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Research Article

Nonsmooth Multiobjective Fractional Programming with Local Lipschitz Exponential B-(p, r)-Invexity

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We study nonsmooth multiobjective fractional programming problem containing local Lipschitz exponential B-(p, r)-invex functions with respect to η and b. We introduce a new concept of nonconvex functions, called exponential B-(p, r)-invex functions. Base on the generalized invex functions, we establish sufficient optimality conditions for a feasible point to be an efficient solution. Furthermore, employing optimality conditions to perform Mond-Weir type duality model and prove the duality theorems including weak duality, strong duality, and strict converse duality theorem under exponential B-(p, r)-invexity assumptions. Consequently, the optimal values of the primal problem and the Mond-Weir type duality problem have no duality gap under the framework of exponential B-(p, r)-invexity.

1. Introduction

Convexity plays an important role in mathematical programming problems, some of which are sufficient optimality conditions or duality theorems. The sufficient optimality conditions and duality theorems are being studied by extending the concept of convexity. One of the most generalizations of convexity of differentiable function in optimality theory was introduced by Hanson [1]. Then the characteristics of invexity—an invariant convexity—were applied in mathematical programming (cf. [1–7]). Besides, the concept of invexity of differentiable functions has been extended to the case of nonsmooth functions (cf. [8–17]). After Clarke [18] defined generalized derivative and subdifferential on local Lipschitz functions, many practical problems are described under nonsmooth functions. For example, Reiland [17] used the generalized gradient of Clarke [18] to define nondifferentiable invexity for Lipschitz real valued functions. Later on, with generalized invex Lipschitz functions, optimality conditions and duality theorems were established in nonsmooth mathematical programming problems (cf. [8–17]). Indeed, problems of multiobjective factional programming have various types of optimization problems, for example, financial and economic problems, game theory, and all optimal decision problems. In multiobjective programming problems, when the necessary optimality conditions are established, the conditions for searching an optimal solution will be employed. That is, extra reasonable assumptions for the necessary optimality conditions are needed in order to prove the sufficient optimality conditions. Moreover, these reasonable assumptions are various (e.g., generalized convexity, generalized invexity, set-value functions, and complex functions). When the existence of optimality solution is approved in the sufficient optimality theorems, the optimality conditions to investigate the duality models could be employed. Then the duality theorems could be proved. The better condition is that there is no duality gap between primal problems and duality problems.

In this paper, we focus a system of nondifferentiable multiobjective nonlinear fractional programming problem as the following form:

$$(P)\ \text{Minimize}\ \phi(x) \equiv \frac{f(x)}{g(x)} \equiv \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \ldots, \frac{f_k(x)}{g_k(x)}\right) \equiv (\phi_1(x), \phi_2(x), \ldots, \phi_k(x)), \tag{1}$$

subject to $x \in X \subset \mathbb{R}^n$ with

$$\mathfrak{F} = \{x \in X \mid h(x) = (h_1, h_2, \ldots, h_m)(x) \in -\mathbb{R}^m_+\}, \quad (2)$$

where X is a separable reflexive Banach space in the Euclidean n-space $\mathbb{R}^n$, f_i, $g_i : X \to \mathbb{R}$, $i = 1, 2, \ldots, k$, and $h : X \to \mathbb{R}^m$ are locally Lipschitz functions on X. Without loss of generality, we may assume that $f_i(x) \geq 0$, $g_i(x) > 0$ for all $x \in X$, $i = 1, 2, \ldots, k$.

In this paper, we introduce a new class of Lipschitz functions, namely, exponential B-(p, r)-invex Lipschitz functions which are motivated from the results of Antczak [3], Clarke [18], and Reiland [17]. We employ this exponential B-(p, r)-invexity and necessary optimality conditions to establish the sufficient optimality conditions on a nondifferentiable multiobjective fractional programming problem (P). Using optimality conditions, we construct Mond-Weir duality model for the primal problem (P) and prove that the duality theorems have the same optimal value as the primal problem involving B-(p, r)-invexity.

2. Definitions and Preliminaries

Let $\mathbb{R}^n$ denote Euclidean space, and let $\mathbb{R}^n_+$ denote the order cone. For cone partial order, if $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, we define:

(1) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \ldots, n$;

(2) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, \ldots, n$;

(3) $x \geqq y$ if and only if $x_i \geq y_i$ for all $i = 1, 2, \ldots, n$;

(4) $x \geq y$ if and only if $x \geqq y$ and $x_i \neq y_i$ for some $i \in \{1, 2, \ldots, n\}$.

Definition 1. Let X be an open subset of $\mathbb{R}^n$. The function $\theta : X \to \mathbb{R}$ is said to be *locally Lipschitz* at $x \in X$ if there exists a positive real constant $\mathscr{C}$ and a neighborhood $\mathscr{N}$ of $x \in X$ such that

$$|\theta(y) - \theta(z)| \leqq \mathscr{C} \|y - z\|, \quad \forall z, y \in \mathscr{N}, \quad (3)$$

where $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^n$.

