Research Article

Generalized Mixed Equilibrium Problems and Fixed Point Problem for a Countable Family of Total Quasi- ϕ -Asymptotically Nonexpansive Mappings in Banach Spaces

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The purpose of this paper is first to introduce the concept of *total quasi- ϕ -asymptotically nonexpansive mapping* which contains many kinds of mappings as its special cases and then to use a hybrid algorithm to introduce a new iterative scheme for finding a common element of the set of solutions for a system of generalized mixed equilibrium problems and the set of common fixed points for a countable family of total quasi- ϕ -asymptotically nonexpansive mappings. Under suitable conditions some strong convergence theorems are established in an uniformly smooth and strictly convex Banach space with Kadec-Klee property. The results presented in the paper improve and extend some recent results.

1. Introduction

Throughout this paper, we denote by $\mathbb{R}$ and $\mathbb{R}^+$ the set of all real numbers and all nonnegative real numbers, respectively. We also assume that E is a real Banach space, E^* is the dual space of E , C is a nonempty closed convex subset of E , and $\langle \cdot, \cdot \rangle$ is the pairing between E and E^* . In the sequel, we denote the strong convergence and weak convergence of a sequence $\{x_n\}$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively, and $J : E \rightarrow 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\| = \|x^*\|\}, \quad x \in E. \quad (1.1)$$

Let $\psi : C \rightarrow \mathbb{R}$ be a proper real-valued function, $A : C \rightarrow E^*$ a nonlinear mapping, and $F : C \times C \rightarrow \mathbb{R}$ a bifunction. The “so called” generalized mixed equilibrium problem for F, A, ψ is to find $x^* \in C$ such that

$$F(x^*, y) + \langle Ax^*, y - x^* \rangle + \psi(y) - \psi(x^*) \geq 0, \quad \forall y \in C. \quad (1.2)$$

We denote the set of solutions of (1.2) by $\text{GMEP}(F, A, \psi)$, that is,

$$\text{GMEP}(F, A, \psi) = \{x^* \in C : F(x^*, y) + \langle Ax^*, y - x^* \rangle + \psi(y) - \psi(x^*) \geq 0, \forall y \in C\}. \quad (1.3)$$

Special Examples

- (i) If $A = 0$, then the problem (1.2) is reduced to the mixed equilibrium problem (MEP), and the set of its solutions is denoted by

$$\text{MEP}(F, \psi) = \{x^* \in C : \Theta(x^*, y) + \psi(y) - \psi(x^*) \geq 0, \forall y \in C\}. \quad (1.4)$$

- (ii) If $\psi \equiv 0$, then the problem (1.2) is reduced to the generalized equilibrium problem (GEP), and the set of its solutions is denoted by

$$\text{GEP}(F, A) = \{x^* \in C : F(x^*, y) + \langle Ax^*, y - x^* \rangle \geq 0, \forall y \in C\}. \quad (1.5)$$

- (iii) If $A = 0, \psi = 0$, then the problem (1.2) is reduced to the equilibrium problem (EP), and the set of its solutions is denoted by

$$\text{EP}(F) = \{x^* \in C : F(x^*, y) \geq 0, \forall y \in C\}. \quad (1.6)$$

- (iv) If $F = 0$, then the problem (1.2) is reduced to the mixed variational inequality of Browder type (VI), and the set of its solutions is denoted by

$$\text{VI}(C, A, \psi) = \{x^* \in C : \langle Ax^*, y - x^* \rangle + \psi(y) - \psi(x^*) \geq 0, \forall y \in C\}. \quad (1.7)$$

These show that the problem (1.2) is very general in the sense that numerous problems in physics, optimization, and economics reduce to finding a solution of (1.2). Recently, some methods have been proposed for the generalized mixed equilibrium problem in Banach space (see, e.g., [1–5]).

A Banach space E is said to be strictly convex if $\|x+y\|/2 < 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. E is said to be uniformly convex if, for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x+y\|/2 < 1 - \delta$ for all $x, y \in U$ with $\|x-y\| \geq \epsilon$. E is said to be smooth if the limit

$$\lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t} \quad (1.8)$$

exists for all $x, y \in U$. E is said to be uniformly smooth if the above limit exists uniformly in $x, y \in U$.

Remark 1.1. The following basic properties for Banach space E and for the normalized duality mapping J can be found in Cioranescu [6].

- (i) If E is an arbitrary Banach space, then J is monotone and bounded;
- (ii) If E is a strictly convex Banach space, then J is strictly monotone;
- (iii) If E is a smooth Banach space, then J is single-valued, and hemicontinuous; that is, J is continuous from the strong topology of E to the weak star topology of E^* ;
- (iv) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E ;
- (v) If E is a reflexive and strictly convex Banach space with a strictly convex dual E^* and $J^* : E^* \rightarrow E$ is the normalized duality mapping in E^* , then $J^{-1} = J^*$, $JJ^* = I_{E^*}$ and $J^*J = I_E$;
- (vi) If E is a smooth, strictly convex and reflexive Banach space, then the normalized duality mapping J is single valued, one to one and onto;
- (vii) A Banach space E is uniformly smooth if and only if E^* is uniformly convex. If E is uniformly smooth, then it is smooth and reflexive.

Recall that a Banach space E has the Kadec-Klee property, if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x \in E$ and $\|x_n\| \rightarrow \|x\|$, then $x_n \rightarrow x$ (as $n \rightarrow \infty$). It is well known that if E is a uniformly convex Banach space, then E has the Kadec-Klee property.

Next we assume that E is a smooth, strictly convex and reflexive Banach space and C is a nonempty closed convex subset of E . In the sequel, we always use $\phi : E \times E \rightarrow \mathbb{R}^+$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \quad (1.9)$$

It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in E. \quad (1.10)$$

Following Alber [7], the generalized projection $\Pi_C : E \rightarrow C$ is defined by

$$\Pi_C(x) = \arg \inf_{y \in C} \phi(y, x), \quad \forall x \in E. \quad (1.11)$$

Let $T : C \rightarrow C$ be a mapping and $F(T)$ be the set of fixed points of T .

Recall that a point $p \in C$ is said to be an *asymptotic fixed point* of T if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \rightarrow 0$. We denoted the set of all asymptotic fixed points of T by $\tilde{F}(T)$. A point $p \in C$ is said to be a *strong asymptotic fixed point* of T , if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightarrow p$ and $\|x_n - Tx_n\| \rightarrow 0$. We denoted the set of all strong asymptotic fixed points of T by $\hat{F}(T)$.

Definition 1.2. (1) A mapping $T : C \rightarrow C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (1.12)$$

(2) A mapping $T : C \rightarrow C$ is said to be *relatively nonexpansive* [8] if $F(T) \neq \emptyset$, $F(T) = \tilde{F}(T)$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, p \in F(T). \quad (1.13)$$

(3) A mapping $T : C \rightarrow C$ is said to be *weak relatively nonexpansive* [9] if $F(T) \neq \emptyset$, $F(T) = \hat{F}(T)$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, p \in F(T). \quad (1.14)$$

(4) A mapping $T : C \rightarrow C$ is said to be *closed*, if for any sequence $\{x_n\} \subset C$ with $x_n \rightarrow x$ and $Tx_n \rightarrow y$, then $Tx = y$.

