Research Article

Note on the Regularity of Nonadditive Measures

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We consider the regularity for nonadditive measures. We prove that the non-additive measures which satisfy Egoroff’s theorem and have pseudometric generating property possess Radon property (strong regularity) on a complete or a locally compact, separable metric space.

1. Introduction

The relations of continuity and regularity of nonadditive measures are considered in several papers [1–4]. In [5], Li et al. investigated the regularity in nonadditive measures. They proved that the null-additive fuzzy measures possess a Radon property (strong regularity) on a complete metric space. In [6], Kawabe also investigated the regularity in fuzzy measures taking value in Riesz spaces. He proved that every weakly null-additive Riesz space valued fuzzy measure on a complete or a locally compact, separable metric space is Radon, provided that the Riesz space has the multiple Egoroff property.

On the other hand Li and Mesiar [7] proved the regularity of nonadditive monotone measures. They proved that the equivalence condition of Egoroff’s theorem implies regularity for the nonadditive measures by using pseudometric generating property of a set function. For information on real valued nonadditive measures, see [8–10].

In this paper, as notes, we prove that Egoroff’s theorem implies Radon property (strong regularity) on nonadditive measures which have pseudometric generating property on a complete or a locally compact, separable metric space.

2. Preliminaries

Let $R$ be the set of real numbers and $N$ the set of natural numbers. In what follows, let $(X, F)$ be a measurable space.

Definition 1. A set function $\mu : F \to R$ is called a nonadditive measure if it satisfies the following two conditions:

1. $\mu(0) = 0$,
2. if $A, B \in F$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

Definition 2. Let $\mu : F \to R$ be a nonadditive measure.

1. $\mu$ is said to be continuous from above if for any $\{A_n\} \subseteq F$ and $A \in F$ satisfying $A_n \searrow A$ and there exists $n_0$ with $\mu(A_{n_0}) < \infty$ it holds that $\lim_{n \to \infty} \mu(A_n) = \mu(A)$.
2. $\mu$ is said to be continuous from below if for any $\{A_n\} \subseteq F$ and $A \in F$ satisfying $A_n \nearrow A$ it holds that $\lim_{n \to \infty} \mu(A_n) = \mu(A)$.
3. $\mu$ is said to be fuzzy measure if it is continuous from above and below.
4. $\mu$ is said to be strongly order continuous if it is continuous from above at measurable sets of measure 0; that is, for any $\{A_n\} \subseteq F$ and $A \in F$ satisfying $A_n \searrow A$ and $\mu(A) = 0$ it holds that $\lim_{n \to \infty} \mu(A_n) = 0$.
5. $\mu$ is said to be weakly null-additive if $\mu(A \cup B) = 0$ whenever $A, B \in F$ and $\mu(A) = \mu(B) = 0$.
6. $\mu$ has property (S) if for any sequence $\{A_n\} \subseteq F$ with $\lim_{n \to \infty} \mu(A_n) = 0$ there exists a subsequence $\{A_{n_k}\}$ such that $\mu(\bigcup_{k=1}^{\infty} A_{n_k}) = 0$; see [11].
(7) \( \mu \) is said to be autocontinuous from above if 
\[ \lim_{n \to \infty} \mu(A \cup B_n) = \mu(A) \] for each \( A \in \mathcal{F} \) and \( \{B_n\} \subset \mathcal{F} \) with \( \lim_{n \to \infty} \mu(B_n) = 0 \).

(8) \( \mu \) is said to be autocontinuous from below if 
\[ \lim_{n \to \infty} \mu(A \setminus B_n) = \mu(A) \] for each \( A \in \mathcal{F} \) and \( \{B_n\} \subset \mathcal{F} \) with \( \lim_{n \to \infty} \mu(B_n) = 0 \).

(9) \( \mu \) is said to be autocontinuous if it is autocontinuous from above and below.

**Definition 3.** Let \( \mu : \mathcal{F} \to R \) be a nonadditive measure.

(1) A double sequence \( \{A_{m,n}\} \subset \mathcal{F} \) is said to be a \( \mu \)-regulator if it satisfies the following two conditions:

\( D1 \) \( A_{m,n} \supset A_{m,n'} \), whenever \( n \leq n' \),

\( D2 \) \( \mu(\bigcup_{m=1}^{\infty} A_{m,m}) = 0 \).

(2) \( \mu \) satisfies the Egoroff condition if for any \( \mu \)-regulator \( \{A_{m,n}\} \) and for every \( \epsilon > 0 \) there exists a sequence \( \{n_m\} \) of natural numbers such that \( \mu(\bigcup_{m=1}^{\infty} A_{m,n_m}) < \epsilon \).

**Remark 4.** A nonadditive measure \( \mu \) satisfies the Egoroff condition if (and only if) for any double sequence \( \{A_{m,n}\} \subset \mathcal{F} \) satisfying \( D2 \) and the following \( D1' \) it holds that for every \( \epsilon > 0 \) there exists a sequence \( \{n_m\} \) of natural numbers such that \( \mu(\bigcup_{m=1}^{\infty} A_{m,n_m}) < \epsilon \):

\( D1' \) \( A_{m,n} \supset A_{m',n'} \), whenever \( m \geq m' \) and \( n \leq n' \).