For any vector ν in $\mathbb{R}^n$, the *generalized directional derivative* of θ at x in the direction $\nu \in \mathbb{R}^n$ in Clarke's sense [18] is defined by

$$\theta^\circ(x; \nu) = \limsup_{\substack{y \to x \\ \lambda \to 0^+}} \frac{\theta(y + \lambda\nu) - \theta(y)}{\lambda}. \quad (4)$$

The *generalized subdifferential* of θ at $x \in X$ is defined by the set

$$\partial^\circ\theta(x) = \{\xi \in X^* : \theta^\circ(x; \nu) \geq \langle \xi, \nu \rangle \ \ \forall \nu \in X\}, \quad (5)$$

where X^* is the dual space of X and $\langle \xi; \nu \rangle$ stands for the dual pair of X and X^*.

Evidently, $\theta^\circ(x; \nu) = \max\{\langle \xi; \nu \rangle : \xi \in \partial^\circ\theta(x)\}$ for any x and ν in X. If θ is a convex function, then $\partial^\circ\theta$ is coincid with usual subdifferential $\partial\theta$.

Definition 2 (see [18]). θ is said to be regular at x if for any $\nu \in X$, the one-side directional derivative $\theta'(x; \nu)$ exists and $\theta'(x; \nu) = \theta^\circ(x; \nu)$.

Lemma 3 (see [18]). *Let f and g be Lipschitz near x, and suppose $g(x) \neq 0$. Then $f(x)/g(x)$ is Lipschitz near x and one has*

$$\partial^\circ\left(\frac{f}{g}\right)(x) \subset \frac{g(x)\partial^\circ f(x) - f(x)\partial^\circ g(x)}{g^2(x)}, \quad \text{provided } f(x) \geqq 0,\ g(x) > 0. \quad (6)$$

If f and $-g$ are regular at x, then equality holds to the above $\subset$, that is, the subdifferential is singleton and f/g is regular at x.

Let $h : X \to \mathbb{R}^m$ be a local Lipschitz function. For $x_0 \in X$, we define

$$J(x_0) = \{j \in J : h_j(x_0) = 0\}, \quad J = \{1, 2, \ldots, m\},$$
$$\Lambda = \{\nu \in X : h_j^\circ(x_0, \nu) < 0,\ j \in J(x_0)\}. \quad (7)$$

If $\Lambda \neq \emptyset$, we say that the problem (P) has *constraint qualification* at x_0 (cf. [19]).

On the basis of the definition for invex functions of Lipschitz functions in Reiland [17], we modified Antczak's generalized B-(p, r)-invex with respect to η and b for differentiable to nondifferentiable case for a class of locally Lipschitz exponential B-(p, r)-invex functions as follows.

Definition 4. Let p, r be arbitrary real numbers. A locally Lipschitz function $\theta : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be *exponential B-(p, r)-invex (strictly)* at $u \in X$ with respect to w.r.t. (for brevity) if there exists a function $\eta : X \times X \to \mathbb{R}^n$ with property $\eta(x, u) = 0$ only if $u = x$ in X and a function $b : X \times X \to \mathbb{R}_+ \setminus \{0\}$ such that for each $x \in X$, the following inequality holds for $\xi \in \partial^\circ f(u)$:

$$\frac{1}{r} b(x, u)\left(e^{r(\theta(x) - \theta(u))} - 1\right) \geq \frac{1}{p}\left\langle \xi, \left(e^{p\eta(x,u)} - \mathbf{1}\right)\right\rangle (> \text{if } x \neq u), \quad \text{for } p \neq 0,\ r \neq 0. \quad (8)$$

If p or r is zero, then (8) can give some modification by using the limit of $p \to 0$ or $r \to 0$.

(i) If $r \neq 0$, $p \to 0$ in (8), then we deduce that

$$\frac{1}{r} b(x, u)\left(e^{r(\theta(x) - \theta(u))} - 1\right) \geq \langle \xi, \eta(x, u)\rangle (> \text{if } x \neq u), \quad \text{for } p = 0,\ r \neq 0. \quad (9)$$

(ii) If $p \neq 0$, $r \to 0$, then (8) becomes

$$b(x, u)(\theta(x) - \theta(u)) \geq \frac{1}{p}\left\langle \xi, \left(e^{p\eta(x,u)} - \mathbf{1}\right)\right\rangle (> \text{if } x \neq u) \quad \text{for } p \neq 0,\ r = 0. \quad (10)$$

(iii) If $r = 0$, $p \to 0$, then (10)

$$b(x,u)\left(f(x)-f(u)\right) \geq \left\langle \xi, \eta(x,u)\right\rangle \ (>\text{if } x \neq u) \quad \text{for } p=0,\ r=0 \tag{11}$$

holds.

