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Research Article

Synchronization between Fractional-Order and Integer-Order Hyperchaotic Systems via Sliding Mode Controller

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The synchronization between fractional-order hyperchaotic systems and integer-order hyperchaotic systems via sliding mode controller is investigated. By designing an active sliding mode controller and choosing proper control parameters, the drive and response systems are synchronized. Synchronization between the fractional-order Chen chaotic system and the integer-order Chen chaotic system and between integer-order hyperchaotic Chen system and fractional-order hyperchaotic Rössler system is used to illustrate the effectiveness of the proposed synchronization approach. Numerical simulations coincide with the theoretical analysis.

1. Introduction

During the past decades, fractional calculus has become a powerful tool to describe the dynamics of complex systems such as power systems, mathematics, biology, medicine, secure communication, and chemical reactors [1–6]. Chaos synchronization has attracted lots of attention in a variety of research fields [7–13] over the last two decades, because it can be applied in vast areas of physics and engineering and secure communication [14, 15]. Moreover, many theoretical analysis and numerical simulation results about the synchronization of chaotic systems are obtained. Wang et al. [16] deal with the finite-time chaos synchronization of the unified chaotic system with uncertain parameters. Chen and Liu [17] propose a simple linear state feedback controller to realize the stability control of a unified chaotic system. The problem of chaos synchronization between two different chaotic systems with fully unknown parameters is investigated in [18]. Moreover, Chen and his partners [19] investigate the chaos control of a class of fractional-order chaotic systems via sliding mode.

All of above articles mainly focus on integer-order chaotic systems or fractional-order chaotic systems. There is little information about the synchronization between fractional-order chaotic systems and integer-order chaotic systems [20, 21]. The study of synchronization between fractional-order hyperchaotic systems and integer-order hyperchaotic systems is also limited.

Motivated by the above discussion, this paper investigates a sliding mode method for synchronization between a class of fractional-order hyperchaotic systems and integer-order hyperchaotic systems. And the integer-order hyperchaotic systems are regarded as response system in the proposed synchronous technique which is simple and theoretically rigorous.

2. System Description and Problem Formulation

Consider the following fractional-order hyperchaotic system as a drive system

$$D^q x = Ax + f(x), \tag{1}$$

where $x(t) \in R^4$ denotes four-dimensional state vector. $A \in R^{4\times 4}$ represents the linear part of the system, and $f : R^4 \to R^4$ is the nonlinear part of the system.

And the response system can be described as

$$Dy = By + g(y), \tag{2}$$

where $y(t) \in R^4$ is four-dimensional state vector, $B \in R^{4\times4}$ and $g : R^4 \to R^4$ imply the same roles as A and f in the drive system, respectively.

Remark 1. A and $f(\cdot)$ in the drive system can be same as B and $g(\cdot)$ in the response system, respectively.

One adds the controller $u(t) \in R^4$ into the response system, which is given by

$$Dy = By + g(y) + u(t). \tag{3}$$

Define the synchronous errors as $e = y - x$. The aim is to choose a suitable controller $u(t) \in R^4$, so that the drive system and response system can achieve chaotic synchronization (i.e., $\lim_{t\to\infty}\|e\| = 0$, where $\|\cdot\|$ is the Euclidean norm).

3. Design of Sliding Mode Controller

Let the controller $u(t)$ be

$$u(t) = u_1(t) + u_2(t), \tag{4}$$

where $u_1(t) \in R^4$ is a compensation controller and $u_1(t) = Dx - B(x) - g(x)$. Here, $x(t) \in R^4$ in the response system belongs to hyperchaotic fractional-order drive system. $u_2(t) \in R^4$ is a vector function, and it will be designed later.