### 3. Compact Measure and Regularity of Measure

In this section, we pick up several results for compact nonadditive measures and regularity of measures.

**Definition 5.** Let \( \mu : \mathcal{F} \to R \) be a nonadditive measure.

(1) A nonempty family \( \mathcal{K} \) of subsets of \( X \) is called a compact system if for any sequence \( \{K_n\} \subset \mathcal{K} \) with \( \bigcap_{n=1}^{\infty} K_n = \emptyset \) there is \( n_0 \in N \) such that \( \bigcap_{n=1}^{n_0} K_n = \emptyset \); see [12].

(2) We say that \( \mu \) is compact if there exists a compact system \( \mathcal{K} \) such that for each \( A \in \mathcal{F} \) there are sequences \( \{K_n\} \subset \mathcal{K} \) and \( \{B_n\} \subset \mathcal{F} \) such that \( B_n \subset K_n \subset A \) for all \( n \in N \) and \( \lim_{n \to \infty} \mu(A \setminus B_n) = 0 \).

**Remark 6.** (1) The family of all compact subsets of a Hausdorff space is a compact system.

(2) The family of all finite unions of sets in a compact system is also compact [13, Lemma 1.4]. Therefore, in (2) of the above definition, the compact system \( \mathcal{K} \) and the sequences \( \{K_n\} \subset \mathcal{K} \) and \( \{B_n\} \subset \mathcal{F} \) may be chosen so that \( \mathcal{K} \) is closed for finite unions and both \( \{K_n\} \) and \( \{B_n\} \) are increasing.

By [6, Theorem 1], the following result follows.

**Theorem 7.** Let \( \mu : \mathcal{F} \to R \) be a nonadditive measure. If \( \mu \) is compact and autocontinuous, then it is continuous from above and below.
sets and \(|G_{n} \cap A| \in G_{n}\) of open sets such that \(K_{n} \subset A \subset G_{n}\) for all \(n \in N\) and \(\lim_{n \to \infty} \mu(G_{n} \setminus K_{n}) = 0\).

(2) \(\mu\) is said to be tight if there is a sequence \(|K_{n}| \in N\) of compact sets such that \(\lim_{n \to \infty} \mu(X \setminus K_{n}) = 0\).

Remark 15. Sequences of sets in the above definition may be chosen so that \(|G_{n}| \) is decreasing, while \(|F_{n}| \) and \(|K_{n}| \) are increasing.

**Proposition 16.** Let \(X\) be a Hausdorff space. Let \(\mu\) be a non-additive Borel measure on \(X\) which is weakly null-additive and strongly order continuous. Then, the following two conditions are equivalent:

(i) \(\mu\) is Radon (strongly regular),

(ii) \(\mu\) is regular and tight.

**Proof.** See [6, Proposition 2].

It is known that every finite Borel measure on a complete or a locally compact, separable metric space is Radon; see [16, Theorem 3.2] and [17, Theorems 6 and 9, Chapter II, Part I]. Its counterpart in nonadditive measure theory can be found in [5, 9, Theorem 1, Lemma 2], which states that every Borel fuzzy measure on a complete separable metric space is tight, so that it is Radon if it is null-additive; see also [3, Theorem 2.3]. The following two theorems contain these previous results; see also [18, Theorem 12].

**Theorem 17.** Let \(X\) be a complete separable metric space and \(\mu : \mathcal{B}(X) \to R\) a nonadditive Borel measure on \(X\). If \(\mu\) is weakly null-additive and satisfies the Egoroff condition, then it is tight. Moreover, if \(\mu\) has pseudometric generating property and satisfies the Egoroff condition, then it is Radon.

To prove the theorem, we need the following; see [7, Proposition 3.7].

**Theorem 18.** Let \(\mu : \mathcal{F} \to R\) be a nonadditive measure. Then (i) implies (ii).

(i) \(\mu\) is weakly null-additive and satisfies the Egoroff condition.

(ii) For each \(\varepsilon > 0\) and double sequence \(|A_{mn}| \subset \mathcal{F}\) satisfying \(A_{mn} \setminus 0\) as \(n \to \infty\) for each \(m \in N\), there exists a sequence \(|n_{m}|\) of natural numbers such that \(\mu(\bigcup_{m=1}^{\infty} A_{mn}) < \varepsilon\).

**Proof of Theorem 17.** Since \(\mu\) satisfies the Egoroff condition, by [19, Proposition 3], it is strongly order continuous. By Proposition 16 and Lemma 11, we have only to prove that \(\mu\) is tight. Let \(|s_{n}| \in N\) be a countable dense subset of \(X\). For each \(m, i \in N\), denote by \(B_{m}(s_{i})\) the closed ball with center \(s_{i}\) and radius \(1/m\). For each \(m, n \in N\), put \(A_{mn} := X \setminus \bigcup_{i=1}^{m} B_{m}(s_{i})\). Then, for any \(\varepsilon > 0\) and \(m \in N\), we have \(A_{mn} \setminus 0\), so that by Proposition 18, there exists a sequence \(|n_{m}|\) of natural numbers such that

\[\mu\left(\bigcup_{m=1}^{\infty} A_{mn}\right) < \varepsilon.\]
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References


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