Definition 1.3. (1) A mapping $T : C \rightarrow C$ is said to be *quasi- ϕ -nonexpansive* [10] if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, p \in F(T). \quad (1.15)$$

(2) A mapping $T : C \rightarrow C$ is said to be *quasi- ϕ -asymptotically nonexpansive* [11], if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x), \quad \forall n \geq 1, x \in C, p \in F(T). \quad (1.16)$$

(3) A mapping $T : C \rightarrow C$ is said to be *uniformly L -Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\|, \quad \forall x, y \in C, \forall n \geq 1. \quad (1.17)$$

Definition 1.4. (1) A mapping $T : C \rightarrow C$ is said to be *total quasi- ϕ -asymptotically nonexpansive* if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{\nu_n\}, \{\mu_n\}$ with $\nu_n \rightarrow 0, \mu_n \rightarrow 0$ (as $n \rightarrow \infty$) and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\zeta(0) = 0$ such that for all $x \in C, p \in F(T)$

$$\phi(p, T^n x) \leq \phi(p, x) + \nu_n \zeta(\phi(p, x)) + \mu_n, \quad \forall n \geq 1. \quad (1.18)$$

(2) A countable family of mappings $\{T_i\} : C \rightarrow C$ is said to be *uniformly total quasi- ϕ -asymptotically nonexpansive*, if $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and there exist nonnegative real sequences $\{\nu_n\}, \{\mu_n\}$ with $\nu_n \rightarrow 0, \mu_n \rightarrow 0$ (as $n \rightarrow \infty$) and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\zeta(0) = 0$ such that for all $x \in C, p \in \bigcap_{i=1}^{\infty} F(T_i)$

$$\phi(p, T_i^n x) \leq \phi(p, x) + \nu_n \zeta(\phi(p, x)) + \mu_n, \quad \forall n \geq 1. \quad (1.19)$$

Remark 1.5. From the definition, it is easy to know that

- (1) each relatively nonexpansive mapping is closed;
- (2) taking $\zeta(t) = t, t \geq 0$, $v_n = (k_n - 1)$ and $\mu_n = 0$, then (1.16) can be rewritten as

$$\phi(p, T_i^n x) \leq \phi(p, x) + v_n \zeta(\phi(p, x)) + \mu_n, \quad \forall n \geq 1, x \in C, p \in F(T). \quad (1.20)$$

This implies that each quasi- ϕ -asymptotically nonexpansive mapping must be a total quasi- ϕ -asymptotically nonexpansive mapping, but the converse is not true;

- (3) the class of quasi- ϕ -asymptotically nonexpansive mappings contains properly the class of quasi- ϕ -nonexpansive mappings as a subclass, but the converse is not true;
- (4) the class of quasi- ϕ -nonexpansive mappings contains properly the class of weak relatively nonexpansive mappings as a subclass, but the converse is not true;
- (5) the class of weak relatively nonexpansive mappings contains properly the class of relatively nonexpansive mappings as a subclass, but the converse is not true.

A mapping $A : C \rightarrow E^*$ is said to be α -inverse strongly monotone, if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2. \quad (1.21)$$

Remark 1.6. If A is an α -inverse strongly monotone mapping, then it is $1/\alpha$ -Lipschitz continuous.

Iterative approximation of fixed points for relatively nonexpansive mappings in the setting of Banach spaces has been studied extensively by many authors. In 2005, Matsushita and Takahashi [12] obtained some weak and strong convergence theorems to approximate a fixed point of a single relatively nonexpansive mapping. Recently, Ofoedu and Malonza [4], Zhang [5], Su et al. [13], Zegeye and Shahzad [14], Wattanawitoon and Kumam [15], Qin et al. [16], Takahashi and Zembayashi [17], Chang et al. [18, 19], Yao et al. [20, 21], Qin et al. [22], and Cho et al. [23, 24] extend the notions from relatively nonexpansive mappings, weakly relatively nonexpansive mappings or quasi- ϕ -nonexpansive mappings to quasi- ϕ -asymptotically nonexpansive mappings and also prove some strong convergence theorems to approximate a common fixed point of quasi- ϕ -nonexpansive mappings or quasi- ϕ -asymptotically nonexpansive mappings.

The purpose of this paper is first to introduce the concept of total quasi- ϕ -asymptotically nonexpansive mapping which contains many kinds of mappings as its special cases, and then by using a hybrid algorithm to introduce a new iterative scheme for finding a common element of the set of solutions for a system of generalized mixed equilibrium problems and the set of common fixed points for a countable family of total quasi- ϕ -asymptotically nonexpansive mappings in a uniformly smooth and strictly convex Banach space with Kadec-Klee property. The results improve and extend the corresponding results in [8, 11–25].

2. Preliminaries

First, we recall some definitions and conclusions.

Lemma 2.1 (see [7, 26]). *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E . Then the following conclusions hold:*

- (a) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$ for all $x \in C$ and $y \in E$;
- (b) if $x \in E$ and $z \in C$, then

$$z = \Pi_C x \iff \langle z - y, Jx - Jz \rangle \geq 0, \quad \forall y \in C; \quad (2.1)$$

- (c) for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.

Remark 2.2. If E is a real Hilbert space H , then $\phi(x, y) = \|x - y\|^2$ and Π_C is the metric projection P_C of H onto C .

Lemma 2.3 (see [18]). *Let E be a uniformly convex Banach space, $r > 0$ a positive number, and $B_r(0)$ a closed ball of E . Then, for any given sequence $\{x_i\}_{i=1}^\infty \subset B_r(0)$ and for any given sequence $\{\lambda_i\}_{i=1}^\infty$ of positive numbers with $\sum_{i=1}^\infty \lambda_i = 1$, then there exists a continuous, strictly increasing and convex function $g : [0, 2r) \rightarrow [0, \infty)$ with $g(0) = 0$ such that for any positive integers i, j with $i < j$,*

$$\left\| \sum_{n=1}^\infty \lambda_n x_n \right\|^2 \leq \sum_{n=1}^\infty \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|). \quad (2.2)$$

Lemma 2.4. *Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and let C be a nonempty closed convex subset of E . Let $T : C \rightarrow C$ be a closed and total quasi- ϕ -asymptotically nonexpansive mapping with nonnegative real sequences $\{\nu_n\}, \{\mu_n\}$ and a strictly increasing continuous functions $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\mu_1 = 0, \nu_n \rightarrow 0, \mu_n \rightarrow 0$ (as $n \rightarrow \infty$) and $\zeta(0) = 0$. Then $F(T)$ is a closed convex subset of C .*

Proof. Letting $\{p_n\}$ be a sequence in $F(T)$ with $p_n \rightarrow p$ (as $n \rightarrow \infty$), we prove that $p \in F(T)$. In fact, from the definition of T , we have

$$\phi(p_n, Tp) \leq \phi(p_n, p) + \nu_1 \zeta(\phi(p_n, p)) + \mu_1 \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \quad (2.3)$$

Therefore we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \phi(p_n, Tp) &= \lim_{n \rightarrow \infty} \left(\|p_n\|^2 - 2\langle p_n, JTp \rangle \right) + \|Tp\|^2 \\ &= \|p\|^2 - 2\langle p, JTp \rangle + \|Tp\|^2 = \phi(p, Tp) = 0, \end{aligned} \quad (2.4)$$

that is, $p \in F(T)$.

Next we prove that $F(T)$ is convex. For any $p, q \in F(T)$, $t \in (0, 1)$, putting $w = tp + (1 - t)q$, we prove that $w \in F(T)$. Indeed, in view of the definition of $\phi(x, y)$, we have

$$\begin{aligned} \phi(w, T^n w) &= \|w\|^2 - 2\langle w, JT^n w \rangle + \|T^n w\|^2 \\ &= \|w\|^2 - 2t\langle p, JT^n w \rangle - 2(1 - t)\langle q, JT^n w \rangle + \|T^n w\|^2 \end{aligned}$$

$$\begin{aligned}
&= \|w\|^2 + t\phi(p, T^n w) + (1-t)\phi(q, T^n w) - t\|p\|^2 - (1-t)\|q\|^2 \\
&\leq \|w\|^2 + t(\phi(p, w) + \nu_n \zeta(\phi(p, w)) + \mu_n) + (1-t)(\phi(q, w) + \nu_n \zeta(\phi(q, w)) + \mu_n) \\
&\quad - t\|p\|^2 - (1-t)\|q\|^2 \\
&= \|w\|^2 + t(\|p\|^2 - 2\langle p, Jw \rangle + \|w\|^2) + t(\nu_n \zeta(\phi(p, w)) + \mu_n) + (1-t) \\
&\quad \times (\|q\|^2 - 2\langle q, Jw \rangle + \|w\|^2) + (1-t)(\nu_n \zeta(\phi(q, w)) + \mu_n) - t\|p\|^2 - (1-t)\|q\|^2 \\
&= \|w\|^2 - 2\langle w, Jw \rangle + \|w\|^2 + t\nu_n \zeta(\phi(p, w)) + (1-t)(\nu_n \zeta(\phi(q, w)) + \mu_n) \\
&= t\nu_n \zeta(\phi(p, w)) + (1-t)(\nu_n \zeta(\phi(q, w)) + \mu_n).
\end{aligned} \tag{2.5}$$

