\sup_{t \in (0, 1)} \Delta (t, t) = 1\). A \(t\)-norm \(\Delta\) is said to be of \(H\)-type if the family of functions \(\{\Delta^m (t)\}_{m=1}^{\infty}\) is equicontinuous at \(t = 1\), where

\[
\Delta^1 (t) = t, \quad \Delta(\Delta^m) = \Delta^{m+1}(t) = t. \tag{2.1}
\]

A \(t\)-norm \(\Delta\) is an \(H\)-type \(t\)-norm if and only if for any \(\lambda \in (0, 1)\), there exists \(\delta (\lambda) \in (0, 1)\) such that \(\Delta^m (t) > (1- \lambda)\) for all \(m \in \mathbb{N}\), when \(t > (1 - \delta)\).

The \(t\)-norm \(\Delta_M = \min\) is an example of \(t\)-norm, of \(H\)-type.

Definition 2.4 (see [2]). The 3-tuple \((X, M, *)\) is said to be a fuzzy metric space if \(X\) is an arbitrary set, \(*\) is a continuous \(t\)-norm and \(M\) is a fuzzy set on \(X^2 \times [0, \infty)\) satisfying the following conditions:

(FM-1) \(M(x, y, 0) > 0\) for all \(x, y \in X,\)

(FM-2) \(M(x, y, t) = 1\) if and only if \(x = y,\) for all \(x, y \in X\) and \(t > 0,\)

(FM-3) \(M(x, y, t) = M(y, x, t)\) for all \(x, y \in X\) and \(t > 0,\)

(FM-4) \(M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)\) for all \(x, y, z \in X\) and \(t, s > 0,\)

(FM-5) \(M(x, y, \cdot) : [0, \infty) \to [0, 1]\) is continuous for all \(x, y \in X.\)

In present paper, we consider \(M\) to be fuzzy metric space with, the following condition:

(FM-6) \(\lim_{t \to \infty} M(x, y, t) = 1,\) for all \(x, y \in X\) and \(t > 0.\)

Definition 2.5 (see [2]). Let \((X, M, *)\) be a fuzzy metric space. A sequence \(\{x_n\} \in X\) is said to be:
(i) convergent to a point \( x \in X \), if for all \( t > 0 \),
\[
\lim_{n \to \infty} M(x_n, x, t) = 1, \tag{2.2}
\]

(ii) a Cauchy sequence, if for all \( t > 0 \) and \( p > 0 \),
\[
\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1. \tag{2.3}
\]

A fuzzy metric space \((X, M, \ast)\) is said to be complete if and only if every Cauchy sequence in \( X \) is convergent.

We note that \( M(x, y, \cdot) \) is nondecreasing for all \( x, y \in X \).

**Lemma 2.6** (see [12]). Let \( x_n \to x \) and \( y_n \to y \), then for all \( t > 0 \):

(i) \( \lim_{n \to \infty} M(x_n, y_n, t) \geq M(x, y, t) \),

(ii) \( \lim_{n \to \infty} M(x_n, y_n, t) = M(x, y, t) \) if \( M(x, y, t) \) is continuous.

**Definition 2.7** (see [7]). Define \( \Phi = \{ \phi : \mathbb{R}^+ \to \mathbb{R}^+ \} \), and each \( \phi \in \Phi \) satisfies the following conditions:

(\( \phi \)-1) \( \phi \) is nondecreasing;

(\( \phi \)-2) \( \phi \) is upper semicontinuous from the right;

(\( \phi \)-3) \( \sum_{n=0}^{\infty} \phi^n(t) < +\infty \) for all \( t > 0 \), where \( \phi^{n+1}(t) = \phi(\phi^n(t)) \), \( n \in \mathbb{N} \).

Clearly, if \( \phi \in \Phi \), then \( \phi(t) < t \) for all \( t > 0 \).