From (4), the system (3) can be rewritten as

$$De = Be + g(y) - g(x) + u_2(t). \tag{5}$$

In accordance with the active control design procedure, the nonlinear part of the error dynamics is eliminated by the following choice of the input vector [22]

$$u_2(t) = g(x) - g(y) + Kw(t). \tag{6}$$

The error system (5) is rewritten as

$$De = Be + Kw(t), \tag{7}$$

where $K = [k_1, k_2, k_3, k_4]^T$ is a constant gain vector and $w(t) \in R$ is the control input which satisfies

$$w(t) = \begin{cases} w^+(t) & s(e) \geq 0 \\ w^-(t) & s(e) < 0. \end{cases} \tag{8}$$

To design a sliding mode controller, one has two steps. First, one constructs a sliding surface that represents a desired system dynamics. Next, one develops a switching control law such that a sliding mode exists on every point of the sliding surface, and any states outside the surface are driven to reach the surface in a finite time [23]. As a choice for the sliding surface, one has

$$\begin{aligned} s_1(t) &= c_1e_1 + c_2e_2, \\ s_2(t) &= c_2e_2 + c_3e_3, \\ s_3(t) &= c_4e_4 + c_3e_3, \\ s_4(t) &= c_4e_4, \end{aligned} \tag{9}$$

which can also be easily given by

$$s(t) = Ce, \quad \text{where } C = \begin{bmatrix} c_1 & c_2 & 0 & 0 \\ 0 & c_2 & c_3 & 0 \\ 0 & 0 & c_3 & c_4 \\ 0 & 0 & 0 & c_4 \end{bmatrix}. \tag{10}$$

In the sliding mode, the sliding surface and its derivative must satisfy

$$s(t) = 0, \qquad \dot{s}(t) = 0. \tag{11}$$

Consider

$$\dot{s}(t) = Ds = 0 \Longrightarrow CDe + e = C(Be + Kw(t)) + e = 0. \tag{12}$$

One can get that

$$w(t) = -(CK)^{-1}(CB - I)e(t). \tag{13}$$

Replacing for $w(t)$ in (7) from $w(t)$ of (13), the error dynamics on the sliding surface are determined by the following relation:

$$De = \left(I - K(CK)^{-1}C\right)Be. \tag{14}$$

To satisfy the sliding condition, the discontinuous reaching law is chosen as follows:

$$Ds = -p\,\mathrm{sign}(s) - rs, \tag{15}$$

where

$$\mathrm{sign}(s) = \begin{cases} +1, & s > 0 \\ 0, & s = 0 \\ -1, & s < 0 \end{cases} \tag{16}$$

and $p > 0$, $r > 0$ are the gains of the controller.

In the sliding phase, it implies that $Ds = \dot{s}(t) = 0$. Considering (12) and (15), one gets

$$w(t) = -(CK)^{-1}\left[C(rI + B)e + p\,\mathrm{sign}(s)\right]. \tag{17}$$

Now, the total control law can be defined as follows:

$$\begin{aligned} u(t) = {} & Dx - Bx - g(y) \\ & - K(CK)^{-1}\left[C(rI + B)e + p\,\mathrm{sign}(s)\right]. \end{aligned} \tag{18}$$

Replacing $w(t)$ in (7) by (17), the error dynamics are determined by

$$De = \left[B - K(CK)^{-1}C(rI + B)\right]e - K(CK)^{-1}p\,\mathrm{sign}(s). \tag{19}$$

Theorem 2 (see [24]). *The following system is as follows:*

$$D^q x = Ax, \qquad x(0) = x_0, \tag{20}$$

where $0 < q \leq 1$, $x \in R^n$ and $A \in R^{n\times n}$. System (20) is asymptotically stable if $|\arg(\lambda_i)| > q\pi/2$, where λ_i are the eigenvalues of matrix A. Also, this system is stable if $|\arg(\lambda_i)| \geq q\pi/2$ and those critical eigenvalues that satisfy $|\arg(\lambda_i)| = q\pi/2$ have geometric multiplicity one.

Theorem 3 (see [24]). *Consider a system given by the following linear state space form with inner dimension n as follows:*

$$D^q x = Ax + Bu,$$
$$y = Cx, \qquad x(0) = x_0, \tag{21}$$

where $0 < q \leq 1$, $x \in R^n$, $y \in R^p$, *and* $A \in R^{n\times n}$. *Assuming that the triplet* (A, B, C) *is minimal, then system* (21) *is stable if* $|\arg(\operatorname{eig}(A))| > q\pi/2$.

According to Theorem 2, the error dynamics on the sliding surface defined by (14) is asymptotically stable, as long as all eigenvalues of $[(I - K(CK)^{-1}C)B]$ satisfy the condition $|\arg(\lambda_i)| > \pi/2$. In the sliding phase, as a linear fractional-order system with bounded inputs ($-K(CK)^{-1}p$ for $s > 0$ and $K(CK)^{-1}p$ for $s > 0$), the error system (19) is stable if $|\arg(\operatorname{eig}([B - K(CK)^{-1}C(rI + B)]))| > \pi/2$. It can be shown that choosing appropriate K, C, and r can make the error dynamics stable; hence, the synchronization is realized.