Remark 5. All theorems in our work will be described only in the case of $p \neq 0$ and $r \neq 0$. We omit the proof of other cases like in (i), (ii), and (iii).

A feasible solution $\overline{x}$ to (P) is said to be an *efficient solution* to (P) if there is no $x \in \mathfrak{F}$ such that $\phi(x) \leq \phi(\overline{x})$.

3. Optimality Conditions

In this section, we establish some sufficient optimality conditions. The necessary optimality conditions to the primal problem (P) given by [20] and the subproblems (SP_i) of (P), for $i \in \{1,2,\ldots,k\}$, given by [8] are used in our theorem.

Lemma 6 (see [8]). *$\overline{x}$ is an optimal solution to problem (P) if and only if $\overline{x}$ solves (SP_i), where (SP_i) is as the following problem:*

$$\begin{aligned}
(SP_i) \quad & \textit{Minimize} \quad \frac{f_i(x)}{g_i(x)} \\
& \textit{subject to} \quad x \in M_i \\
& = \left\{x \in X : \frac{f_p(x)}{g_p(x)} \leqq \frac{f_p(\overline{x})}{g_p(\overline{x})} = \phi_p(\overline{x}),\ p \neq i,\ p = 1,2,\ldots,k,\ h(x) \in -\mathbb{R}^m_+\right\} \\
& = \left\{x \in X : f_p(x) - \phi_p(\overline{x})\, g_p(x) \leqq 0,\ p \neq i,\ p = 1,2,\ldots,k,\ h(x) \in -\mathbb{R}^m_+\right\}.
\end{aligned} \tag{12}$$

Theorem 7 (see [20], necessary optimality conditions). *If $\overline{x}$ is an optimal solution of (P) and has a constraint qualification, for (SP_i), $i = 1,2,\ldots,k$, then, there exist $\alpha^* \in \mathbb{R}^k$ and $z^* \in \mathbb{R}^m$ such that*

$$0 \in \sum_{i=1}^{k} \alpha_i^* \partial^\circ f_i(\overline{x}) + \phi_i(\overline{x})\, \partial^\circ(-g_i)(\overline{x}) + \left\langle z^*, \partial^\circ h(\overline{x})\right\rangle_m, \tag{13}$$

$$z_j^* h_j(\overline{x}) = 0 \quad \forall j = 1,2,\ldots,m, \tag{14}$$

$$f_i(\overline{x}) - \phi_i(\overline{x})\, g_i(\overline{x}) = 0, \quad \forall i = 1,2,\ldots,k, \tag{15}$$

$$\alpha^* \in \mathbb{I}^k_+ \setminus \{0\}, \qquad z^* \in \mathbb{R}^m_+, \tag{16}$$

where

$$\mathbb{I}^k_+ = \left\{\alpha^* \in \mathbb{R}^k_+ \mid \alpha^* = \left(\alpha_1^*, \alpha_2^*, \ldots, \alpha_k^*\right),\ \sum_{i=1}^{k} \alpha_i^* = 1\right\}, \qquad \left\langle z^*, \partial^\circ h(\overline{x})\right\rangle_m \equiv \sum_{j=1}^{m} z_j^* \partial^\circ h_j(\overline{x}). \tag{17}$$

For convenience, let

$$\left\langle z^*, h(x)\right\rangle_m \equiv \sum_{j=1}^{m} z_j^* h_j(x), \qquad \left\langle z^*, \rho\right\rangle_m \equiv \sum_{j=1}^{m} z_j^* \rho_j, \tag{18}$$

where $z^* \in \mathbb{R}^m_+ \rho_j \in \partial^\circ h_j(\overline{x})$.

Now, we give a useful lemma whose simple proof is omitted in this paper.

Lemma 8. *If $(1/r)(e^{r\theta(x)}-1) \geq 0$, where $\theta(x)$ is a real function, then $\theta(x) \geq 0$.*

The sufficient optimality conditions can be deduced from the converse of necessary optimality conditions with extra assumptions. Since the sufficient optimality theorem is various depending on extra assumptions, the duality model is also various. We establish the sufficient optimality conditions and duality theorems involving the exponential B-(p,r)-invexity.

Theorem 9. *Let $\overline{x} \in \mathfrak{F}$ be a feasible solution of (P) such that there exist y^*, z^* satisfying the conditions* (13)~(16) *at $\overline{x}$. Furthermore, suppose that any one of the conditions (a) and (b) hold:*

(a) *$A_1(x) = \sum_{i=1}^{k} \alpha_i^* [f_i(x) - \phi_i(\overline{x}) g_i(x)] + \langle z^*, h(x)\rangle_m$ is an exponential B-(p,r)-invex function at $\overline{x}$ in $\mathfrak{F}$ w.r.t. η and b_1,*

(b) *$A_2(x) = \sum_{i=1}^{k} \alpha_i^* [f_i(x) - \phi_i(\overline{x}) g_i(x)]$ is an exponential B-(p,r)-invex function at $\overline{x}$ in $\mathfrak{F}$ w.r.t. η and b_2, and $A_3(x) = \langle z^*, h(x)\rangle_m$ is an exponential B-(p,r)-invex function at $\overline{x}$ in $\mathfrak{F}$ w.r.t. the same function η and b_3 but not necessarily, equal to b_2.*

Then, $\overline{x}$ is an efficient solution to problem (P).