**Definition 2.8** (see [4]). An element \((x, y) \in X \times X\) is called:

(i) a coupled fixed point of the mapping \( f : X \times X \to X \) if \( f(x, y) = x, f(y, x) = y \),

(ii) a coupled coincidence point of the mappings \( f : X \times X \to X \) and \( g : X \to X \) if \( f(x, y) = g(x), f(y, x) = g(y) \),

(iii) a common coupled fixed point of the mappings \( f : X \times X \to X \) and \( g : X \to X \) if \( x = f(x, y) = g(x), y = f(y, x) = g(y) \).

**Definition 2.9** (see [6]). An element \( x \in X \) is called a common fixed point of the mappings \( f : X \times X \to X \) and \( g : X \to X \) if \( x = f(x, x) = g(x) \).

**Definition 2.10** (see [6]). The mappings \( f : X \times X \to X \) and \( g : X \to X \) are called:

(i) commutative if \( gf(x, y) = fg(x, y) \) for all \( x, y \in X \),

(ii) compatible if
\[
\lim_{n \to \infty} M(gf(x_n, y_n), f(g(x_n), g(y_n)), t) = 1, \tag{2.4}
\]
\[
\lim_{n \to \infty} M(gf(y_n, x_n), f(g(y_n), g(x_n)), t) = 1,
\]
for all \( t > 0 \) whenever \([x_n]\) and \([y_n]\) are sequences in \( X \), such that \( \lim_{n \to \infty} f(x_n, y_n) = \lim_{n \to \infty} g(x_n) = x \), and \( \lim_{n \to \infty} f(y_n, x_n) = \lim_{n \to \infty} g(y_n) = y \), for some \( x, y \in X \).
Definition 2.11 (see [13]). The maps \( f : X \times X \to X \) and \( g : X \to X \) are called \( \omega \)-compatible if \( gf(x, y) = f(gx, gy) \) whenever \( f(x, y) = g(x), f(y, x) = g(y) \).

We note that the maps \( f : X \times X \to X \) and \( g : X \to X \) are called weakly compatible if

\[
f(x, y) = g(x), \quad f(y, x) = g(y),
\]

implies \( gf(x, y) = f(gx, gy), gf(y, x) = f(gy, gx) \), for all \( x, y \in X \).

There exist pair of mappings that are neither compatible nor weakly compatible, as shown in the following example.

Example 2.12. Let \((X, M, \ast)\) be a fuzzy metric space, \(\ast\) being a continuous norm with \(X = [0, 1]\). Define \(M(x, y, t) = \frac{t}{t + |x - y|}\) for all \(t > 0, x, y \in X\). Also define the maps \(f : X \times X \to X\) and \(g : X \to X\) by \(f(x, y) = (x^2/2) + (y^2/2)\) and \(g(x) = x/2\), respectively. Note that \((0, 0)\) is the coupled coincidence point of \(f\) and \(g\) in \(X\). It is clear that the pair \((f, g)\) is weakly compatible on \(X\).

We next show that the pair \((f, g)\) is not compatible.

Consider the sequences \(\{x_n\} = \{(1/2) + (1/n)\}\) and \(\{y_n\} = \{(1/2) - (1/n)\}, n \geq 3\), then

\[
\lim_{n \to \infty} f(x_n, y_n) = \frac{1}{4} = \lim_{n \to \infty} g(x_n),
\]

\[
\lim_{n \to \infty} f(y_n, x_n) = \frac{1}{4} = \lim_{n \to \infty} g(y_n),
\]

but

\[
M(f(gx_n, gy_n), gf(x_n, y_n), t) = \frac{t}{t + |f(gx_n, gy_n) - gf(x_n, y_n)|} = \frac{t}{t + (1/8)(1/2 + 2/n^2)},
\]

which is not convergent to 1 as \(n \to \infty\).

Hence the pair \((f, g)\) is not compatible.

We note that, if \(f\) and \(g\) are compatible then they are weakly compatible. But the converse need not be true, as shown in the following example.