4. Numerical Simulation

This section presents two illustrative examples to verify and demonstrate the effectiveness of the proposed control scheme. Case 1 is the synchronization between the same structure hyperchaotic systems. Case 2 is the synchronization between the different structure hyperchaotic systems.

Case 1. Synchronization between fractional-order and integer-order hyperchaotic Chen systems.

Consider Chen hyperchaotic system which is written as [25]

$$\begin{aligned}
\frac{d^{q_1}x_1}{dt^{q_1}} &= a_1(x_2 - x_1) + x_4,\\
\frac{d^{q_2}x_2}{dt^{q_2}} &= \gamma x_1 - x_1x_3 + c_1x_2,\\
\frac{d^{q_3}x_3}{dt^{q_3}} &= x_1x_2 - b_1x_3,\\
\frac{d^{q_4}x_4}{dt^{q_4}} &= x_2x_3 + d_1x_4.
\end{aligned} \tag{22}$$

When $q_1 = q_2 = q_3 = q_4 = 1$, the system is integer-order system; otherwise we call the system (22) a fractional-order system.

Take the fractional-order system with fractional-order $q_1 = q_2 = q_3 = q_4 = 0.95$ as a drive system, and the integer-order Chen hyperchaotic system as a response system with the following initial conditions: $[x_1, x_2, x_3, x_4]^T = [0.1, 0.5, -0.9, 1]^T$ and $[y_1, y_2, y_3, y_4]^T = [0.1, 0, 0.9, 0]^T$, and the system parameters are $(a_1, b_1, c_1, d_1, \gamma) = (35, 3, 28, 7)$.

The controller parameters are chosen as $K = \operatorname{diag}(-2, -8, -2, -2)$, $C = [4, 4, 0, 0; 0, 4, 4, 0; 0, 0, 4, 4; 0, 0, 0, 4]$, $r = 10$, and $p = 2$. This selection of parameters results in eigenvalues $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-10, -10, -10, -10)$, which are located in the stable region. According to (18), the control inputs are taken as follows:

$$\begin{aligned}
u_1 &= \frac{dx_1}{dt} - a_1(x_2 - x_1) - x_4 + 25e_1 - 35e_2 - e_4\\
&\quad - \frac{1}{2}(\operatorname{sign}(s_1) - \operatorname{sign}(s_2) + \operatorname{sign}(s_3) - \operatorname{sign}(s_4)),\\
u_2 &= \frac{dx_2}{dt} - \gamma x_1 - c_1x_2 + y_1y_3 - 38e_2 - 7e_1\\
&\quad + \frac{1}{2}(\operatorname{sign}(s_3) - \operatorname{sign}(s_2) - \operatorname{sign}(s_4)),\\
u_3 &= \frac{dx_3}{dt} + b_1x_3 - y_1y_2 - 7e_3 + \frac{1}{2}(\operatorname{sign}(s_4) - \operatorname{sign}(s_3)),\\
u_4 &= \frac{dx_4}{dt} - d_1x_4 - y_2y_3 - \frac{21}{2}e_4 - \frac{1}{2}\operatorname{sign}(s_4).
\end{aligned} \tag{23}$$

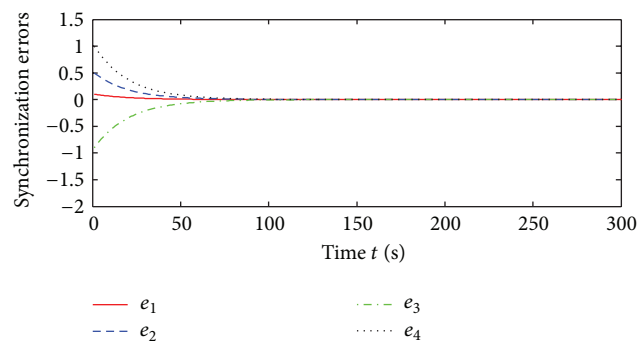

Figure 1: Synchronization errors between Chen systems.

The simulation results are given in Figure 1. As we can see, the errors converge to zero which implies that synchronization between the two systems is realized.

