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Research Article

Optimal Lower Generalized Logarithmic Mean Bound for the Seiffert Mean

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We present the greatest value p such that the inequality $P(a,b) > L_p(a,b)$ holds for all $a,b > 0$ with $a \neq b$, where $P(a,b)$ and $L_p(a,b)$ denote the Seiffert and pth generalized logarithmic means of a and b, respectively.

1. Introduction

For $p \in \mathbb{R}$, the pth generalized logarithmic mean $L_p : (0,\infty)^2 \to (0,\infty)$ is defined by

$$L_p(a,b) = \begin{cases} \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & a \neq b,\ p \neq 0,\ -1, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & a \neq b,\ p = 0, \\ \dfrac{b-a}{\log b - \log a}, & a \neq b,\ p = -1, \\ a, & a = b, \end{cases} \tag{1}$$

and the Seiffert mean $P : (0,\infty)^2 \to (0,\infty)$ [1] is defined by

$$P(a,b) = \begin{cases} \dfrac{a-b}{4\arctan\left(\sqrt{a/b}\right)-\pi}, & a \neq b, \\ a, & a = b. \end{cases} \tag{2}$$

It is well known that the generalized logarithmic mean $L_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b > 0$ with $a \neq b$. The special cases of the generalized logarithmic mean are, for example, $G(a,b) = \sqrt{ab} = L_{-2}(a,b)$ is the geometric mean, $L(a,b) = (b-a)/(\log b - \log a) = L_{-1}(a,b)$ is the logarithmic mean, $I(a,b) = 1/e(b^b/a^a)^{1/(b-a)} = L_0(a,b)$ is the identric mean, and $A(a,b) = (a+b)/2 = L_1(a,b)$ is the arithmetic mean. The Seiffert mean $P(a,b)$ can be rewritten as (see [2, equation (2.4)])

$$P(a,b) = \begin{cases} \dfrac{a-b}{2\arcsin\left[(a-b)/(a+b)\right]}, & a \neq b, \\ a, & a = b. \end{cases} \tag{3}$$

Recently, the bivariate means have been the subject of intensive research. In particular, many remarkable inequalities and properties for the generalized logarithmic and the Seiffert means can be found in the literature [3–13].

In [1, 11], Seiffert proved that the inequalities

$$\begin{gathered} L_{-1}(a,b) < P(a,b) < L_0(a,b) = I(a,b), \\ \frac{2}{\pi}A(a,b) < P(a,b) < A(a,b), \\ P(a,b) > \frac{G(a,b)A(a,b)}{L(a,b)}, \\ P(a,b) > \frac{3A(a,b)G(a,b)}{A(a,b)+2G(a,b)} \end{gathered} \tag{4}$$

hold for all $a,b > 0$ with $a \neq b$.

Sándor [14] presented the bounds for the Seiffert mean $P(a,b)$ in terms of the arithmetic mean $A(a,b)$ and geometric mean $G(a,b)$ as follows:

$$\frac{1}{2}A(a,b)+\frac{1}{2}G(a,b)<P(a,b)<\frac{2}{3}A(a,b)+\frac{1}{3}G(a,b) \tag{5}$$

for all $a,b>0$ with $a\neq b$.

Hästö [15] proved that the double inequality $M_\alpha(a,b)<P(a,b)<M_\beta(a,b)$ holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq\log 2/\log\pi$ and $\beta\geq 2/3$, where $M_p(a,b)=((a^p+b^p)/2)^{1/p}$ is the pth power mean of a and b.

In [16], the authors found the greatest value α and least value β such that the double inequality $\alpha C(a,b)+(1-\alpha)G(a,b)<P(a,b)<\beta C(a,b)+(1-\beta)G(a,b)$ holds for all $a,b>0$ with $a\neq b$, where $C(a,b)=(a^2+b^2)/(a+b)$ is the contraharmonic mean of a and b.

Motivated by the first inequality in (4), Gao [17] gave the best possible constants λ and μ such that the double inequality $\lambda I(a,b)<P(a,b)<\mu I(a,b)$ holds for all $a,b>0$ with $a\neq b$.

In [18], the author solved the following open problem proposed by Long and Chu [19]: what is the smallest p (largest q) such that the inequality $\alpha A(a,b)+(1-\alpha)G(a,b)<L_p(a,b)(>L_q(a,b))$ holds for $\alpha\in(0,1/2)(\alpha\in(1/2,1))$ and all $a,b>0$ with $a\neq b$?

Chu et al. [20] proved that the double inequality $L_p(a,b)<T(a,b)<L_q(a,b)$ holds for all $a,b>0$ with $a\neq b$ if and only if $p\leq 3$ and $q\geq q_0=3.152\ldots$, where $T(a,b)=(a-b)/[2\arctan((a-b)/(a+b))]$ is the second Seiffert mean of a and b, and $p_0=3.152\ldots$ is the unique solution of the equation $(x+1)^{-1/x}=2/\pi$.