Example 2.13. Let \((X, M, \ast)\) be a fuzzy metric space, \(\ast\) being a continuous norm with \(X = [2, 20]\). Define \(M(x, y, t) = t/(t + |x - y|)\) for all \(t > 0, x, y \in X\). Define the maps \(f : X \times X \to X\) and \(g : X \to X\) by

\[
f(x, y) = \begin{cases} 
2, & \text{if } x = 2 \text{ or } x > 5, \ y \in X, \\
6, & \text{if } 2 < x \leq 5, \ y \in X,
\end{cases}
\]

\[
g(x) = \begin{cases} 
2, & \text{if } x = 2, \\
12, & \text{if } 2 < x \leq 5, \\
x - 3, & x > 5.
\end{cases}
\]
The only coupled coincidence point of the pair \((f, g)\) is \((2, 2)\). The mappings \(f\) and \(g\) are noncompatible, since for the sequences \(\{x_n\} = \{y_n\} = \{5 + (1/n)\}, n \geq 1\) we have \(f(x_n, y_n) = 2, g(x_n) \to 2, f(y_n, x_n) = 2, g(y_n) \to 2, M(f(gx_n, gy_n), g(f(x_n, y_n)), t) = t/(t + 4) \to 1\) as \(n \to \infty\). But they are weakly compatible since they commute at their coupled coincidence point \((2, 2)\).

Now we introduce our notions.

Aamri and El Moutawakil [8] introduced the concept of E.A. property in a metric space as follows.

Let \((X, d)\) be a metric space. Self mappings \(f : X \to X\) and \(g : X \to X\) are said to satisfy E.A. property if there exists a sequence \(\{x_n\} \in X\) such that

\[
\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = t
\]

for some \(t \in X\).

Now we extend this notion for a pair of coupled maps as follows.

**Definition 2.14.** Let \((X, d)\) be a metric space. Two mappings \(f : X \times X \to X\) and \(g : X \to X\) are said to satisfy E.A. property if there exists sequences \(\{x_n\}, \{y_n\} \in X\) such that

\[
\begin{align*}
\lim_{n \to \infty} f(x_n, y_n) &= \lim_{n \to \infty} g(x_n) = x, \\
\lim_{n \to \infty} f(y_n, x_n) &= \lim_{n \to \infty} g(y_n) = y,
\end{align*}
\]

for some \(x, y \in X\).

In a similar mode, we state E.A. property for coupled mappings in fuzzy metric spaces as follows.

Let \((X, M, \ast)\) be a FM space. Two maps \(f : X \times X \to X\) and \(g : X \to X\) satisfy E.A. property if there exists sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that \(f(x_n, y_n), g(x_n)\) converges to \(x\) and \(f(y_n, x_n), g(y_n)\) converges to \(y\) in the sense of Definition 2.5.

**Example 2.15.** Let \((\mathbb{R}, \mathbb{R})\) be a usual metric space. Define mappings \(f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) and \(g : \mathbb{R} \to \mathbb{R}\) by \(f(x, y) = x^2 + y^2\) and \(g(x) = 2x\) for all \(x, y \in X\). Consider the sequences \(\{x_n\} = \{1/n\}\) and \(\{y_n\} = \{-1/n\}\). Since

\[
\begin{align*}
\lim_{n \to \infty} f(x_n, y_n) &= \lim_{n \to \infty} f\left(\frac{1}{n}, -\frac{1}{n}\right) = 0 = \lim_{n \to \infty} g\left(\frac{1}{n}\right) = \lim_{n \to \infty} g(x_n), \\
\lim_{n \to \infty} f(y_n, x_n) &= \lim_{n \to \infty} f\left(-\frac{1}{n}, \frac{1}{n}\right) = 0 = \lim_{n \to \infty} g\left(-\frac{1}{n}\right) = \lim_{n \to \infty} g(y_n),
\end{align*}
\]

therefore, \(f\) and \(g\) satisfy E.A. property, since \(0 \in X\).
Remark 2.16. It is to be noted that property E.A. need not imply compatibility, since in Example 2.12, the maps $f$ and $g$ defined are not compatible, but satisfy property E.A., since for the sequences $\{x_n\} = \{(1/2) + (1/n)\}$ and $\{x_n\} = \{(1/2) - (1/n)\}$ we have

$$
\lim_{n \to \infty} f(x_n, y_n) = \frac{1}{4} = \lim_{n \to \infty} g(x_n),
$$