Case 2. Synchronization between integer-order hyperchaotic Chen system and fractional-order hyperchaotic Rössler system.

Consider hyperchaotic Rössler system which is written as [26]

$$\begin{aligned}
\frac{d^{q_1}x_1}{dt^{q_1}} &= -x_2 - x_3,\\
\frac{d^{q_2}x_2}{dt^{q_2}} &= x_1 + a_2x_2 + x_4,\\
\frac{d^{q_3}x_3}{dt^{q_3}} &= b_2 + x_1x_3,\\
\frac{d^{q_3}x_4}{dt^{q_3}} &= -c_2x_3 + d_2x_4.
\end{aligned} \tag{24}$$

Similarly, take the fractional-order Rössler hyperchaotic system with fractional-order $q_1 = q_2 = q_3 = q_4 = 0.95$ as a drive system, and take the integer-order Chen

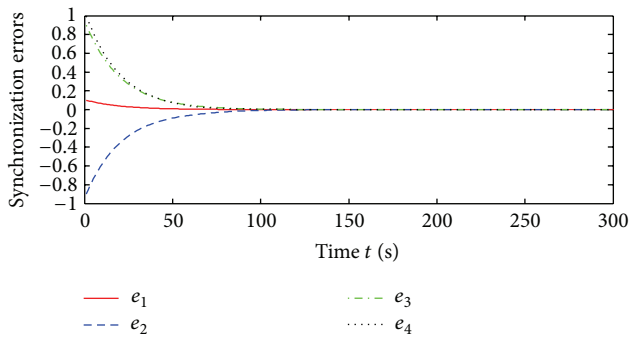

Figure 2: Synchronization between integer-order Chen system and fractional-order Rössler system.

hyperchaotic system as a response system with the following initial conditions: $[x_1, x_2, x_3, x_4]^T = [-0.1, -0.9, 0.9, 1]^T$ and $[y_1, y_2, y_3, y_4]^T = [-0.9, 0.1, 0.9, 1.9]^T$, and the system parameters are $(a_2, b_2, c_2, d_2) = (0.25, 3, 0.5, 0.05)$.

We choose the design parameters in the simulations as $K = \mathrm{diag}(-3, -5, -6, -3)$, $C = [3, 3, 0, 0; 0, 3, 3, 0; 0, 0, 3, 3; 0, 0, 0, 3]$, $r = 5$, and $p = 0.2$. This selection of parameters results in eigenvalues, $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-5, -5, -5, -5)$, which are located in the stable region. According to (19), we can yield the response system easily, that is,

$$\begin{aligned}
u_1 &= \frac{dx_1}{dt} - a_1(x_2 - x_1) - x_4 + 30e_1 - 35e_2 - e_4 \\
&\quad - \frac{1}{15}\left(\mathrm{sign}(s_1) - \mathrm{sign}(s_2) + \mathrm{sign}(s_3) - \mathrm{sign}(s_4)\right), \\
u_2 &= \frac{dx_2}{dt} - \gamma x_1 - c_1 x_2 + y_1 y_3 - 7e_1 - 33e_2 \\
&\quad + \frac{1}{15}\left(\mathrm{sign}(s_3) - \mathrm{sign}(s_2) - \mathrm{sign}(s_4)\right), \\
u_3 &= \frac{dx_3}{dt} + b_1 x_3 - y_1 y_2 - 2e_3 + \frac{1}{15}\left(\mathrm{sign}(s_4) - \mathrm{sign}(s_3)\right), \\
u_4 &= \frac{dx_4}{dt} - d_1 x_4 - y_2 y_3 - \frac{11}{2}e_4 - \frac{1}{15}\mathrm{sign}(s_4).
\end{aligned} \tag{25}$$

The synchronization errors are shown in Figure 2, which show that the proposed method is succeeded in synchronizing the two different structure systems.

5. Conclusion

In this paper, the problem of synchronization between fractional-order hyperchaotic systems and integer-order hyperchaotic systems is investigated. The integer-order hyperchaotic system is regarded as the response system. A sliding mode controller is designed to synchronize two systems with different orders successfully. It is rigorously proven that the proposed synchronization approach can be achieved between two different order hyperchaotic systems. Some numerical simulations are presented to show the applicability and feasibility of the proposed scheme.

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