Research Article

Hermite-Hadamard-Type Inequalities for r -Preinvex Functions

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We aim to find Hermite-Hadamard inequality for r -preinvex functions. Also, it is investigated for the product of an r -preinvex function and s -preinvex function.

1. Introduction

Several interesting generalizations and extensions of classical convexity have been studied and investigated in recent years. Hanson [1] introduced the invex functions as a generalization of convex functions. Later, subsequent works inspired from Hanson's result have greatly found the role and applications of invexity in nonlinear optimization and other branches of pure and applied sciences. The basic properties and role of preinvex functions in optimization, equilibrium problems and variational inequalities were studied by Noor [2, 3] and Weir and Mond [4]. The well known Hermite-Hadamard inequality has been extensively investigated for the convex functions and their variant forms. Noor [5] and Pachpatte [6] studied these inequalities and related types for preinvex and log-preinvex functions.

The Hermite-Hadamard inequality was investigated in [7] to an r -convex positive function which is defined on an interval $[a, b]$. Certain refinements of the Hadamard inequality for r -convex functions were studied in [8, 9]. Recently, Zabandan et al. [10] extended and refined the work of [8]. Bessenyei [11] studied Hermite-Hadamard-type inequalities for generalized 3-convex functions. In this paper, we introduce the r -preinvex functions and establish Hermite-Hadamard inequality for such functions by using the method of [10].

2. Preliminaries

Let $f : K \rightarrow \mathbb{R}$ and $\eta(\cdot, \cdot) : K \times K \rightarrow \mathbb{R}$ be continuous functions, where $K \subset \mathbb{R}^n$ is a nonempty closed set. We use

the notations, $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, for inner product and norm, respectively. We require the following well known concepts and results which are essential in our investigations.

Definition 1 (see [2, 4]). Let $u \in K$. Then, the set K is said to be invex at $u \in K$ with respect to $\eta(\cdot, \cdot)$ if

$$u + t\eta(v, u) \in K, \quad \forall u, v \in K, t \in [0, 1]. \quad (1)$$

K is said to be invex set with respect to $\eta(\cdot, \cdot)$ if it is invex at every $u \in K$. The invex set K is also called a η connected set.

Geometrically, Definition 1 says that there is a path starting from the point u which is contained in K . The point v should not be one of the end points of the path in general, see [12]. This observation plays a key role in our study. If we require that v should be an end point of the path for every pair of points $u, v \in K$, then $\eta(v, u) = v - u$, and consequently, invexity reduces to convexity. Thus, every convex set is also an invex set with respect to $\eta(v, u) = v - u$, but the converse is not necessarily true; see [4, 13] and the references therein. For the sake of simplicity, we assume that $K = [a, a + \eta(b, a)]$, unless otherwise specified.

Definition 2 (see [4]). The function f on the invex set K is said to be preinvex with respect to η if

$$f(u + t\eta(v, u)) \leq (1 - t)f(u) + tf(v), \quad \forall u, v \in K, t \in [0, 1]. \quad (2)$$

Note that every convex function is a preinvex function, but the converse is not true. For example, the function $f(u) =$

$-|u|$ is not a convex function, but it is a preinvex function with respect to η , where

$$\eta(v, u) = \begin{cases} v - u, & \text{if } v \leq 0, u \leq 0 \text{ or } v \geq 0, u \geq 0, \\ u - v, & \text{otherwise.} \end{cases} \quad (3)$$

The concepts of the invex and preinvex functions have played very important roles in the development of generalized convex programming, see [14–17]. For more characterizations and applications of invex and preinvex functions, we refer to [3, 15, 18–25].

Antczak [26, 27] introduced and studied the concept of r -invex and r -preinvex functions. Here, we define the following.

Definition 3. A positive function f on the invex set K is said to be r -preinvex with respect to η if, for each $u, v \in K$, $t \in [0, 1]$:

$$f(u + t\eta(v, u)) \leq \begin{cases} ((1-t)f^r(u) + tf^r(v))^{1/r}, & r \neq 0 \\ (f(u))^{1-t}(f(v))^t, & r = 0. \end{cases} \quad (4)$$

Note that 0-preinvex functions are logarithmic preinvex and 1-preinvex functions are classical preinvex functions. It should be noted that if f is r -preinvex function, then f^r is preinvex function ($r > 0$).

The well known Hermite-Hadamard inequality for a convex function defined on the interval $[a, b]$ is given by

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}; \quad (5)$$

see [6, 28, 29].

Hermite-Hadamard inequalities for log-convex functions were proved by Dragomir and Mond [30]. Pachpatte [6, 17] also gave some other refinements of these inequalities related to differentiable log-convex functions. Noor [5] proved the following Hermite-Hadamard inequalities for the preinvex and log-preinvex functions, respectively.