Proof. Suppose that $\overline{x}$ is (P)-feasible. By expression (13), there exist $\xi_i \in \partial^\circ f_i(\overline{x})$, $\zeta_i \in \partial^\circ(-g_i)(\overline{x})$, $i = 1,2,\ldots,k$ and $\rho_j \in \partial^\circ h_j(\overline{x})$, $j = 1,2,\ldots,m$ such that

$$\langle \tilde{a}_1 \rangle \equiv \sum_{i=1}^{k} \alpha_i^* \left(\xi_i + \phi_i(\overline{x})\, \zeta_i\right) + \left\langle z^*, \rho\right\rangle_m = 0 \quad \text{in } X^* \tag{19}$$

and that $\langle \tilde{a}_1 \rangle$ is a zero vector of X^*.

From the above expression, the dual pair of $\langle X^*, X\rangle$

$$\left\langle \langle \tilde{a}_1 \rangle, \left(e^{p\eta(x,\overline{x})} - \mathbf{1}\right)\right\rangle = 0. \tag{20}$$

If $\overline{x}$ is not an efficient solution to problem (P), then there exists $x \in (P)$-feasible such that

$$\begin{aligned} &\frac{f_i(x)}{g_i(x)} \leqq \frac{f_i(\overline{x})}{g_i(\overline{x})} \quad \text{for } i = 1, 2, \ldots, k, \\ &\frac{f_t(x)}{g_t(x)} < \frac{f_t(\overline{x})}{g_t(\overline{x})} \quad \text{for some } t \in \underline{k} = \{1, 2, \ldots, k\}; \end{aligned} \tag{21}$$

that is,

$$\begin{aligned} &f_i(x) - \phi_i(\overline{x}) g_i(x) \\ &\quad \leqq f_i(\overline{x}) - \phi_i(\overline{x}) g_i(\overline{x}) \quad \text{for } i = 1, 2, \ldots, k, \\ &f_t(x) - \phi_t(\overline{x}) g_t(x) \\ &\quad < f_t(\overline{x}) - \phi_t(\overline{x}) g_t(\overline{x}) \quad \text{for some } t \in \underline{k}. \end{aligned} \tag{22}$$

Thus, we have

$$\begin{aligned} A_2(x) &= \sum_{i=1}^{k} \alpha_i^* \left[f_i(x) - \phi_i(\overline{x}) g_i(x) \right] \\ &< \sum_{i=1}^{k} \alpha_i^* \left[f_i(\overline{x}) - \phi_i(\overline{x}) g_i(\overline{x}) \right] = A_2(\overline{x}). \end{aligned} \tag{23}$$

From relations $h(x) \in -\mathbb{R}_+^m$, (14), and (16), we obtain

$$A_3(x) = \langle z^*, h(x) \rangle_m \leqq \langle z^*, h(\overline{x}) \rangle_m = A_3(\overline{x}), \tag{24}$$

where $\langle z^*, h(\overline{x}) \rangle_m \equiv \sum_{j=1}^{m} z_j^* h(\overline{x})$.

If hypothesis (a) holds, $A_1(x)$ is an exponential B-(p, r)-invexity w.r.t. η and b_1 at $\overline{x}$ for all $x \in \mathfrak{F}$. Then by Definition 4, we have that the following inequality

$$\begin{aligned} &\frac{1}{r} b_1(x, \overline{x}) \left(e^{r(A_1(x) - A_1(\overline{x}))} - 1 \right) \\ &\quad \geqq \frac{1}{p} \left\langle \langle \widetilde{\mathbb{a}}_1 \rangle, \left(e^{p\eta(x, \overline{x})} - \mathbf{1} \right) \right\rangle \end{aligned} \tag{25}$$

holds. Because of equality (20) and inequality (25), we obtain

$$\frac{1}{r} b_1(x, \overline{x}) \left(e^{r(A_1(x) - A_1(\overline{x}))} - 1 \right) \geqq 0. \tag{26}$$

According to Lemma 8 and $b_1(x, \overline{x}) \in \mathbb{R}_+ \setminus \{0\}$, we have

$$A_1(x) \geqq A_1(\overline{x}). \tag{27}$$

Equation (23) along with (24) yields

$$\begin{aligned} A_1(x) &= \sum_{i=1}^{k} \alpha_i^* \left[f_i(x) - \phi_i(\overline{x}) g_i(x) \right] + \langle z^*, h(\overline{x}) \rangle_m \\ &< \sum_{i=1}^{k} \alpha_i^* \left[f_i(\overline{x}) - \phi_i(\overline{x}) g_i(\overline{x}) \right] + \left\langle z^*, h\left(\overline{\overline{x}}\right) \right\rangle_m \\ &= A_1(\overline{x}) \end{aligned} \tag{28}$$

which contradicts inequality (27).