In [21], the authors answered the question: what are the greatest value $p_1=p_1(q)$ and the least value $p_2=p_2(q)$ such that the double inequality $L_{p_1}(a,b)<[L(a^q,b^q)]^{1/q}<L_{p_2}(a,b)$ holds for any $q>0$ with $q\neq 1$ and all $a,b>0$ with $a\neq b$?

Motivated by the first inequality in (4), it is natural to ask what are the best possible generalized logarithmic mean bounds for the Seiffert mean $P(a,b)$? It is the aim of this paper to answer this question.

2. Preliminaries

In order to prove our main results, we need two lemmas, which we present in this section.

Lemma 1. *Let the function $E:(-1,+\infty)\to\mathbb{R}_+$ be defined with*

$$\begin{aligned}&(x+1)^{-1/x}, \quad x\in(-1,0)\cup(0,+\infty),\\ &\frac{1}{e}, \quad x=0.\end{aligned} \tag{6}$$

Then, E is a continuous and strictly increasing function.

Proof. From (6), we clearly see that

$$\lim_{x\longrightarrow 0}E(x)=\frac{1}{e}=E(0), \tag{7}$$

$$\begin{aligned}&\lim_{x\longrightarrow -1^+}E(x)=0,\\ &\lim_{x\longrightarrow +\infty}E(x)=1.\end{aligned} \tag{8}$$

If $x\in(-1,0)\cup(0,+\infty)$, then simple computation yields

$$E'(x)=\frac{(1+x)\log(1+x)-x}{x^2(1+x)}E(x). \tag{9}$$

If we define

$$F(x)=(1+x)\log(1+x)-x, \tag{10}$$

then

$$F'(x)=\log(1+x). \tag{11}$$

Equation (11) implies that

$$\begin{aligned}&F'(x)<0, \quad x\in(-1,0),\\ &F'(x)>0, \quad x\in(0,+\infty).\end{aligned} \tag{12}$$

Equations (10) and (12) lead to

$$F(x)>\lim_{x\longrightarrow 0}F(x)=0 \tag{13}$$

for $x\in(-1,0)\cup(0,+\infty)$.

From (9) and (10) together with (13), we clearly see that

$$E'(x)>0 \tag{14}$$

for $x\in(-1,0)\cup(0,+\infty)$.

Therefore, the continuity of E follows from (6) and (7), and the strict monotonicity of E follows from (8), (14) and the continuity of E. □

Remark 2. From Lemma 1, we clearly see that for any fixed $\lambda\in(0,1)$, there exists a unique $x\in(-1,\infty)$ such $E(x)=\lambda$. In particular, for $\lambda=1/\pi$, making use of Mathematica software, we get

$$\begin{aligned}&E(-0.241)-\frac{1}{\pi}=0.000166\cdots>0,\\ &E(-0.242)-\frac{1}{\pi}=-0.000062\cdots<0.\end{aligned} \tag{15}$$

Therefore, the unique solution of the equation $E(x)=1/\pi$ belongs to the interval $(-0.242,-0.241)$.

Lemma 3. *Let $p\in(-0.242,-0.241)$ and let $g:[1,+\infty)\to\mathbb{R}$ be defined with $g(t)=pt^{2p+3}+2(p-1)t^{2p+2}-(3p+2)t^{2p+1}-p(2p+1)t^{p+3}+(2p^2+p+4)t^{p+2}+(2p^2+p+4)t^{p+1}-p(2p+1)t^p-(3p+2)t^2+2(p-1)t+p$. Then, there exists $\lambda\in(1,+\infty)$ such that $g(t)<0$ for $t\in(1,\lambda)$ and $g(t)>0$ for $t\in(\lambda,+\infty)$.*

Proof. Simple computations lead to

$$g(1) = 0, \tag{16}$$

$$\lim_{t\longrightarrow+\infty} g(t) = +\infty, \tag{17}$$

$$\begin{aligned} g'(t) = {} & p(2p+3)t^{2p+2} + 4(p-1)(p+1)t^{2p+1} \\ & - (2p+1)(3p+2)t^{2p} - p(p+3)(2p+1)t^{p+2} \\ & + (p+2)\left(2p^2+p+4\right)t^{p+1} + (p+1) \\ & \times \left(2p^2+p+4\right)t^{p} - p^2(2p+1)t^{p-1} \\ & - 2(3p+2)t + 2(p-1), \end{aligned} \tag{18}$$