(2.12)

$$
\lim_{n \to \infty} f(y_n, x_n) = \frac{1}{4} = \lim_{n \to \infty} g(y_n),
$$

since $1/4 \in X$.

Recently, Sintunavarat and Kuman [9] introduced a new concept of the common limit in the range of $g$, (CLRg) property, as follows.

Definition 2.17. Let $(X, d)$ be a metric space. Two mappings $f : X \to X$ and $g : X \to X$ are said to satisfy (CLRg) property if there exists a sequence $\{x_n\} \in X$ such that $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = g(p)$ for some $p \in X$.

Now we extend this notion for a pair of coupled mappings as follows.

Definition 2.18. Let $(X, d)$ be a metric space. Two mappings $f : X \times X \to X$ and $g : X \to X$ are said to satisfy (CLRg) property if there exists sequences $\{x_n\}, \{y_n\} \in X$ such that

$$
\lim_{n \to \infty} f(x_n, y_n) = \lim_{n \to \infty} g(x_n) = g(p),
$$

(2.13)

$$
\lim_{n \to \infty} f(y_n, x_n) = \lim_{n \to \infty} g(y_n) = g(q),
$$

for some $p, q \in X$.

Similarly, we state (CLRg) property for coupled mappings in fuzzy metric spaces.

Let $(X, M, *)$ be an FM space. Two maps $f : X \times X \to X$ and $g : X \to X$ satisfy (CLRg) property if there exists sequences $\{x_n\}, \{y_n\} \in X$ such that $f(x_n, y_n), g(x_n)$ converge to $g(p)$ and $f(y_n, x_n), g(y_n)$ converge to $g(q)$, in the sense of Definition 2.5.

Example 2.19. Let $X = [0, \infty)$ be a metric space under usual metric. Define mappings $f : X \times X \to X$ and $g : X \to X$ by $f(x, y) = x + y + 2$ and $g(x) = 2(1 + x)$ for all $x, y \in X$. We consider the sequences $\{x_n\} = \{1 + (1/n)\}$ and $\{x_n\} = \{1 - (1/n)\}$. Since

$$
\lim_{n \to \infty} f(x_n, y_n) = \lim_{n \to \infty} f\left(1 + \frac{1}{n}, 1 - \frac{1}{n}\right) = 4 = g(1) = \lim_{n \to \infty} g\left(1 + \frac{1}{n}\right) = \lim_{n \to \infty} g(x_n),
$$

(2.14)

$$
\lim_{n \to \infty} f(y_n, x_n) = \lim_{n \to \infty} f\left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right) = 4 = g(1) = \lim_{n \to \infty} g\left(1 - \frac{1}{n}\right) = \lim_{n \to \infty} g(y_n),
$$

therefore, the maps $f$ and $g$ satisfy (CLRg) property.

In the next example, we show that the maps satisfying (CLRg) property need not be continuous, that is, continuity is not the necessary condition for self maps to satisfy (CLRg) property.
Example 2.20. Let $X = [0, \infty)$ be a metric space under usual metric. Define mappings $f : X \times X \to X$ and $g : X \to X$ by

\[
f(x, y) = \begin{cases} 
  x + y, & \text{if } x \in [0,1), \ y \in X, \\
  x + \frac{y}{2}, & \text{if } x \in [1,\infty), \ y \in X,
\end{cases}
\]

\[g(x) = \begin{cases} 
  1 + x, & \text{if } x \in [0,1), \\
  \frac{x}{2}, & \text{if } x \in [1,\infty).
\end{cases}
\]  