If hypothesis (b) holds, $A_3(x)$ is an exponential B-(p, r)-invex function w.r.t. η and b_3 at $\overline{x}$ for all x, that is, (P)-feasible. Then by Definition 4, we have the following inequality:

$$\begin{aligned} &\frac{1}{r} b_3(x, \overline{x}) \left(e^{r(A_3(x) - A_3(\overline{x}))} - 1 \right) \\ &\quad \geqq \frac{1}{p} \left\langle \langle z^*, \rho \rangle_m, \left(e^{p\eta(x, \overline{x})} - \mathbf{1} \right) \right\rangle. \end{aligned} \tag{29}$$

From inequalities (24) and (29), we have

$$\frac{1}{p} \left\langle \langle z^*, \rho \rangle_m, \left(e^{p\eta(x, \overline{x})} - \mathbf{1} \right) \right\rangle \leq 0. \tag{30}$$

By inequality (30) and multiplying (20) by $1/p$, it yields that

$$\frac{1}{p} \left\langle \sum_{i=1}^{k} \alpha_i^* \left(\xi_i + \phi_i(\overline{x}) \zeta_i \right), \left(e^{p\eta(x, \overline{x})} - \mathbf{1} \right) \right\rangle \geq 0. \tag{31}$$

Since $A_2(x)$ is an exponential B-(p, r)-invex function w.r.t. η and b_2 at $\overline{x}$ for all x, that is, (P)-feasible then by Definition 4, we have

$$\begin{aligned} &\frac{1}{r} b_2(x, \overline{x}) \left(e^{r(A_2(x) - A_2(\overline{x}))} - 1 \right) \\ &\quad \geqq \frac{1}{p} \left\langle \sum_{i=1}^{k} \alpha_i^* \left(\xi_i + \phi_i(\overline{x}) \zeta_i \right), \left(e^{p\eta(x, \overline{x})} - \mathbf{1} \right) \right\rangle. \end{aligned} \tag{32}$$

From inequalities (31) and (32), we obtain

$$\frac{1}{r} b_2(x, \overline{x}) \left(e^{r(A_2(x) - A_2(\overline{x}))} - 1 \right) \geqq 0. \tag{33}$$

By Lemma 8 and $b_2(x, \overline{x}) \in \mathbb{R}_+ \setminus \{0\}$, we get

$$A_2(x) \geqq A_2(\overline{x}). \tag{34}$$

If $\overline{x}$ is not an efficient solution to problem (P), then we reduce inequality (23) in the same way. But inequality (34) contradicts inequality (23). Hence, the proof is complete. □

4. Mond-Weir Type Duality Model

In order to propose Mond-Weir type duality model, it is convenient to restate the necessary conditions in Theorem 7 as the following form. Mainly, we use the expressions (13) and (15) to get

$$0 \in \sum_{i=1}^{k} \alpha_i^* \left[\partial^\circ f_i(\overline{x}) + \frac{f_i(\overline{x})}{g_i(\overline{x})} \partial^\circ (-g_i)(\overline{x}) \right] + \langle z^*, \partial^\circ h(\overline{x}) \rangle_m. \tag{35}$$

Then putting $\alpha^* = \overline{\alpha}^* g(\overline{x}) \in \mathbb{I}_+^k$ in the above expression, we obtain

$$\begin{aligned} 0 \in &\sum_{i=1}^{k} \overline{\alpha}_i^* g_i(\overline{x}) \left[\partial^\circ f_i(\overline{x}) + \langle z^*, \partial^\circ h(\overline{x}) \rangle_m \right] \\ &+ \sum_{i=1}^{k} \overline{\alpha}_i^* f_i(\overline{x}) \partial^\circ (-g_i)(\overline{x}). \end{aligned} \tag{36}$$

Consequently, from inequality (14), it yields that

$$0 \in \sum_{i=1}^{k} \overline{\alpha}_i^* g_i(\overline{x}) \left[\partial^\circ f_i(\overline{x}) + \langle z^*, \partial^\circ h(\overline{x})\rangle_m\right] + \sum_{i=1}^{k} \overline{\alpha}_i^* \left[f_i(\overline{x}) + \langle z^*, h(\overline{x})\rangle_m\right] \partial^\circ(-g_i)(\overline{x}), \tag{37}$$

where $\langle z^*, h(\overline{x})\rangle_m \equiv \sum_{j=1}^{m} z_j^* h_j(\overline{x})$. For simplicity, we write $\overline{\alpha}_i^*$ still by α_i^*. Then the result of Theorem 7 can be restated as the following theorem.