Since $\mu_n \rightarrow 0$ and $\nu_n \rightarrow 0$, we have $\phi(w, T^n w) \rightarrow 0$ (as $n \rightarrow \infty$). From (1.10) we have $\|T^n w\| \rightarrow \|w\|$. Consequently $\|JT^n w\| \rightarrow \|Jw\|$. This implies that $\{JT^n w\}$ is a bounded sequence. Since E is reflexive, E^* is also reflexive. So we can assume that

$$JT^n w \rightharpoonup f_0 \in E^*. \tag{2.6}$$

Again since E is reflexive, we have $J(E) = E^*$. Therefore there exists $x \in E$ such that $Jx = f_0$. By virtue of the weakly lower semicontinuity of norm $\|\cdot\|$, we have

$$\begin{aligned}
0 &= \liminf_{n \rightarrow \infty} \phi(w, T^n w) = \liminf_{n \rightarrow \infty} (\|w\|^2 - 2\langle w, J(T^n w) \rangle + \|T^n w\|^2) \\
&= \liminf_{n \rightarrow \infty} (\|w\|^2 - 2\langle w, J(T^n w) \rangle + \|J(T^n w)\|^2) \\
&\geq \|w\|^2 - 2\langle w, f_0 \rangle + \|f_0\|^2 \\
&= \|w\|^2 - 2\langle w, Jx \rangle + \|Jx\|^2 \\
&= \|w\|^2 - 2\langle w, Jx \rangle + \|x\|^2 = \phi(w, x),
\end{aligned} \tag{2.7}$$

that is, $w = x$ which implies that $f_0 = Jw$. Hence from (2.6) we have $JT^n w \rightharpoonup Jw \in E^*$. Since $\|JT^n w\| \rightarrow \|w\|$ and E^* has the Kadec-Klee property, we have $JT^n w \rightarrow Jw$. Since E is uniformly smooth, E^* is uniformly convex, which in turn implies that E^* is smooth. From Remark 1.1(iii) it yields that $J^{-1} : E^* \rightarrow E$ is hemi-continuous. Therefore we have $T^n w \rightharpoonup w$. Again since $\|T^n w\| \rightarrow \|w\|$, by using the Kadec-Klee property of E , we have $T^n w \rightarrow w$. This implies that $TT^n w = T^{n+1}w \rightarrow w$. Since T is closed, we have $w = Tw$.

This completes the proof of Lemma 2.4. $\square$

Lemma 2.5. *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E . Let $f : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying the following conditions:*

- (A1) $f(x, x) = 0$, for all $x \in C$,
- (A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$, for all $x, y \in C$,

(A3) $\limsup_{t \downarrow 0} f(x + t(z - x), y) \leq f(x, y)$ for all $x, z, y \in C$,

(A4) The function $y \mapsto f(x, y)$ is convex and lower semi-continuous.

Then the following conclusions hold:

(1) (Blum and Oettli [27]) for any given $r > 0$ and $x \in E$, there exists a unique $z \in C$ such that

$$f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C; \quad (2.8)$$

(2) (Takahashi and Zembayashi [28]) for any given $r > 0$ and $x \in E$, define a mapping $K_r^f : E \rightarrow C$ by

$$K_r^f(x) = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C \right\}, \quad x \in E. \quad (2.9)$$

Then, the following conclusions hold:

(a) K_r^f is single-valued;

(b) K_r^f is firmly nonexpansive-type mapping, that is, for all $z, y \in E$,

$$\langle K_r^f z - K_r^f y, JK_r^f z - JK_r^f y \rangle \leq \langle K_r^f z - K_r^f y, Jz - Jy \rangle; \quad (2.10)$$

(c) $F(K_r^f) = \text{EP}(f)$ and K_r^f is quasi- ϕ -nonexpansive;

(d) $\text{EP}(f)$ is closed and convex;

(e) $\phi(q, K_r^f x) + \phi(K_r^f x, x) \leq \phi(q, x)$, for all $q \in F(K_r^f)$.

For solving the generalized mixed equilibrium problem (1.2), let us assume that the following conditions are satisfied:

(1) E is a smooth, strictly convex, and reflexive Banach space and C is a nonempty closed convex subset of E ;

(2) $A : C \rightarrow E^*$ is β -inverse strongly monotone mapping;

(3) $F : C \times C \rightarrow \mathbb{R}$ is bifunction satisfying the conditions (A1), (A3), (A4) in Lemma 2.5 and the following condition (A2)':

(A2)' for some $\gamma \geq 0$ with $\gamma \leq \beta$

$$F(x, y) + F(y, x) \leq \gamma \|Ax - Ay\|^2, \quad \forall x, y \in C; \quad (2.11)$$

(4) $\psi : C \rightarrow \mathbb{R}$ is a lower semicontinuous and convex function.

Under the assumptions as above, we have the following results.

Lemma 2.6. Let E, C, A, F, ψ satisfy the above conditions (1)–(4). Denote by

$$\Gamma(x, y) = F(x, y) + \psi(y) - \psi(x) + \langle Ax, y - x \rangle, \quad \forall x, y \in C. \quad (2.12)$$

For any given $r > 0$ and $x \in E$, define a mapping $K_r^\Gamma : E \rightarrow C$ by

$$K_r^\Gamma(x) = \left\{ z \in C : \Gamma(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C \right\}. \quad (2.13)$$

Then, the following hold:

- (a) K_r^Γ is single-valued;
- (b) K_r^Γ is a firmly nonexpansive-type mapping, that is, for all $z, y \in E$,

$$\langle K_r^\Gamma(z) - K_r^\Gamma(y), JK_r^\Gamma(z) - JK_r^\Gamma(y) \rangle \leq \langle K_r^\Gamma(z) - K_r^\Gamma(y), Jz - Jy \rangle; \quad (2.14)$$

- (c) $F(K_r^\Gamma) = \text{EP}(\Gamma) = \text{GMEP}(F, A, \psi)$;
- (d) $\text{GMEP}(F, A, \psi)$ is closed and convex;
- (e)

$$\phi(q, K_r^\Gamma x) + \phi(K_r^\Gamma x, x) \leq \phi(q, x), \quad \forall q \in F(K_r^\Gamma). \quad (2.15)$$

Proof. It follows from Lemma 2.5 that in order to prove the conclusions of Lemma 2.6 it is sufficient to prove that the function $\Gamma : C \times C \rightarrow \mathbb{R}$ satisfies the conditions (A1)–(A4) in Lemma 2.5.

In fact, by the similar method as given in the proof of Lemma 2.4 in [1], we can prove that the function Γ satisfies the conditions (A1), (A3), and (A4). Now we prove that Γ also satisfies the conditions (A2).

Indeed, for any $x, y \in C$, by condition (A2)', we have

$$\begin{aligned} \Gamma(x, y) + \Gamma(y, x) &= F(x, y) + \psi(y) - \psi(x) + \langle Ax, y - x \rangle \\ &\quad + F(y, x) + \psi(x) - \psi(y) + \langle Ay, x - y \rangle \\ &= F(x, y) + F(y, x) - \langle Ax - Ay, x - y \rangle \\ &\leq (\gamma - \beta) \|Ax - Ay\|^2 \leq 0. \end{aligned} \quad (2.16)$$

This implies that the function Γ satisfies the conditions (A2). Therefore the conclusions of Lemma 2.6 can be obtained from Lemma 2.3 immediately. $\square$

Remark 2.7. It follows from Lemma 2.5 that the mapping K_r^Γ is a relatively nonexpansive mapping. Thus, it is quasi- ϕ -nonexpansive.

3. Main Results

In this section, we shall use the hybrid method to prove some strong convergence theorems for finding a common element of the set of solutions for a system of the generalized mixed equilibrium problems (1.2) and the set of common fixed points of a countable family of total quasi- ϕ -asymptotically nonexpansive mappings in Banach spaces.

In the sequel, we assume that $E, C, \{S_i\}_{i=1}^\infty, \{A_i\}_{i=1}^M, \{F_i\}_{i=1}^M, \{\psi_i\}_{i=1}^M$ satisfy the following conditions.