(i) Let $f : K = [a, a + \eta(b, a)] \rightarrow (0, \infty)$ be a preinvex function on the interval of real numbers K^0 (interior of K) and $a, b \in K^0$ with $a < a + \eta(b, a)$. Then, the following inequalities hold:

$$f\left(\frac{2a + \eta(b, a)}{2}\right) \leq \frac{1}{\eta(b, a)} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \quad (6)$$

(ii) Let f be a log-preinvex function defined on the interval $[a, a + \eta(b, a)]$. Then,

$$\begin{aligned} \frac{1}{\eta(b, a)} \int_a^b f(x) dx &\leq \frac{f(a) - f(b)}{\log f(a) - \log f(b)} \\ &= L(f(a), f(b)), \end{aligned} \quad (7)$$

where $L(p, q)$ ($p \neq q$) is the logarithmic mean of positive real numbers p, q .

3. Main Results

Theorem 4. Let $f : K = [a, a + \eta(b, a)] \rightarrow (0, \infty)$ be an r -preinvex function on the interval of real numbers K^0 (interior of K) and $a, b \in K^0$ with $a < a + \eta(b, a)$. Then, the following inequalities hold:

$$\frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \leq \left[\frac{f^r(a) + f^r(b)}{2} \right]^{1/r}, \quad r \geq 1. \quad (8)$$

Proof. From Jensen's inequality, we obtain

$$\left(\frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right)^r \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f^r(x) dx. \quad (9)$$

Since f^r is preinvex, then, using Hermite-Hadamard inequality for preinvex functions (see [5]), we have

$$\frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f^r(x) dx \leq \frac{f^r(a) + f^r(b)}{2}. \quad (10)$$

Hence,

$$\frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \leq \left[\frac{f^r(a) + f^r(b)}{2} \right]^{1/r}. \quad (11)$$

This completes the proof. $\square$

Corollary 5. Let $f : K = [a, a + \eta(b, a)] \rightarrow (0, \infty)$ be a 1-preinvex function on the interval of real numbers K^0 (interior of K) and $a, b \in K^0$ with $a < a + \eta(b, a)$. Then,

$$\frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \leq \frac{f(a) + f(b)}{2}. \quad (12)$$

Theorem 6. Let $f : K = [a, a + \eta(b, a)] \rightarrow (0, \infty)$ be an r -preinvex function (with $r \geq 0$) on the interval of real numbers K^0 (interior of K) and $a, b \in K^0$ with $a < a + \eta(b, a)$. Then, the following inequalities hold:

$$\begin{aligned} &\frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \\ &\leq \begin{cases} \frac{r}{r+1} \left[\frac{f^{r+1}(a) - f^{r+1}(b)}{f^r(a) - f^r(b)} \right], & r \neq 0 \\ \frac{f(a) - f(b)}{\log f(a) - \log f(b)}, & r = 0. \end{cases} \end{aligned} \quad (13)$$

Proof. For $r = 0$, Noor [5] proved this result. We proceed for the case $r > 0$. Since f is r -preinvex function, for all $t \in [0, 1]$, we have

$$f(a + t\eta(b, a)) \leq ((1-t)f^r(a) + tf^r(b))^{1/r}. \quad (14)$$

Therefore,

$$\begin{aligned} & \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \\ &= \int_0^1 f(a + t\eta(b, a)) dt \\ &\leq \int_0^1 (f^r(a) + t(f^r(b) - f^r(a)))^{1/r} dt. \end{aligned} \quad (15)$$

Putting $u = f^r(a) + t(f^r(b) - f^r(a))$, we have

$$\begin{aligned} \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx &\leq \frac{1}{f^r(b) - f^r(a)} \int_{f^r(a)}^{f^r(b)} (u)^{1/r} dt \\ &= \frac{r}{r+1} \frac{f^{r+1}(b) - f^{r+1}(a)}{f^r(b) - f^r(a)}, \end{aligned} \quad (16)$$

which completes the proof. $\square$

Note that for $r = 1$, in Theorem 6, we have the same inequality again as in Corollary 5.

Theorem 7. Let $f : K = [a, a + \eta(b, a)] \rightarrow (0, \infty)$ be an r -preinvex function with $0 \leq r \leq s$ on the interval of real numbers K^0 (interior of K) and $a, b \in K^0$ with $a < a + \eta(b, a)$. Then, f is s -preinvex function.