Theorem 10 (necessary optimality conditions). *If $\overline{x}$ is an efficient solution to (P) and satisfyies constraint qualification in (SP_i), $i = 1, 2, \ldots, k$, then, there exist $\alpha^* \in \mathbb{R}^k$, $z^* \in \mathbb{R}^m$ such that*

$$0 \in \sum_{i=1}^{k} \alpha_i^* g_i(\overline{x}) \left[\partial^\circ f_i(\overline{x}) + \langle z^*, \partial^\circ h(\overline{x})\rangle_m\right] + \sum_{i=1}^{k} \alpha_i^* \left[f_i(\overline{x}) + \langle z^*, h(\overline{x})\rangle_m\right] \partial^\circ(-g_i)(\overline{x}), \tag{38}$$

$$z_j^* h_j(\overline{x}) = 0 \quad \forall j = 1, 2, \ldots, m, \tag{39}$$

$$\alpha^* \in \mathbb{I}_+^k, \qquad z^* \in \mathbb{R}_+^m. \tag{40}$$

For any $u \in \mathfrak{F}$, if we use $(\alpha, z) \in \mathbb{R}^k \times \mathbb{R}^m$ instead of $(\alpha^*, z^*) \in \mathbb{R}^k \times \mathbb{R}^m$ satisfying the necessary conditions (38)~(40) as the constraints of a new dual problem, namely, Mond-Weir type dual (D), then it constitutes by a maximization programming problem with the same objective function as the problem (P), and we use the necessary optimality conditions of (P) as the constraint of the new problem (D). Precisely, we can state this dual problem as the maximization problem as the following form:

$$(D) \text{ Maximize } \quad \Phi(u) \equiv \left(\frac{f_1(u)}{g_1(u)}, \frac{f_2(u)}{g_2(u)}, \ldots, \frac{f_k(u)}{g_k(u)}\right) \equiv \left(\Phi_1(u), \Phi_2(u), \ldots, \Phi_k(u)\right), \tag{41}$$

subject to the resultant of necessary condition in Theorem 10:

$$0 \in \sum_{i=1}^{k} \alpha_i g_i(u) \left[\partial^\circ f_i(u) + \langle z, \partial^\circ h(u)\rangle_m\right] + \sum_{i=1}^{k} \alpha_i \partial^\circ(-g_i)(u) \left[f_i(u) + \langle z, h(u)\rangle_m\right], \tag{42}$$

$$\langle z, h(u)\rangle_m \equiv \sum_{j=1}^{m} z_j h_j(u) = 0, \tag{43}$$

$$u \in X, \qquad \alpha \in \mathbb{I}_+^k, \qquad z \in \mathbb{R}_+^m. \tag{44}$$

Let $\mathscr{D}$ be the constraint set $\{u; \alpha, z\}$ of (D) satisfying (42)~(44) which are the necessary optimality conditions of (P). For convenience, we denote the projective-like set by:

$$pr_{\mathfrak{F}}\mathscr{D} = \{u \in \mathfrak{F} \mid (u; \alpha, z) \in \mathscr{D}\}. \tag{45}$$

Then we can derive the following weak duality theorem between (P) and (D).

Theorem 11 (weak duality). *Let x and $(u; \alpha, z)$ be (P)-feasible and (D)-feasible, respectively. Denote a function $A_4 : X \to \mathbb{R}$ by*

$$A_4(\cdot) = \sum_{i=1}^{k} \alpha_i g_i(u) \left[f_i(\cdot) + \langle z, h(\cdot)\rangle_m\right] - \sum_{i=1}^{k} \alpha_i g_i(\cdot) \left[f_i(u) + \langle z, h(u)\rangle_m\right], \tag{46}$$

with $A_4(u) = 0$. Suppose that $A_4(\cdot)$ is an exponential B-(p, r)-invex function at $u \in pr_{\mathfrak{F}}\mathscr{D}$ w.r.t. η and b_4.

Then $\phi(x) \nleq \Phi(u)$.

Proof. Let x and $(u; \alpha, z)$ be (P)- and (D)-feasible, respectively. From expression (38), there exist $\xi_i \in \partial^\circ f_i(u)$, $\zeta_i \in \partial^\circ(-g_i)(u)$, $i = 1, 2, \ldots, k$ and $\rho_j \in \partial^\circ h_j(u)$, $j = 1, 2, \ldots, m$ to satisfy

$$\langle \widetilde{a}_4 \rangle \equiv \sum_{i=1}^{k} \alpha_i g_i(u) \left[\xi_i + \langle z, \rho\rangle_m\right] + \sum_{i=1}^{k} \alpha_i \left[f_i(u) + \langle z, h(u)\rangle_m\right] \zeta_i = 0 \in X^*, \tag{47}$$

where $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$.