(2.15)

We consider the sequences $\{x_n\} = \{1/n\}$ and $\{y_n\} = \{1 + (1/n)\}$. Since

\[
\lim_{n \to \infty} f(x_n, y_n) = \lim_{n \to \infty} f\left(\frac{1}{n}, 1 + \frac{1}{n}\right) = 1 = g(0) = \lim_{n \to \infty} g\left(\frac{1}{n}\right) = \lim_{n \to \infty} g(x_n),
\]

\[
\lim_{n \to \infty} f(y_n, x_n) = \lim_{n \to \infty} f\left(1 + \frac{1}{n}, \frac{1}{n}\right) = \frac{1}{2} = g(1) = \lim_{n \to \infty} g\left(1 + \frac{1}{n}\right) = \lim_{n \to \infty} g(y_n),
\]

(2.16)

therefore, the maps $f$ and $g$ satisfy $(CLRg)$ property but the maps are not continuous.

We next show that the pair of maps satisfying $(CLRg)$ property may not be compatible.

Example 2.21. Let $(X, M, *)$ be a fuzzy metric space, * being a continuous norm, $X = [0, 1/2)$, and $M(x, y, t) = t/(t + |x - y|)$ for all $x, y \in X$ and $t > 0$.

Define the maps $f : X \times X \to X$ and $g : X \to X$ by $f(x, y) = (x^2 + y^2)/2$ and $g(x) = x/3$, respectively.

Consider the sequences $\{x_n\} = \{(1/3) + (1/n)\}$ and $\{y_n\} = \{(1/3) - (1/n)\}$, $n > 7$. Then

\[
\lim_{n \to \infty} f(x_n, y_n) = \frac{1}{9} = \lim_{n \to \infty} g(x_n),
\]

\[
\lim_{n \to \infty} f(y_n, x_n) = \frac{1}{9} = \lim_{n \to \infty} g(y_n).
\]  

(2.17)

Further there exists the point $1/3$ in $X$ such that $g(1/3) = 1/9$, so that the pair $(f, g)$ satisfies $(CLRg)$ property. But,

\[
M(f(gx_n, gy_n), gf(x_n, y_n), t)
\]

\[= \frac{t}{t + |f(gx_n, gy_n) - gf(x_n, y_n)|} = \frac{t}{t + (1/18)((1/9) + (1/n^2))}
\]

(2.18)

does not converge to 1 as $n \to \infty$.

Hence, the pair $(f, g)$ is not compatible.
3. Main Results

For convenience, we denote

\[ [M(x,y,t)]^n = \frac{M(x,y,t) \ast M(x,y,t) \ast \cdots \ast M(x,y,t)}{n}, \]

for all \( n \in \mathbb{N} \).

Hu [7] proved the following result.

**Theorem 3.1.** Let \((X,M,\ast)\) be a complete fuzzy metric space where \(\ast\) is a continuous t-norm of H-type. Let \(f : X \times X \to X\) and \(g : X \to X\) be two mappings and there exists \(\phi \in \Phi\) such that

\[ M(f(x,y),f(u,v),\phi(t)) \geq M(gx,gu,t) \ast M(gy,gv,t), \]

for all \(x,y,u,v \in X\) and \(t > 0\). Suppose that \(f(X \times X) \subseteq g(X)\), \(g\) is continuous, \(f\) and \(g\) are compatible maps. Then there exists a unique point \(x \in X\) such that \(x = g(x) = f(x,x)\), that is, \(f\) and \(g\) have a unique common fixed point in \(X\).

We now give our main result which provides a generalization of Theorem 3.1.