- (1) Let E be a uniformly smooth and strictly convex Banach space with Kleac-Klee property and C a nonempty closed convex subset of E .
- (2) Let $S_i : C \rightarrow C$ be a countable family of closed and uniformly total quasi- ϕ -asymptotically nonexpansive mappings with nonnegative real sequences $\{\nu_n\}, \{\mu_n\}$ and a strictly increasing continuous functions $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\nu_n \rightarrow 0, \mu_n \rightarrow 0$ (as $n \rightarrow \infty$) and $\mu_1 = 0, \zeta(0) = 0$. Suppose further that for each $i \geq 1, S_i$ is a uniformly L_i -Lipschitz mapping, that is, there exists a constant $L_i > 0$ such that

$$\|S_i^n x - S_i^n y\| \leq L_i \|x - y\|, \quad \forall x, y \in C, \forall n \geq 1. \quad (3.1)$$

- (3) Let $A_i : C \rightarrow E^*$ ($i = 1, 2, \dots, M$) be a finite family of β_i -inverse strongly monotone mappings.
- (4) Let $F_i : C \rightarrow \mathbb{R}$ ($i = 1, 2, \dots, M$) be a finite family of bifunction satisfying the conditions (A1), (A3), (A4), and the following condition (A2)':
- (A2)' For each $i = 1, 2, \dots, M$ there exists $\gamma_i \geq 0$ with $\gamma_i \leq \beta_i$ such that

$$F_i(x, y) + F_i(y, x) \leq \gamma_i \|A_i x - A_i y\|^2, \quad \forall x, y \in C; \quad (3.2)$$

- (5) Let $\psi_i : C \rightarrow \mathbb{R}$ ($i = 1, 2, \dots, M$) be a finite family of lower semicontinuous and convex functions.

Theorem 3.1. Let $E, C, \{S_i\}_{i=1}^\infty, \{A_i\}_{i=1}^M, \{F_i\}_{i=1}^M, \{\psi_i\}_{i=1}^M$ be the same as above. Suppose that

$$\mathcal{F} := \bigcap_{i=1}^\infty F(T_i) \bigcap \bigcap_{j=1}^M \text{GMEP}(F_j, A_j, \psi_j) \quad (3.3)$$

is a nonempty and bounded subset of C . For any given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by

$$x_0 \in C_0 = C,$$

$$z_n = J^{-1} \left(\alpha_{n,0} Jx_n + \sum_{i=1}^\infty \alpha_{n,i} JS_i^n x_n \right),$$

$$\begin{aligned}
y_n &= J^{-1}(\alpha_n Jz_n + (1 - \alpha_n)Jx_n), \\
u_n &= K_{r_{M,n}}^{\Gamma_M} K_{r_{M-1,n}}^{\Gamma_{M-1}} \cdots K_{r_{2,n}}^{\Gamma_2} K_{r_{1,n}}^{\Gamma_1} y_n, \\
C_{n+1} &= \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n) + \eta_n\}, \\
x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0,
\end{aligned} \tag{3.4}$$

where

$$\eta_n = v_n \sup_{u \in \mathcal{F}} \zeta(\phi(u, x_n)) + \mu_n, \quad \forall n \geq 1, \tag{3.5}$$

$K_{r_{i,n}}^{\Gamma_i} : E \rightarrow C, i = 1, 2, \dots, M$ is the mapping defined by (2.13) with $\Gamma = \Gamma_i, r = r_{i,n}$, and

$$\Gamma_i(x, y) = F_i(x, y) + \langle A_i x, y - x \rangle + \psi_i(y) - \psi_i(x), \quad \forall x, y \in C. \tag{3.6}$$

$r_{k,n} \in [d, \infty), k = 1, 2, \dots, M, n \geq 1$ for some $d > 0$, $\Pi_{C_{n+1}}$ is the generalized projection of E onto the set C_{n+1} , and $\{\alpha_{n,i}\}, \{\alpha_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:

- (a) $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ for all $n \geq 0$;
- (b) $\liminf_{n \rightarrow \infty} \alpha_{n,0} \cdot \alpha_{n,i} > 0$ for all $i \geq 1$;
- (c) $0 < \alpha < \alpha_n < 1$ for some $\alpha \in (0, 1)$.

Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_0$, where $\Pi_{\mathcal{F}}$ is the generalized projection from E onto $\mathcal{F}$.

Proof. We divide the proof of Theorem 3.1 into five steps.

- (i) We first prove that $\mathcal{F}$ and C_n both are closed and convex subset of C for all $n \geq 0$.

In fact, it follows from Lemmas 2.4 and 2.6 that $F(S_i), i \geq 1$ and $\text{GMEP}(F_j, A_j, \psi_j)$ ($j = 1, 2, \dots, M$) both are closed and convex. Therefore $\mathcal{F}$ is a closed and convex subset in C . Furthermore, it is obvious that $C_0 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \geq 1$. Since the inequality $\phi(v, u_n) \leq \phi(v, x_n) + \eta_n$ is equivalent to

$$2\langle v, Jx_n - Ju_n \rangle \leq \|x_n\|^2 - \|u_n\|^2 + \eta_n, \tag{3.7}$$

therefore, we have

$$C_{n+1} = \left\{ v \in C_n : 2\langle v, Jx_n - Ju_n \rangle \leq \|x_n\|^2 - \|u_n\|^2 + \eta_n \right\}. \tag{3.8}$$

This implies that C_{n+1} is closed and convex. The desired conclusions are proved. These in turn show that $\Pi_{\mathcal{F}} x_0$ and $\Pi_{C_n} x_0$ are well defined.

- (ii) We prove that $\{x_n\}$ and $\{S_i^n x_n\}_{n=0}^{\infty}$ for all $i \geq 1$ are both bounded sequences in C .

By the definition of C_n , we have $x_n = \Pi_{C_n} x_0$ for all $n \geq 0$. It follows from Lemma 2.1 (a) that

$$\begin{aligned}\phi(x_n, x_0) &= \phi(\Pi_{C_n} x_0, x_0) \leq \phi(u, x_0) - \phi(u, \Pi_{C_n} x_0) \\ &\leq \phi(u, x_0), \quad \forall n \geq 0, u \in \mathcal{F}.\end{aligned}\tag{3.9}$$

This implies that $\{\phi(x_n, x_0)\}$ is bounded. By virtue of (1.10), $\{x_n\}$ is bounded. Since $\phi(u, S_i^n x_n) \leq \phi(u, x_n) + v_n \zeta(\phi(u, x_n)) + \mu_n$ for all $u \in \mathcal{F}$ and $i \geq 1$, $\{S_i^n x_n\}$ is bounded for all $i \geq 1$, and so $\{z_n\}$ is bounded in E . Denote M by

$$M = \sup_{n \geq 0, i \geq 1} \{\|x_n\|, \|S_i^n x_n\|, \|z_n\|\} < \infty.\tag{3.10}$$

In view of the structure of $\{C_n\}$, we have $C_{n+1} \subset C_n$, $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0$. This implies that $x_{n+1} \in C_n$ and

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 1.\tag{3.11}$$

Therefore $\{\phi(x_n, x_0)\}$ is convergent. Without loss of generality, we can assume that

$$\lim_{n \rightarrow \infty} \phi(x_n, x_0) = r \geq 0.\tag{3.12}$$

(iii) Next, we prove that $\mathcal{F} := \bigcap_{i=1}^{\infty} F(S_i) \cap \bigcap_{i=1}^M \text{GMEP}(F_i, A_i, \psi_i) \subset C_n$ for all $n \geq 0$.