Proof. To prove this, we need the following inequality:

$$u^{1-t} v^t \leq ((1-t)u^r + tv^r)^{1/r} \leq ((1-t)u^s + tv^s)^{1/s}, \quad (17)$$

(see [10]) where $0 \leq t \leq 1$, $0 \leq r \leq s$. Since f is r -preinvex, by inequality (17), for all $u, v \in K^0$, $t \in [0, 1]$, we obtain

$$\begin{aligned} & f(u + t\eta(v, u)) \\ &\leq \begin{cases} ((1-t)f^r(u) + tf^r(v))^{1/r} \\ \leq ((1-t)f^s(u) + tf^s(v))^{1/s}, & 0 < r \leq s \\ (f(u))^{1-t} (f(v))^t \\ \leq ((1-t)f^s(u) + tf^s(v))^{1/s}, & 0 = r \leq s. \end{cases} \end{aligned} \quad (18)$$

Hence, f is s -preinvex function. $\square$

As a special case of Theorem 7, we deduce the following result.

Corollary 8. Let $f : K = [a, a + \eta(b, a)] \rightarrow (0, \infty)$ be an r -preinvex function with $0 \leq r \leq 1$ on the interval of real numbers K^0 (interior of K) and $a, b \in K^0$ with $a < a + \eta(b, a)$. Then, f is preinvex function.

Theorem 9. Let $f : K = [a, a + \eta(b, a)] \rightarrow (0, \infty)$ be an r -preinvex function with $0 \leq r \leq s$ on the interval of real

numbers K^0 (interior of K) and $a, b \in K^0$ with $a < a + \eta(b, a)$. Then, the following inequalities hold:

$$\begin{aligned} & \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \\ &\leq \begin{cases} \frac{r}{r+1} \left[\frac{f^{r+1}(a) - f^{r+1}(b)}{f^r(a) - f^r(b)} \right] \\ \leq \frac{s}{s+1} \left[\frac{f^{s+1}(a) - f^{s+1}(b)}{f^s(a) - f^s(b)} \right], & 0 < r \leq s \\ \frac{f(a) - f(b)}{\log f(a) - \log f(b)} \\ \leq \frac{s}{s+1} \left[\frac{f^{s+1}(a) - f^{s+1}(b)}{f^s(a) - f^s(b)} \right], & 0 = r \leq s. \end{cases} \end{aligned} \quad (19)$$

Proof. The left side of these inequalities is clear from Theorem 7. For the right hand side, we have from inequality (17)

$$((1-t)f^r(a) + tf^r(b))^{1/r} \leq ((1-t)f^s(a) + tf^s(b))^{1/s}. \quad (20)$$

Integrating both sides with respect to t , we obtain

$$\frac{r}{r+1} \frac{f^{r+1}(a) - f^{r+1}(b)}{f^r(a) - f^r(b)} \leq \frac{s}{s+1} \left[\frac{f^{s+1}(a) - f^{s+1}(b)}{f^s(a) - f^s(b)} \right]. \quad (21)$$

Again, using the other part of (17) and integrating, one can easily deduce the second part of the required inequalities. $\square$

Corollary 10. Let $f : K = [a, a + \eta(b, a)] \rightarrow (0, \infty)$ be an r -preinvex function with $0 \leq r \leq 1$ on the interval of real numbers K^0 (interior of K) and $a, b \in K^0$ with $a < a + \eta(b, a)$. Then, the following inequalities hold:

$$\begin{aligned} & f\left(\frac{2a + \eta(b, a)}{2}\right) \\ &\leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \leq [f(a) - f(b)] \ln \frac{f(a)}{f(b)} \\ &\leq \frac{r}{r+1} \left[\frac{f^{r+1}(a) - f^{r+1}(b)}{f^r(a) - f^r(b)} \right] \leq \frac{f(a) + f(b)}{2}. \end{aligned} \quad (22)$$

Theorem 11. Let $f, g : K = [a, a + \eta(b, a)] \rightarrow (0, \infty)$ be r -preinvex and s -preinvex functions, respectively, with $r, s > 0$

on the interval of real numbers K^0 (interior of K) and $a, b \in K^0$ with $a < a + \eta(b, a)$. Then, the following inequalities hold:

$$\begin{aligned} & \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) g(x) dx \\ & \leq \frac{1}{2} \frac{r}{r+2} \left[\frac{f^{r+2}(a) - f^{r+2}(b)}{f^r(a) - f^r(b)} \right] \\ & \quad + \frac{1}{2} \frac{s}{s+2} \left[\frac{g^{s+2}(a) - g^{s+2}(b)}{g^s(a) - g^s(b)} \right] \\ & \quad (f(a) \neq f(b), g(a) \neq g(b)). \end{aligned} \quad (23)$$

Proof. Since f is r -preinvex and g is s -preinvex, for all $t \in [0, 1]$, we have

$$\begin{aligned} f(a + t\eta(b, a)) & \leq ((1-t)f^r(a) + tf^r(b))^{1/r}, \\ g(a + t\eta(b, a)) & \leq ((1-t)g^s(a) + tg^s(b))^{1/s}. \end{aligned} \quad (24)$$