**Theorem 3.2.** Let \((X,M,\ast)\) be a Fuzzy Metric Space, \(\ast\) being continuous t-norm of H-type. Let \(f : X \times X \to X\) and \(g : X \to X\) be two mappings and there exists \(\phi \in \Phi\) satisfying (2) with the following conditions:

(3) the pair \((f,g)\) is weakly compatible,

(4) the pair \((f,g)\) satisfy (CLRg) property.

Then \(f\) and \(g\) have a coupled coincidence point in \(X\). Moreover, there exists a unique point \(x \in X\) such that \(x = f(x,x) = g(x)\).

**Proof.** Since \(f\) and \(g\) satisfy (CLRg) property, there exists sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[ \lim_{n \to \infty} f(x_n,y_n) = \lim_{n \to \infty} g(x_n) = g(p), \quad \lim_{n \to \infty} f(y_n,x_n) = \lim_{n \to \infty} g(y_n) = g(q), \]

for some \(p,q \in X\).

**Step 1.** To show that \(f\) and \(g\) have a coupled coincidence point. From (2),

\[ M(f(x_n,y_n),f(p,q),t) \geq M(f(x_n,y_n),f(p,q),\phi(t)) \geq M(gx_n,g(p),t) \ast M(gy_n,g(q),t). \]

Taking limit \(n \to \infty\), we get \(M(g(p),f(p,q),t) = 1\), that is, \(f(p,q) = g(p) = x\).

Similarly, \(f(q,p) = g(q) = y\).
Step 2. To show that \( f(x) = x \) and \( g(y) = y \). Since \( * \) is a \( t \)-norm of \( H \)-type, for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that
\[
(1 - \delta) \cdot \cdots \cdot (1 - \delta) \geq (1 - \epsilon),
\]
for all \( p \in \mathbb{N} \).

Since \( \lim_{t \to \infty} M(x, y, t) = 1 \) for all \( x, y \in X \), there exists \( t_0 > 0 \) such that
\[
M(gx, x, t_0) \geq (1 - \delta), \quad M(gy, y, t_0) \geq (1 - \delta).
\]
(3.6)

Also since \( \phi \in \Phi \) using condition \((\phi - 3)\), we have \( \sum_{n=1}^{\infty} \phi^n(t_0) < \infty \).

Then for any \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( t > \sum_{k=n_0}^{\infty} \phi^k(t_0) \). From (2), we have
\[
M(gx, x, \phi(t_0)) = M(f(x, y), f(p, q), \phi(t_0)) \geq M(gx, gp, t_0) \ast M(gy, gq, t_0)
\]
\[
= M(gx, x, t_0) \ast M(gy, y, t_0),
\]
(3.7)

Similarly, we can also get
\[
M(gx, x, \phi^2(t_0)) = M(f(x, y), f(p, q), \phi^2(t_0))
\]
\[
\geq M(gx, gp, \phi(t_0)) \ast M(gy, gq, \phi(t_0))
\]
\[
= M(gx, x, \phi(t_0)) \ast M(gy, y, \phi(t_0))
\]
\[
\geq [M(gx, x, t_0)]^2 \ast [M(gy, y, t_0)]^2,
\]
(3.8)

Continuing in the same way, we can get for all \( n \in \mathbb{N} \),
\[
M(gx, x, \phi^n(t_0)) = M(gx, x, \phi^{n-1}(t_0)) \ast M(gy, y, \phi^{n-1}(t_0))
\]
\[
\geq M(gx, x, t_0)^{2^{n-1}} \ast M(gy, y, t_0)^{2^{n-1}},
\]
(3.9)

\[
M(gy, y, \phi^n(t_0)) \geq [M(gy, y, t_0)]^{2^{n-1}} \ast [M(gx, x, t_0)]^{2^{n-1}}.
\]
Then, we have

\[ M(gx, x, t) \geq M\left( gx, x, \sum_{k=n_0}^{\infty} \phi^k(t_0) \right) \]

\[ \geq M(gx, x, \phi^{n_0}t_0) \]

\[ \geq \left[ M(gx, x, t) \right]^{2^{n_0-1}} * \left[ M(gy, y, t_0) \right]^{2^{n_0-1}} \]

\[ \geq \left( 1 - \delta \right) \cdot \cdots \cdot \left( 1 - \delta \right) \geq (1 - \varepsilon). \]  \( (3.10) \)

So, for any \( \varepsilon > 0 \), we have \( M(gx, x, t) \geq (1 - \varepsilon) \) for all \( t > 0 \).