Then, the dual pair of $\langle X^*, X\rangle$ yields

$$\left\langle \langle \widetilde{a}_4 \rangle, \left(e^{p\eta(x,u)} - \mathbf{1}\right)\right\rangle = 0. \tag{48}$$

Since A_4 is an exponential B-(p, r)-invex function w.r.t. η and b_4 at $u \in pr_{\mathfrak{F}}\mathscr{D}$, we have the following inequality:

$$\frac{1}{r} b_4(x, u) \left(e^{r(A_4(x) - A_4(u))} - 1\right) \geq \frac{1}{p} \left\langle \langle \widetilde{a}_4 \rangle, \left(e^{p\eta(x,u)} - \mathbf{1}\right)\right\rangle = 0. \tag{49}$$

By the above inequality and equality (48), we obtain

$$\frac{1}{r} b_4(x, u) \left(e^{r(A_4(x) - A_4(u))} - 1\right) \geq 0. \tag{50}$$

According to Lemma 8 and $b_4 \in \mathbb{R} \setminus \{0\}$, we have

$$A_4(x) \geq A_4(u) = 0. \tag{51}$$

We want to prove that $\phi(x) \nleq \Phi(u)$.

Suppose on the contrary that $\phi(x) \leq \Phi(u)$. Then

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(u)}{g_i(u)} \quad \forall i = 1, 2, \ldots, k, \tag{52}$$

and there is some index $t \in \underline{k}$ such that

$$\frac{f_t(x)}{g_t(x)} < \frac{f_t(u)}{g_t(u)}. \tag{53}$$

Then by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{I}_+^k$, we have

$$\sum_{i=1}^{k} \alpha_i f_i(x) g_i(u) < \sum_{i=1}^{k} \alpha_i g_i(x) f_i(u). \tag{54}$$

Since $h(x) \in -\mathbb{R}_+^m$, it follows from (43), (44), and (54) that

$$\begin{aligned} &\sum_{i=1}^{k} \alpha_i g_i(u) \left[f_i(x) + \langle z, h(x) \rangle_m \right] \\ &\quad < \sum_{i=1}^{k} \alpha_i g_i(x) \left[f_i(u) + \langle z, h(u) \rangle_m \right]. \end{aligned} \tag{55}$$

This implies that

$$\begin{aligned} A_4(x) = &\sum_{i=1}^{k} \alpha_i g_i(u) \left[f_i(x) + \langle z, h(x) \rangle_m \right] \\ &- \sum_{i=1}^{k} \alpha_i g_i(x) \left[f_i(u) + \langle z, h(u) \rangle_m \right] < 0, \end{aligned} \tag{56}$$

which contradicts inequality (51), and the proof of theorem is complete. □

Theorem 12 (strong duality). *Let $\overline{x}$ be the efficient solution of problem (P) satisfying the constraint qualification at $\overline{x}$ in (SP_i), $i = 1, 2, \ldots, k$. Then there exist $\alpha^* \in \mathbb{R}^k$ and $z^* \in \mathbb{R}^m$ such that $(\overline{x}; \alpha^*, z^*) \in (D)$-feasible. If the hypotheses of Theorem 11 are fulfilled, then $(\overline{x}; \alpha^*, z^*)$ is an efficient solution to problem (D). Furthermore, the efficient values of (P) and (D) are equal.*

Proof. Let $\overline{x}$ be an efficient solution to problem (P). Then there exist α^*, z^* such that $(\overline{x}; \alpha^*, z^*)$ satisfies (42)~(44) that is, $(\overline{x}; \alpha^*, z^*) \in \mathscr{D}$ is a feasible solution for the problem (D). Actually, $(\overline{x}; \alpha^*, z^*)$ is also an efficient solution of (D).

Suppose on the contrary that if $(\overline{x}; \alpha^*, z^*)$ were not an efficient solution to (D), then there exists a feasible solution $(x; \alpha, z)$ of (D) such that

$$\frac{f_i(\overline{x})}{g_i(\overline{x})} \leqq \frac{f_i(x)}{g_i(x)} \quad \forall i = 1, 2, \ldots, k, \tag{57}$$

and there is a $t \in \underline{k}$,

$$\frac{f_t(\overline{x})}{g_t(\overline{x})} < \frac{f_t(x)}{g_t(x)}. \tag{58}$$

It follows that $\phi(\overline{x}) \leq \Phi(x)$ which contradicts the weak duality Theorem 11. Hence, $(\overline{x}; \alpha^*, z^*)$ is an efficient solution of (D) and the efficient values of (P) and (D) are clearly equal. □

Theorem 13 (strict converse duality). *Let $\overline{x}$ and $(u^*; \alpha^*, z^*)$ be the efficient solutions of (P) and (D), respectively. Denote a function $A_5 : X \to \mathbb{R}$ by*

$$\begin{aligned} A_5(\cdot) = &\sum_{i=1}^{k} \alpha_i^* g_i(u^*) \left[f_i(\cdot) + \langle z^*, h(\cdot) \rangle_m \right] \\ &- \sum_{i=1}^{k} \alpha_i^* g_i(\cdot) \left[f_i(u^*) + \langle z^*, h(u^*) \rangle_m \right], \end{aligned} \tag{59}$$

with $A_5(u^*) = 0$. If $A_5(\cdot)$ is a strictly exponential B-(p, r)-invex function at $u^* \in pr_{\mathfrak{F}}\mathscr{D}$ w.r.t. η and b_5 for all optimal vectors $\overline{x}$ in (P) and $(u^*; \alpha^*, z^*)$ in (D), respectively, then $\overline{x} = u^*$ and the efficient values of (P) and (D) are equal.