Therefore,

$$\begin{aligned} & \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) g(x) dx \\ & = \int_0^1 f(a + t\eta(b, a)) g(a + t\eta(b, a)) dt \\ & \leq \int_0^1 [(1-t)f^r(a) + tf^r(b)]^{1/r} \\ & \quad \times [(1-t)g^s(a) + tg^s(b)]^{1/s} dt \\ & = \int_0^1 [f^r(a) + t(f^r(b) - f^r(a))]^{1/r} \\ & \quad \times [g^s(a) + t(g^s(b) - g^s(a))]^{1/s} dt. \end{aligned} \quad (25)$$

Now, applying Cauchy's inequality, we obtain

$$\begin{aligned} & \int_0^1 [f^r(a) + t(f^r(b) - f^r(a))]^{1/r} \\ & \quad \times [g^s(a) + t(g^s(b) - g^s(a))]^{1/s} dt \\ & \leq \frac{1}{2} \int_0^1 [f^r(a) + t(f^r(b) - f^r(a))]^{2/r} dt \\ & \quad + \frac{1}{2} \int_0^1 [g^s(a) + t(g^s(b) - g^s(a))]^{2/s} dt \\ & = \frac{1}{2} \frac{r}{r+2} \left[\frac{f^{r+2}(a) - f^{r+2}(b)}{f^r(a) - f^r(b)} \right] \\ & \quad + \frac{1}{2} \frac{s}{s+2} \left[\frac{g^{s+2}(a) - g^{s+2}(b)}{g^s(a) - g^s(b)} \right], \end{aligned} \quad (26)$$

which leads us to the required result. $\square$

Note. By putting $f = g$, $r = s = 2$, in Theorem 11, we have the following inequality:

$$\frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f^2(x) dx \leq \frac{f^2(a) + f^2(b)}{2}. \quad (27)$$

Theorem 11, for $r = s = 0$, was proved by Noor [31].

Theorem 12. Let $f, g : K = [a, a + \eta(b, a)] \rightarrow (0, \infty)$ be r -preinvex ($r > 0$) and 0-preinvex functions, respectively, on the interval of real numbers K^0 (interior of K) and $a, b \in K^0$ with $a < a + \eta(b, a)$. Then, the following inequalities hold:

$$\begin{aligned} & \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) g(x) dx \\ & \leq \frac{1}{2} \frac{r}{r+2} \left[\frac{f^{r+2}(a) - f^{r+2}(b)}{f^r(a) - f^r(b)} \right] \\ & \quad + \frac{1}{4} \left[\frac{g^2(a) - g^2(b)}{\log g(a) - \log g(b)} \right] \\ & \quad (f(a) \neq f(b)). \end{aligned} \quad (28)$$

Proof. Since f is r -preinvex and g is 0-preinvex, for all $t \in [0, 1]$, we have

$$\begin{aligned} f(a + t\eta(b, a)) & \leq ((1-t)f^r(a) + tf^r(b))^{1/r}, \\ g(a + t\eta(b, a)) & \leq (g(a))^{1-t} (g(b))^t. \end{aligned} \quad (29)$$

Therefore,

$$\begin{aligned} & \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) g(x) dx \\ & = \int_0^1 f(a + t\eta(b, a)) g(a + t\eta(b, a)) dt \\ & \leq \int_0^1 [(1-t)f^r(a) + tf^r(b)]^{1/r} \\ & \quad \times [(g(a))^{1-t} (g(b))^t] dt \\ & = \int_0^1 [f^r(a) + t(f^r(b) - f^r(a))]^{1/r} \\ & \quad \times [(g(a))^{1-t} (g(b))^t] dt. \end{aligned} \quad (30)$$

Now, applying Cauchy's inequality, we obtain

$$\begin{aligned} & \int_0^1 [f^r(a) + t(f^r(b) - f^r(a))]^{1/r} [(g(a))^{1-t} (g(b))^t] dt \\ & \leq \frac{1}{2} \int_0^1 [f^r(a) + t(f^r(b) - f^r(a))]^{2/r} dt \\ & \quad + \frac{1}{2} \int_0^1 [(g(a))^{2-2t} (g(b))^{2t}] dt \\ & = \frac{1}{2} \frac{r}{r+2} \left[\frac{f^{r+2}(a) - f^{r+2}(b)}{f^r(a) - f^r(b)} \right] + \frac{1}{4} \left[\frac{g^2(a) - g^2(b)}{\log(g(a)/g(b))} \right], \end{aligned} \quad (31)$$

which is the required result. $\square$

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