Indeed, it is obvious that $\mathcal{F} \subset C_0 = C$. Suppose that $\mathcal{F} \subset C_n$ for some $n \geq 0$. Since $u_n = K_{r_{M,n}}^{\Gamma_M} K_{r_{M-1,n}}^{\Gamma_{M-1}} \cdots K_{r_{2,n}}^{\Gamma_2} K_{r_{1,n}}^{\Gamma_1} y_n$, by Lemma 2.6 and Remark 2.7, $K_{r_{i,n}}^{\Gamma_i}$ is quasi- ϕ -nonexpansive. Again since E is uniformly smooth, E^* is uniformly convex. Hence, For any given $u \in \mathcal{F} \subset C_n$ and for any positive integer $j > 0$, from Lemma 2.3 we have

$$\begin{aligned}\phi(u, u_n) &= \phi\left(u, K_{r_{M,n}}^{\Gamma_M} K_{r_{M-1,n}}^{\Gamma_{M-1}} \cdots K_{r_{2,n}}^{\Gamma_2} K_{r_{1,n}}^{\Gamma_1} y_n\right) \leq \phi(u, y_n) \\ &= \phi\left(u, J^{-1}(\alpha_n J z_n + (1 - \alpha_n) J x_n)\right) \\ &\leq \|u\|^2 - 2\langle u, \alpha_n J z_n + (1 - \alpha_n) J x_n \rangle + \|\alpha_n J z_n + (1 - \alpha_n) J x_n\|^2 \\ &\leq \|u\|^2 - 2\langle u, \alpha_n J z_n + (1 - \alpha_n) J x_n \rangle + \alpha_n \|J z_n\|^2 + (1 - \alpha_n) \|J x_n\|^2 \\ &= \|u\|^2 - 2\langle u, \alpha_n J z_n + (1 - \alpha_n) J x_n \rangle + \alpha_n \|z_n\|^2 + (1 - \alpha_n) \|x_n\|^2 \\ &= \alpha_n \phi(u, z_n) + (1 - \alpha_n) \phi(u, x_n) \\ &= \alpha_n \phi\left(u, J^{-1}\left(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J S_i^n x_n\right)\right) + (1 - \alpha_n) \phi(u, x_n)\end{aligned}$$

$$\begin{aligned}
&= \alpha_n \left(\|u\|^2 - 2\alpha_{n,0} \langle u, Jx_n \rangle - 2 \sum_{i=1}^{\infty} \alpha_{n,i} \langle u, JS_i^n x_n \rangle + \left\| \alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n x_n \right\|^2 \right) \\
&\quad + (1 - \alpha_n) \phi(u, x_n) \\
&\leq \alpha_n \left(\|u\|^2 - 2\alpha_{n,0} \langle u, Jx_n \rangle - 2 \sum_{i=1}^{\infty} \alpha_{n,i} \langle u, JS_i^n x_n \rangle + \alpha_{n,0} \|Jx_n\|^2 + \sum_{i=1}^{\infty} \alpha_{n,i} \|JS_i^n x_n\|^2 \right. \\
&\quad \left. - \alpha_{n,0} \alpha_{n,j} g(\|Jx_n - JS_j^n x_n\|) \right) + (1 - \alpha_n) \phi(u, x_n) \\
&\leq \alpha_n \left(\|u\|^2 - 2\alpha_{n,0} \langle u, Jx_n \rangle - 2 \sum_{i=1}^{\infty} \alpha_{n,i} \langle u, JS_i^n x_n \rangle + \alpha_{n,0} \|x_n\|^2 + \sum_{i=1}^{\infty} \alpha_{n,i} \|S_i^n x_n\|^2 \right. \\
&\quad \left. - \alpha_{n,0} \alpha_{n,j} g(\|Jx_n - JS_j^n x_n\|) \right) + (1 - \alpha_n) \phi(u, x_n) \\
&= \alpha_n \left(\alpha_{n,0} \phi(u, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} \phi(u, S_i^n x_n) - \alpha_{n,0} \alpha_{n,j} g(\|Jx_n - JS_j^n x_n\|) \right) \\
&\quad + (1 - \alpha_n) \phi(u, x_n) \\
&\leq \alpha_n \left(\alpha_{n,0} \phi(u, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} \{ \phi(u, x_n) + v_n \zeta(\phi(u, x_n)) + \mu_n \} \right. \\
&\quad \left. - \alpha_{n,0} \alpha_{n,j} g(\|Jx_n - JS_j^n x_n\|) \right) + (1 - \alpha_n) \phi(u, x_n) \\
&\leq \alpha_n \left(\phi(u, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} (v_n \zeta(\phi(u, x_n)) + \mu_n) - \alpha_{n,0} \alpha_{n,j} g(\|Jx_n - JS_j^n x_n\|) \right) \\
&\quad + (1 - \alpha_n) \phi(u, x_n) \\
&\leq \phi(u, x_n) + \alpha_n \left(v_n \sup_{u \in \mathcal{F}} \zeta(\phi(u, x_n)) + \mu_n \right) - \alpha_n \alpha_{n,0} \alpha_{n,j} g(\|Jx_n - JS_j^n x_n\|) \\
&= \phi(u, x_n) + \alpha_n \eta_n - \alpha_n \alpha_{n,0} \alpha_{n,j} g(\|Jx_n - JS_j^n x_n\|) \\
&\leq \phi(u, x_n) + \eta_n.
\end{aligned} \tag{3.13}$$

Hence $u \in C_{n+1}$ and so $\mathcal{F} \subset C_n$ for all $n \geq 0$. By the way, from the definition of $\{\eta_n\}$ and ζ and (3.10), it is easy to see that

$$\eta_n = v_n \sup_{u \in \mathcal{F}} \zeta(\phi(u, x_n)) + \mu_n \leq v_n \sup_{u \in \mathcal{F}} \zeta((\|u\| + M)^2) + \mu_n \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{3.14}$$

(IV) Now, we prove that $\{x_n\}$ converges strongly to some point

$$p \in \mathcal{F} := \bigcap_{i=1}^{\infty} F(S_i) \cap \bigcap_{j=1}^{\infty} \text{GMEP}(F_j, A_j, \psi_j). \quad (3.15)$$

First, we prove that $\{x_n\}$ converges strongly to some point $p \in \bigcap_{i=1}^{\infty} F(S_i)$.

In fact, since $\{x_n\}$ is bounded in C and E is reflexive, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup p$. Again since C_n is closed and convex for each $n \geq 1$, it is weakly closed, and so $p \in C_n$ for each $n \geq 0$. Since $x_n = \Pi_{C_n} x_0$, from the definition of Π_{C_n} , we have

$$\phi(x_{n_i}, x_0) \leq \phi(p, x_0), \quad n \geq 0. \quad (3.16)$$

Since

$$\begin{aligned} \liminf_{n_i \rightarrow \infty} \phi(x_{n_i}, x_0) &= \liminf_{n_i \rightarrow \infty} \left\{ \|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_0 \rangle + \|x_0\|^2 \right\} \\ &\geq \|p\|^2 - 2\langle p, Jx_0 \rangle + \|x_0\|^2 = \phi(p, x_0), \end{aligned} \quad (3.17)$$

we have

$$\phi(p, x_0) \leq \liminf_{n_i \rightarrow \infty} \phi(x_{n_i}, x_0) \leq \limsup_{n_i \rightarrow \infty} \phi(x_{n_i}, x_0) \leq \phi(p, x_0). \quad (3.18)$$

This implies that $\lim_{n_i \rightarrow \infty} \phi(x_{n_i}, x_0) = \phi(p, x_0)$, that is, $\|x_{n_i}\| \rightarrow \|p\|$. In view of the Kadec-Klee property of E , we obtain that $\lim_{n \rightarrow \infty} x_{n_i} = p$.

Now we prove that $x_n \rightarrow p$ ($n \rightarrow \infty$). In fact, if there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightarrow q$, then we have

$$\begin{aligned} \phi(p, q) &= \lim_{n_i \rightarrow \infty, n_j \rightarrow \infty} \phi(x_{n_i}, x_{n_j}) \leq \lim_{n_i \rightarrow \infty, n_j \rightarrow \infty} \phi(x_{n_i}, x_0) - \phi(\Pi_{C_{n_j}} x_0, x_0) \\ &= \lim_{n_i \rightarrow \infty, n_j \rightarrow \infty} \phi(x_{n_i}, x_0) - \phi(x_{n_j}, x_0) = 0 \quad (\text{by (3.12)}). \end{aligned} \quad (3.19)$$

Therefore we have $p = q$. This implies that

$$\lim_{n \rightarrow \infty} x_n = p. \quad (3.20)$$

Now we prove that $p \in \bigcap_{i=1}^{\infty} F(S_i)$. In fact, by the construction of C_n , we have that $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi_{C_{n+1}} x_0$. Therefore by Lemma 2.1(a) we have

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0) \\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \end{aligned} \quad (3.21)$$

In view of $x_{n+1} \in C_n$ and noting the construction of C_{n+1} we obtain that

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + \eta_n \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \quad (3.22)$$

From (1.10) it yields $(\|x_{n+1}\| - \|u_n\|)^2 \rightarrow 0$. Since $\|x_{n+1}\| \rightarrow \|p\|$, we have

$$\|u_n\| \longrightarrow \|p\| \quad (\text{as } n \longrightarrow \infty). \quad (3.23)$$

Hence we have

$$\|Ju_n\| \longrightarrow \|Jp\| \quad (\text{as } n \longrightarrow \infty). \quad (3.24)$$