This implies \( g(x) = x \). Similarly, \( g(y) = y \).

**Step 3.** Next we shall show that \( x = y \). Since \( * \) is a \( t \)-norm of \( H \)-type, for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[ \left( 1 - \delta \right) \cdot \cdots \cdot \left( 1 - \delta \right) \geq (1 - \varepsilon), \]

for all \( p \in \mathbb{N} \).

Since \( \lim_{t \to \infty} M(x, y, t) = 1 \) for all \( x, y \in X \), there exists \( t_0 > 0 \) such that \( M(x, y, t_0) \geq (1 - \delta) \).

Also since \( \phi \in \Phi \), using condition (\( \phi \)-3), we have \( \sum_{n=1}^{\infty} \phi^n(t_0) < \infty \). Then for any \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that

\[ t > \sum_{k=n_0}^{\infty} \phi^k(t_0). \]  \( (3.12) \)

Using condition (2), we have

\[ M(x, y, \phi(t_0)) = M(f(p, q), f(q, p), \phi(t_0)) \geq M(gp, gq, t_0) * M(gq, gp, t_0) \]

\[ = M(x, y, t_0) * M(y, x, t_0). \]  \( (3.13) \)

Continuing in the same way, we can get for all \( n_0 \in \mathbb{N} \),

\[ M(x, y, \phi^n(t_0)) \geq \left[ M(x, y, t_0) \right]^{2^{n_0-1}} * \left[ M(y, x, t_0) \right]^{2^{n_0-1}}. \]  \( (3.14) \)
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Then we have

\[
M(x, y, t) \geq M\left(x, y, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\
\geq M(x, y, \phi^{n_0}t_0) \\
\geq [M(x, y, t_0)]^{2^{n_0-1}} \ast [M(y, x, t_0)]^{2^{n_0-1}} \\
\geq (1 - \delta) \ast \cdots \ast (1 - \delta) \geq (1 - \epsilon),
\]

(3.15)

which implies that \(x = y\). Thus, we have proved that \(f\) and \(g\) have a common fixed point \(x \in X\).

**Step 4.** We now prove the uniqueness of \(x\). Let \(z\) be any point in \(X\) such that \(z \neq x\) with \(g(z) = z = f(z, z)\). Since \(\ast\) is a \(t\)-norm of \(H\)-type, for any \(\epsilon > 0\), there exists \(\delta > 0\) such that

\[
\left(1 - \delta\right) \ast \cdots \ast (1 - \delta) \geq (1 - \epsilon),
\]

(3.16)

for all \(p \in \mathbb{N}\). Since \(\lim_{t \to \infty} M(x, y, t) = 1\) for all \(x, y \in X\), there exists \(t_0 > 0\) such that \(M(x, z, t_0) \geq (1 - \delta)\). Also since \(\phi \in \Phi\) and using condition (\(\phi\)-3), we have \(\sum_{n=1}^{\infty} \phi^n(t_0) < \infty\). Then for any \(t > 0\), there exists \(n_0 \in \mathbb{N}\) such that

\[
t > \sum_{k=n_0}^{\infty} \phi^k(t_0).
\]

(3.17)

Using condition (2), we have

\[
M(x, z, \phi(t_0)) = M(f(x, x), f(z, z), \phi(t_0)) \\
\geq M(g(x), g(z), t_0) \ast M(g(x), g(z), t_0) \\
\geq M(x, z, t_0) \ast M(x, z, t_0) = [M(x, z, t_0)]^2.
\]