Proof. Suppose that $\overline{x} \neq u^*$. From expression (42), there exist $\xi_i \in \partial^\circ f_i(u^*)$, $\zeta_i \in \partial^\circ(-g_i)(u^*)$, $i = 1, 2, \ldots, k$ and $\rho_j \in \partial^\circ h_j(u^*)$, $j = 1, 2, \ldots, m$ such that

$$\begin{aligned} \langle \widetilde{\mathfrak{a}}_5 \rangle \equiv &\sum_{i=1}^{k} \alpha_i^* g_i(u^*) \left[\xi_i + \langle z^*, \rho \rangle_m \right] \\ &+ \sum_{i=1}^{k} \alpha_i^* \left[f_i(u^*) + \langle z^*, h(u^*) \rangle_m \right] \zeta_i = 0 \in X^*, \end{aligned} \tag{60}$$

where $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$.

It follows that the dual pair in $\langle X^*, X \rangle$ becomes

$$\frac{1}{p} \left\langle \langle \widetilde{\mathfrak{a}}_5 \rangle, \left(e^{p\eta(\overline{x}, u^*)} - \mathbf{1} \right) \right\rangle = 0. \tag{61}$$

From Theorem 12, we see that there exist $\overline{\alpha}$ and $\overline{z}$ such that $(\overline{x}; \overline{\alpha}, \overline{z})$ is the efficient solution of (D) and

$$\frac{f_i(\overline{x})}{g_i(\overline{x})} = \frac{f_i(u^*)}{g_i(u^*)} \quad \forall i = 1, 2, \ldots, k. \tag{62}$$

By inequality (43) and equality (62), it becomes

$$\frac{f_i(\overline{x})}{g_i(\overline{x})} = \frac{f_i(u^*) + \langle z^*, h(u^*) \rangle_m}{g_i(u^*)}. \tag{63}$$

Eliminating the dominators in (63), we get

$$f_i(\overline{x}) g_i(u^*) = \left[f_i(u^*) + \langle z^*, h(u^*) \rangle_m \right] g_i(\overline{x}) \tag{64}$$

or

$$f_i(\overline{x}) g_i(u^*) - \left[f_i(u^*) + \langle z^*, h(u^*) \rangle_m \right] g_i(\overline{x}) = 0. \tag{65}$$

According to the above equality and by the property (44), $A_5(\overline{x})$ reduces to

$$\begin{aligned} &\sum_{i=1}^{k} \alpha_i^* g_i(u^*) \left[f_i(\overline{x}) + \langle z^*, h(\overline{x}) \rangle_m \right] \\ &\quad - \sum_{i=1}^{k} \alpha_i^* g_i(\overline{x}) \left[f_i(u^*) + \langle z^*, h(u^*) \rangle_m \right] \\ &\quad = A_5(\overline{x}) = \sum_{i=1}^{k} \alpha_i^* g_i(u^*) \langle z^*, h(\overline{x}) \rangle_m. \end{aligned} \tag{66}$$

From relations $h(\overline{x}) \in -\mathbb{R}_+^m$, (44), (66), and $g_i(u^*) > 0$, we obtain

$$A_5(\overline{x}) \leqq 0 = A_5(u^*). \tag{67}$$

Hence, we reduce

$$\frac{1}{r} b_5(\overline{x}, u^*)\left(e^{r(A_5(\overline{x})-A_5(u^*))} - 1\right) \leqq 0 \quad \text{for any } r \neq 0. \tag{68}$$

Since A_5 is a strictly exponential B-(p,r)-invex function w.r.t. η and b_5 at $u^* \in pr_{\mathfrak{F}}\mathcal{D}$, we have

$$\begin{aligned} &\frac{1}{r} b_5(\overline{x}, u^*)\left(e^{r(A_5(\overline{x})-A_5(u^*)} - 1\right) \\ &\quad > \left\langle \langle \widetilde{\mathrm{a}}_5 \rangle, \left(e^{p\eta(\overline{x},u^*)} - \mathbf{1}\right)\right\rangle. \end{aligned} \tag{69}$$

From (68) and (69), we obtain

$$\frac{1}{p}\left\langle \langle \widetilde{\mathrm{a}}_5 \rangle, \left(e^{p\eta(x^*,u^*)} - \mathbf{1}\right)\right\rangle < 0. \tag{70}$$

This contradicts equality (61). Hence, the proof of theorem is complete. □

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