This implies that $\{Ju_n\}$ is bounded in E^* . Since E is reflexive, and so E^* is reflexive, there exists a subsequence $\{Ju_{n_i}\} \subset \{Ju_n\}$ such that $Ju_{n_i} \rightharpoonup p_0 \in E^*$. In view of the reflexive of E , we see that $J(E) = E^*$. Hence there exists $x \in E$ such that $Jx = p_0$. Since

$$\begin{aligned} \phi(x_{n_i+1}, u_{n_i}) &= \|x_{n_i+1}\|^2 - 2\langle x_{n_i+1}, Ju_{n_i} \rangle + \|u_{n_i}\|^2 \\ &= \|x_{n_i+1}\|^2 - 2\langle x_{n_i+1}, Ju_{n_i} \rangle + \|Ju_{n_i}\|^2 \end{aligned} \quad (3.25)$$

taking $\liminf_{n \rightarrow \infty}$ on the both sides of above equality and in view of the weak lower semicontinuity of norm $\|\cdot\|$, then it yields that

$$\begin{aligned} 0 &\geq \|p\|^2 - 2\langle p, p_0 \rangle + \|p_0\|^2 = \|p\|^2 - 2\langle p, Jx \rangle + \|Jx\|^2 \\ &= \|p\|^2 - 2\langle p, Jx \rangle + \|x\|^2 = \phi(p, x). \end{aligned} \quad (3.26)$$

That is $p = x$. This implies that $p_0 = Jp$, and so $Ju_n \rightharpoonup Jp$. It follows from (3.24) and the Kadec-Klee property of E^* that $Ju_{n_i} \rightarrow Jp$ (as $n \rightarrow \infty$). Note that $J^{-1} : E^* \rightarrow E$ is hemi-continuous, it yields that $u_{n_i} \rightarrow p$. It follows from (3.23) and the Kadec-Klee property of E that $\lim_{n_i \rightarrow \infty} u_{n_i} = p$.

By the similar way as given in the proof of (3.20), we can also prove that

$$\lim_{n \rightarrow \infty} u_n = p. \quad (3.27)$$

From (3.20) and (3.27) we have that

$$\|x_n - u_n\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \quad (3.28)$$

Since J is uniformly continuous on any bounded subset of E , we have

$$\|Jx_n - Ju_n\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \quad (3.29)$$

For any $j \geq 1$ and any $u \in \mathcal{F}$, it follows from (3.13), (3.20), and (3.27) that

$$\alpha_n \alpha_{n,0} \alpha_{n,j} g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \leq \phi(u, x_n) - \phi(u, u_n) + \alpha_n \eta_n. \quad (3.30)$$

Since

$$\begin{aligned} \phi(u, x_n) - \phi(u, u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle u, Jx_n - Ju_n \rangle \\ &\leq \left| \|x_n\|^2 - \|u_n\|^2 \right| + 2\|u\| \cdot \|Jx_n - Ju_n\| \\ &\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\| \cdot \|Jx_n - Ju_n\|, \end{aligned} \quad (3.31)$$

from (3.28) and (3.29), it follows that

$$\phi(u, x_n) - \phi(u, u_n) \longrightarrow 0 \quad (n \longrightarrow \infty). \quad (3.32)$$

In view of condition (b) and condition (c), we have that

$$g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \quad (3.33)$$

It follows from the property of g that

$$\left\|Jx_n - JS_j^n x_n\right\| \longrightarrow 0, \quad (\text{as } n \longrightarrow \infty). \quad (3.34)$$

Since $x_n \rightarrow p$ and J is uniformly continuous, it yields $Jx_n \rightarrow Jp$. Hence from (3.34) we have

$$JS_j^n x_n \longrightarrow Jp \quad (\text{as } n \longrightarrow \infty). \quad (3.35)$$

Since $J^{-1} : E^* \rightarrow E$ is hemicontinuous, it follows that

$$S_j^n x_n \rightharpoonup p \quad (\forall j \geq 1). \quad (3.36)$$

On the other hand, for each $j \geq 1$ we have

$$\left| \|S_j^n x_n\| - \|p\| \right| = \left| \|J(S_j^n x_n)\| - \|Jp\| \right| \leq \|JS_j^n x_n - Jp\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \quad (3.37)$$

This together with (3.36) shows that

$$S_j^n x_n \longrightarrow p \quad (\text{for each } j \geq 1). \quad (3.38)$$

Furthermore, by the assumption that for each $j \geq 1$, S_j is uniformly L_j -Lipschitz continuous, hence we have

$$\begin{aligned} \|S_j^{n+1}x_n - S_j^n x_n\| &\leq \|S_j^{n+1}x_n - S_j^{n+1}x_{n+1}\| + \|S_j^{n+1}x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| + \|x_n - S_j^n x_n\| \\ &\leq (L_j + 1)\|x_{n+1} - x_n\| + \|S_j^{n+1}x_{n+1} - x_{n+1}\| + \|x_n - S_j^n x_n\|. \end{aligned} \quad (3.39)$$

This together with (3.20) and (3.38), yields $\|S_j^{n+1}x_n - S_j^n x_n\| \rightarrow 0$ (as $n \rightarrow \infty$). Hence from (3.36) we have $S_j^{n+1}x_n \rightarrow p$, that is, $S_j S_j^n x_n \rightarrow p$. In view of (3.38) and the closeness of S_j , it yields that $S_j p = p$, for all $j \geq 1$. This implies that $p \in \bigcap_{i=1}^{\infty} F(S_i)$.

Next, we prove that $p \in \bigcap_{i=1}^M \text{GMEP}(F_i, A_i, \varphi_i)$. Denote that

$$u_n^{(m)} = K_{r_{m,n}}^{\Gamma_m} K_{r_{m-1,n}}^{\Gamma_{m-1}} \cdots K_{r_{2,n}}^{\Gamma_2} K_{r_{1,n}}^{\Gamma_1} y_n, \quad m = 1, 2, \dots, M-1, \quad u_n^{(M)} = u_n. \quad (3.40)$$

By the similar method as in the proof of (3.13), we can prove that

$$\phi(u, u_n^{(m)}) \leq \phi(u, x_n) + \eta_n, \quad m = 1, 2, \dots, M, \quad u \in \mathcal{F}, \quad \forall n \geq 1. \quad (3.41)$$

It follows from Lemma 2.6, (2.15), (3.32) that for any $u \in \mathcal{F}$,

$$\begin{aligned} \phi(u_n^{(M)}, u_n^{(M-1)}) &= \phi(K_{r_n}^{\Gamma_M} u_n^{(M-1)}, u_n^{(M-1)}) \\ &\leq \phi(u, u_n^{(M-1)}) - \phi(u, K_{r_n}^{\Gamma_M} u_n^{(M-1)}) \\ &\leq \phi(u, x_n) + \eta_n - \phi(u, \Gamma_{r_n}^M u_n^{(M-1)}) \\ &= \phi(u, x_n) + \eta_n - \phi(u, u_n^{(M)}) \\ &= \phi(u, x_n) + \eta_n - \phi(u, u_n) \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \end{aligned} \quad (3.42)$$

From (1.10) it yields $(\|u_n^{(M)}\| - \|u_n^{(M-1)}\|)^2 \rightarrow 0$. Since $\|u_n^{(M)}\| = \|u_n\| \rightarrow \|p\|$, we have

$$\|u_n^{(M-1)}\| \rightarrow \|p\| \quad (\text{as } n \rightarrow \infty). \quad (3.43)$$

Hence we have

$$\|Ju_n^{(M-1)}\| \rightarrow \|Jp\| \quad (\text{as } n \rightarrow \infty). \quad (3.44)$$