(3.18)

Continuing in the same way, we can get for all \(n \in \mathbb{N}\),

\[
M(x, z, \phi^n(t_0)) \geq \left([M(x, z, t_0)]^{2^{n_0-1}}\right)^2.
\]

(3.19)
Then we have

\[
M(x, z, t) \geq M\left(x, z, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\
\geq M(x, z, \phi^{n_0}(t_0)) \\
\geq \left([M(x, z, t_0)]^{2^{n_0-1}}\right)^2 = [M(x, z, t_0)]^{2n_0} \\
\geq \left((1 - \delta) \cdots \cdots \cdots (1 - \delta)\right)_{2^{n_0}} \geq (1 - \epsilon),
\]

which implies that \(x = z\).

Hence \(f\) and \(g\) have a unique common fixed point in \(X\).

Remark 3.3. We still get a unique common fixed point if weakly compatible notion is replaced by \(w\)-compatible notion.

Now we give another generalization of Theorem 3.1.

**Corollary 3.4.** Let \((X, M, \ast)\) be a fuzzy metric space where \(\ast\) is a continuous \(t\)-norm of \(H\)-type. Let \(f : X \times X \to X\) and \(g : X \to X\) be two mappings and there exists \(\phi \in \Phi\) satisfying (2) and (3) with the following condition:

(5) the pair \((f, g)\) satisfy E.A. property.

If \(g(X)\) is a closed subspace of \(X\), then \(f\) and \(g\) have a unique common fixed point in \(X\).

**Proof.** Since \(f\) and \(g\) satisfy E.A. property, there exists sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[
\lim_{n \to \infty} f(x_n, y_n) = \lim_{n \to \infty} g(x_n) = x, \\
\lim_{n \to \infty} f(y_n, x_n) = \lim_{n \to \infty} g(y_n) = y,
\]

for some \(x, y \in X\).

It follows from \(g(X)\) being a closed subspace of \(X\) that \(x = g(p)\), \(y = g(q)\) for some \(p, q \in X\) and then \(f\) and \(g\) satisfy the (CLRg) property. By Theorem 3.2, we get that \(f\) and \(g\) have a unique common fixed point in \(X\). \(\square\)

**Corollary 3.5.** Let \((X, M, \ast)\) be a fuzzy metric space where \(\ast\) is a continuous \(t\)-norm of \(H\)-type. Let \(f : X \times X \to X\) and \(g : X \to X\) be two mappings and there exists \(\phi \in \Phi\) satisfying (2), (3), and (5).

Suppose that \(f(X \times X) \subseteq g(X)\), if range of one of the maps \(f\) or \(g\) is a closed subspace of \(X\), then \(f\) and \(g\) have a unique common fixed point in \(X\).

**Proof.** It follows immediately from Corollary 3.5. \(\square\)

Taking \(g = I_X\) in Theorem 3.2, the Corollary 3.6 follows immediately the following.
Corollary 3.6. Let \((X, M, \ast)\) be a fuzzy metric space where \(\ast\) is a continuous \(t\)-norm of \(H\)-type. Let \(f : X \times X \to X\) and \(g : X \to X\) be two mappings and there exists \(\phi \in \Phi\) satisfying the following conditions, for all \(x, y, u, v \in X\) and \(t > 0\):

\[
(6) \quad M(f(x, y), f(u, v), \phi(t)) \geq M(x, u, t) \ast M(y, v, t),
\]

\[
(7) \quad \text{there exists sequences } \{x_n\} \text{ and } \{y_n\} \text{ in } X \text{ such that}
\]

\[
\lim_{n \to \infty} f(x_n, y_n) = \lim_{n \to \infty} x_n = x,
\]

\[
\lim_{n \to \infty} f(y_n, x_n) = \lim_{n \to \infty} y_n = y,
\]

for some \(x, y \in X\).

Then, there exists a unique \(z \in X\) such that \(z = f(z, z)\).

References