This implies that $\{Ju_n^{(M-1)}\}$ is bounded in E^* . Since E is reflexive, and so E^* is reflexive, there exists a subsequence $\{Ju_{n_i}^{(M-1)}\} \subset \{Ju_n^{(M-1)}\}$ such that $Ju_{n_i}^{(M-1)} \rightharpoonup p_0 \in E^*$. In view of the reflexive of E , we see that $J(E) = E^*$. Hence there exists $x \in E$ such that $Jx = p_0$. Since

$$\begin{aligned}\phi(u_{n_i}^{(M)}, u_{n_i}^{(N-1)}) &= \|u_{n_i}^{(M)}\|^2 - 2\langle u_{n_i}^{(M)}, Ju_{n_i}^{(M-1)} \rangle + \|u_{n_i}^{(M-1)}\|^2 \\ &= \|u_{n_i}^{(M)}\|^2 - 2\langle u_{n_i}^{(M)}, Ju_{n_i}^{(M-1)} \rangle + \|Ju_{n_i}^{(M-1)}\|^2\end{aligned}\quad (3.45)$$

taking $\liminf_{n_i \rightarrow \infty}$ on the both sides of above equality and in view of the weak lower semi-continuity of norm $\|\cdot\|$, it yields that

$$\begin{aligned}0 &\geq \|p\|^2 - 2\langle p, p_0 \rangle + \|p_0\|^2 = \|p\|^2 - 2\langle p, Jx \rangle + \|Jx\|^2 \\ &= \|p\|^2 - 2\langle p, Jx \rangle + \|x\|^2 = \phi(p, x).\end{aligned}\quad (3.46)$$

This is, $p = x$. This implies that $p_0 = Jp$, and so $Ju_{n_i}^{(M-1)} \rightharpoonup Jp$. It follows from (3.44) and the Kadec-Klee property of E^* that $Ju_{n_i}^{(M-1)} \rightarrow Jp$ (as $n_i \rightarrow \infty$). Note that $J^{-1} : E^* \rightarrow E$ is hemi-continuous it yields that $u_{n_i}^{(M-1)} \rightharpoonup p$. It follows from (3.43) and the Kadec-Klee property of E that $\lim_{n_i \rightarrow \infty} u_{n_i}^{(M-1)} = p$.

By the similar way as given in the proof of (3.20), we can also prove that

$$\lim_{n \rightarrow \infty} u_n^{(M-1)} = p. \quad (3.47)$$

From (3.27) and (3.47) we have that

$$\|u_n^{(M)} - u_n^{(M-1)}\| \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \quad (3.48)$$

Since J is uniformly continuous on any bounded subset of E , we have

$$\|Ju_n^{(M)} - Ju_n^{(M-1)}\| \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \quad (3.49)$$

Since

$$u_n^{(i)} = K_{r_n}^{\Gamma_i} u_n^{(i-1)}, \quad i = 2, 3, \dots, M, u_n^{(0)} = y_n, u_n^{(M)} = u_n. \quad (3.50)$$

By the similar way as above, we can also prove that

$$\begin{aligned}u_n^{(i)} &\rightarrow p, \quad \|u_n^{(i)} - u_n^{(i-1)}\| \rightarrow 0, \quad \|Ju_n^{(i)} - Ju_n^{(i-1)}\| \rightarrow 0, \quad i = 2, 3, \dots, M \\ \|y_n - u_n^{(1)}\| &\rightarrow 0, \quad \|Jy_n - Ju_n^{(1)}\| \rightarrow 0 \quad (\text{as } n \rightarrow \infty).\end{aligned}\quad (3.51)$$

From (3.51) and the assumption that $r_n \geq d$, $\forall n \geq 0$, we have

$$\lim_{n \rightarrow \infty} \frac{\|Ju_n^{(i)} - Ju_n^{(i-1)}\|}{r_{i,n}} = 0, \quad i = 2, 3, \dots, M; \quad \lim_{n \rightarrow \infty} \frac{\|Jy_n - Ju_n^{(1)}\|}{r_{1,n}} = 0. \quad (3.52)$$

In the proof of Lemma 2.6 we have proved that the function $\Gamma_i, i = 1, 2, \dots, M$ defined by (3.6) satisfies the condition (A1)–(A4) and

$$\Gamma_i(u_n^{(i)}, y) + \frac{1}{r_{i,n}} \langle y - u_n^{(i)}, Ju_n^{(i)} - Ju_n^{(i-1)} \rangle \geq 0, \quad \forall y \in C. \quad (3.53)$$

Therefore for any $y \in C$ we have

$$\frac{1}{r_{i,n}} \langle y - u_n^{(i)}, Ju_n^{(i)} - Ju_n^{(i-1)} \rangle \geq -\Gamma_i(u_n^{(i)}, y) \geq \Gamma_i(y, u_n^{(i)}). \quad (3.54)$$

This implies that

$$\begin{aligned} \Gamma_i(y, u_n^{(i)}) &\leq \frac{1}{r_{i,n}} \langle y - u_n^{(i)}, Ju_n^{(i)} - Ju_n^{(i-1)} \rangle \\ &\leq (M_1 + \|y\|) \frac{\|Ju_n^{(i)} - Ju_n^{(i-1)}\|}{r_{i,n}} \end{aligned} \quad (3.55)$$

for some constant $M_1 > 0$. Since the function $y \mapsto \Gamma_i(x, y)$ is convex and lower semi-continuous, letting $n \rightarrow \infty$ in (3.55), from (3.52) and (3.55), for each i , we have $\Gamma_i(y, p) \leq 0$, for all $y \in C$.

For $t \in (0, 1]$ and $y \in C$, letting $y_t = ty + (1-t)p$, there are $y_t \in C$ and $\Gamma_i(y_t, p) \leq 0$. By condition (A1) and (A4), we have

$$0 = \Gamma_i(y_t, y_t) \leq t\Gamma_i(y_t, y) + (1-t)\Gamma_i(y_t, p) \leq t\Gamma_i(y_t, y). \quad (3.56)$$

Dividing both sides of the above equation by t , we have $\Gamma_i(y_t, y) \geq 0$, for all $y \in C$. Letting $t \downarrow 0$, from condition (A3), we have $\Gamma_i(p, y) \geq 0$, for all $y \in C$, for all $i = 1, 2, \dots, M$, that is, for each $i = 1, 2, \dots, M$, we have

$$F_i(p, y) + \langle A_i p, y - p \rangle + \psi_i(y) - \psi_i(p) \geq 0, \quad \forall y \in C. \quad (3.57)$$

This implies that $p \in \bigcap_{j=1}^M \text{GMEP}(F_j, A_j, \psi_j)$. Therefore, we have that

$$p \in \mathcal{F}. \quad (3.58)$$

(V) Now, we prove $x_n \rightarrow \Pi_{\mathcal{F}} x_0$.

Let $w = \Pi_{\mathcal{F}}x_0$. From $w \in \mathcal{F} \subset C_{n+1}$ and $x_{n+1} = \Pi_{C_{n+1}}x_0$, we have $\phi(x_{n+1}, x_0) \leq \phi(w, x_0)$, for all $n \geq 0$. This implies that

$$\phi(p, x_0) = \lim_{n \rightarrow \infty} \phi(x_n, x_0) \leq \phi(w, x_0). \quad (3.59)$$

By the definition of $\Pi_{\mathcal{F}}x_0$ and (3.59), we have $p = w$. Therefore, $x_n \rightarrow \Pi_{\mathcal{F}}x_0$. This completes the proof of Theorem 3.1. $\square$

Theorem 3.2. Let $E, C, \{K_{r_i,n}^{\Gamma_i}\}_{i=1}^M, \{A_i\}_{i=1}^M, \{F_i\}_{i=1}^M, \{\psi_i\}_{i=1}^M, \{\text{GMEP}(F_j, A_j, \psi_j)\}_{j=1}^M$ be the same as above. Let $\{S_i\}_{i=1}^\infty : C \rightarrow C$ be an infinite family of closed and uniformly quasi- ϕ -asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \rightarrow 1$. Suppose that for each $i \geq 1, S_i$ is uniformly L_i -Lipschitz continuous and that

$$G := \bigcap_{i=1}^\infty F(S_i) \bigcap \bigcap_{j=1}^M \text{GMEP}(F_j, A_j, \psi_j) \quad (3.60)$$

is a nonempty and bounded subset of C . For any given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_0 &\in C_0 = C, \\ z_n &= J^{-1} \left(\alpha_{n,0} Jx_n + \sum_{i=1}^\infty \alpha_{n,i} JS_i^n x_n \right), \\ y_n &= J^{-1} (\alpha_n Jz_n + (1 - \alpha_n) Jx_n), \\ u_n &= K_{r_{M,n}}^{\Gamma_M} K_{r_{M-1,n}}^{\Gamma_{M-1}} \cdots K_{r_{2,n}}^{\Gamma_2} K_{r_{1,n}}^{\Gamma_1} y_n, \\ C_{n+1} &= \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n) + \xi_n\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{aligned} \quad (3.61)$$

where $\xi_n = \sup_{u \in G} (k_n - 1) \phi(u, x_n)$, $r_n \in [d, \infty)$ for some $d > 0$, and for $i \geq 0$, $\{\alpha_{n,i}\}, \{\alpha_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:

- (a) $\sum_{i=0}^\infty \alpha_{n,i} = 1$ for all $n \geq 0$;
- (b) $\liminf_{n \rightarrow \infty} \alpha_{n,0} \cdot \alpha_{n,i} > 0$ for all $i \geq 1$;
- (c) $0 < \alpha \leq \alpha_n < 1$ for some $\alpha \in (0, 1)$.

Then $\{x_n\}$ converges strongly to $\Pi_G x_0$.

Proof. Since $\{S_i\}_{i=1}^\infty : C \rightarrow C$ is an infinite family of closed quasi- ϕ -asymptotically nonexpansive mappings, it is an infinite family of closed and uniformly total quasi- ϕ -asymptotically nonexpansive mappings with sequence $\zeta(t) = t, t \geq 0, v_n = k_n - 1, \mu_n = 0$. Hence $\zeta_n = v_n \sup_{u \in G} \zeta(\phi(u, x_n)) + \mu_n = \sup_{u \in G} (k_n - 1) \phi(u, x_n) \rightarrow 0$. Therefore all conditions in Theorem 3.1 are satisfied. The conclusion of Theorem 3.2 is obtained from Theorem 3.1 immediately. $\square$

Remark 3.3. Theorems 3.1 and 3.2 improve and extend the corresponding results in [8, 11, 15, 16, 18–24, 28] and others in the following aspects.

- (a) For the framework of spaces, we extend the space from a uniformly smooth and uniformly convex Banach space to a uniformly smooth and strictly convex Banach space with the Kadec-Klee property (note that each uniformly convex Banach space must have Kadec-Klee property).
- (b) For the mappings, we extend the mappings from nonexpansive mappings, relatively nonexpansive mappings, quasi- ϕ -nonexpansive mapping or quasi- ϕ -asymptotically nonexpansive mappings to a countable family of total quasi- ϕ -asymptotically nonexpansive mappings.
- (c) We extend a single generalized mixed equilibrium problem to a system of generalized mixed equilibrium problems.

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References

- [1] S.-S. Chang, H. W. J. Lee, and C. K. Chan, "A new hybrid method for solving a generalized equilibrium problem, solving a variational inequality problem and obtaining common fixed points in Banach spaces, with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 7, pp. 2260–2270, 2010.
- [2] L.-C. Ceng and J.-C. Yao, "A hybrid iterative scheme for mixed equilibrium problems and fixed point problems," *Journal of Computational and Applied Mathematics*, vol. 214, no. 1, pp. 186–201, 2008.
- [3] J. Tang and S.-S. Chang, "Strong convergence theorem for a generalized mixed equilibrium problem and fixed point problem for a family of infinitely nonexpansive mappings in Hilbert spaces," *Panamerican Mathematical Journal*, vol. 19, no. 2, pp. 75–86, 2009.
- [4] E. U. Ofoedu and D. M. Malonza, "Hybrid approximation of solutions of nonlinear operator equations and application to equation of Hammerstein-type," *Applied Mathematics and Computation*, vol. 217, no. 13, pp. 6019–6030, 2011.
- [5] S.-s. Zhang, "Generalized mixed equilibrium problem in Banach spaces," *Applied Mathematics and Mechanics. English Edition*, vol. 30, no. 9, pp. 1105–1112, 2009.
- [6] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, vol. 62 of *Mathematics and Its Applications*, Kluwer Academic Publishers Group, Dordrecht, The Netherlands, 1990.
- [7] Y. I. Alber, "Metric and generalized projection operators in Banach spaces: properties and applications," in *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, vol. 178 of *Lecture Notes in Pure and Appl. Math.*, pp. 15–50, Dekker, New York, NY, USA, 1996.
- [8] S.-y. Matsushita and W. Takahashi, "A strong convergence theorem for relatively nonexpansive mappings in a Banach space," *Journal of Approximation Theory*, vol. 134, no. 2, pp. 257–266, 2005.
- [9] Y. Su, H.-k. Xu, and X. Zhang, "Strong convergence theorems for two countable families of weak relatively nonexpansive mappings and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 12, pp. 3890–3906, 2010.
- [10] W. Nilsrakoo and S. Saejung, "Strong convergence to common fixed points of countable relatively quasi-nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2008, Article ID 312454, 19 pages, 2008.

- [11] H. Zhou, G. Gao, and B. Tan, "Convergence theorems of a modified hybrid algorithm for a family of quasi- ϕ -asymptotically nonexpansive mappings," *Journal of Applied Mathematics and Computing*, vol. 32, no. 2, pp. 453–464, 2010.
- [12] S.-Y. Matsushita and W. Takahashi, "A strong convergence theorem for relatively nonexpansive mappings in a Banach space," *Journal of Approximation Theory*, vol. 134, no. 2, pp. 257–266, 2005.
- [13] Y. Su, H.-K. Xu, and X. Zhang, "Strong convergence theorems for two countable families of weak relatively nonexpansive mappings and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 12, pp. 3890–3906, 2010.
- [14] H. Zegeye and N. Shahzad, "Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 7, pp. 2707–2716, 2009.
- [15] K. Wattanawitoon and P. Kumam, "Strong convergence theorems by a new hybrid projection algorithm for fixed point problems and equilibrium problems of two relatively quasi-nonexpansive mappings," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 1, pp. 11–20, 2009.
- [16] X. Qin, S. Y. Cho, and S. M. Kang, "On hybrid projection methods for asymptotically quasi- ϕ -nonexpansive mappings," *Applied Mathematics and Computation*, vol. 215, no. 11, pp. 3874–3883, 2010.
- [17] W. Takahashi and K. Zembayashi, "Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 1, pp. 45–57, 2009.
- [18] S.-S. Chang, J. K. Kim, and X. R. Wang, "Modified block iterative algorithm for solving convex feasibility problems in Banach spaces," *Journal of Inequalities and Applications*, vol. 2010, Article ID 869684, 14 pages, 2010.
- [19] S.-s. Chang, X. R. Wang, H. W. J. Lee, and C. K. Chan, "Strong convergence theorems for asymptotically relatively nonexpansive mappings with application," *Communications on Applied Nonlinear Analysis*, vol. 18, no. 1, pp. 45–60, 2011.
- [20] Y. Yao, Y. J. Cho, and Y.-C. Liou, "Iterative algorithms for variational inclusions, mixed equilibrium and fixed point problems with application to optimization problems," *Central European Journal of Mathematics*, vol. 9, no. 3, pp. 640–656, 2011.
- [21] Y. Yao, Y. J. Cho, and Y.-C. Liou, "Algorithms of common solutions for variational inclusions, mixed equilibrium problems and fixed point problems," *European Journal of Operational Research*, vol. 212, no. 2, pp. 242–250, 2011.
- [22] X. Qin, S.-S. Chang, and Y. J. Cho, "Iterative methods for generalized equilibrium problems and fixed point problems with applications," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 4, pp. 2963–2972, 2010.
- [23] Y. J. Cho, I. K. Argyros, and N. Petrot, "Approximation methods for common solutions of generalized equilibrium, systems of nonlinear variational inequalities and fixed point problems," *Computers & Mathematics with Applications*, vol. 60, no. 8, pp. 2292–2301, 2010.
- [24] Y. J. Cho, X. Qin, and J. I. Kang, "Convergence theorems based on hybrid methods for generalized equilibrium problems and fixed point problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 9, pp. 4203–4214, 2009.
- [25] H. Zhang and Y. Su, "Strong convergence of modified hybrid algorithm for quasi- ϕ -asymptotically nonexpansive mappings," *Korean Mathematical Society. Communications*, vol. 24, no. 4, pp. 539–551, 2009.
- [26] S. Kamimura and W. Takahashi, "Strong convergence of a proximal-type algorithm in a Banach space," *SIAM Journal on Optimization*, vol. 13, no. 3, pp. 938–945, 2002.
- [27] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.
- [28] W. Takahashi and K. Zembayashi, "Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 1, pp. 45–57, 2009.