$$g'(1) = 0, \tag{19}$$

$$\lim_{t\longrightarrow+\infty} g'(t) = +\infty, \tag{20}$$

$$\begin{aligned} g''(t) = {} & 2p(p+1)(2p+3)t^{2p+1} \\ & + 4(p-1)(p+1)(2p+1)t^{2p} - 2p \\ & \times (2p+1)(3p+2)t^{2p-1} \\ & - p(p+2)(p+3)(2p+1)t^{p+1} \\ & + (p+1)(p+2)\left(2p^2+p+4\right)t^{p} \\ & + p(p+1)\left(2p^2+p+4\right)t^{p-1} \\ & - p^2(p-1)(2p+1)t^{p-2} - 2(3p+2), \end{aligned} \tag{21}$$

$$g''(1) = 0, \tag{22}$$

$$\lim_{t\longrightarrow+\infty} g''(t) = +\infty. \tag{23}$$

Let $h(t) = t^{3-p}g'''(t)$. Further computations lead to

$$\begin{aligned} h(t) = {} & 2p(p+1)(2p+1)(2p+3)t^{p+3} \\ & + 8p(p-1)(p+1)(2p+1)t^{p+2} \\ & - 2p(2p-1)(2p+1)(3p+2)t^{p+1} \\ & - p(p+1)(p+2)(p+3) \\ & \times (2p+1)t^3 + p(p+1)(p+2) \\ & \times \left(2p^2+p+4\right)t^2 + p(p-1) \\ & \times (p+1)\left(2p^2+p+4\right)t \\ & - p^2(p-2)(p-1)(2p+1), \end{aligned} \tag{24}$$

$$h(1) = 0, \tag{25}$$

$$\lim_{t\longrightarrow+\infty} h(t) = +\infty, \tag{26}$$

$$\begin{aligned} h'(t) = {} & 2p(p+1)(p+3)(2p+1)(2p+3)t^{p+2} \\ & + 8p(p-1)(p+1)(p+2)(2p+1)t^{p+1} \\ & - 2p(p+1)(2p-1)(2p+1)(3p+2)t^{p} \\ & - 3p(p+1)(p+2)(p+3)(2p+1)t^2 \\ & + 2p(p+1)(p+2)\left(2p^2+p+4\right)t + p \\ & \times (p-1)(p+1)\left(2p^2+p+4\right), \end{aligned} \tag{27}$$

$$h'(1) = 24p^3(p+1) < 0, \tag{28}$$

$$\lim_{t\longrightarrow+\infty} h'(t) = +\infty, \tag{29}$$

$$\begin{aligned} h''(t) = {} & 2p(p+1)(p+2)(p+3)(2p+1)(2p+3)t^{p+1} \\ & + 8p(p-1)(p+1)^2(p+2)(2p+1)t^{p} \\ & - 2p^2(p+1)(2p-1)(2p+1)(3p+2)t^{p-1} \\ & - 6p(p+1)(p+2)(p+3)(2p+1)t \\ & + 2p(p+1)(p+2)\left(2p^2+p+4\right), \end{aligned} \tag{30}$$

$$h''(1) = p^3\left(72p^2+156p+84\right) < 0, \tag{31}$$

$$\lim_{t\longrightarrow+\infty} h''(t) = +\infty, \tag{32}$$

$$\begin{aligned} h'''(t) = {} & 2p(p+1)^2(p+2)(p+3)(2p+1) \\ & \times (2p+3)t^{p} + 8p^2(p-1)(p+1)^2 \\ & \times (p+2)(2p+1)t^{p-1} - 2p^2(p-1) \\ & \times (p+1)(2p-1)(2p+1) \\ & \times (3p+2)t^{p-2} - 6p(p+1) \\ & \times (p+2)(p+3)(2p+1), \end{aligned} \tag{33}$$

$$h'''(1) = p^2\left(112p^4+312p^3+352p^2+192p+40\right) > 0, \tag{34}$$

$$\begin{aligned} h^{(4)}(t) = {} & 2p^2(p+1)^2(p+2)(p+3) \\ & \times (2p+1)(2p+3)t^{p-1} \\ & + 8p^2(p-1)^2(p+1)^2(p+2)(2p+1)t^{p-2} \\ & + 2p^2(1-p)(2-p)(1-2p) \\ & \times (p+1)(2p+1)(3p+2)t^{p-3} \\ > {} & 0. \end{aligned} \tag{35}$$

Inequalities (34) and (35) imply that

$$h'''(t) > 0 \tag{36}$$

for $t \in (1, \infty)$.

From (31) and (32) together with (36), we clearly see that there exists $t_0 \in (1, \infty)$ such that $h''(t) < 0$ for $t \in (1, t_0)$ and $h''(t) > 0$ for $t \in (t_0, \infty)$. Hence, h' is strictly decreasing on $(1, t_0)$ and strictly increasing on (t_0, ∞).

It follows from (28) and (29) together with the monotonicity of h' that there exists $t_1 \in (1, \infty)$ such that h is strictly decreasing on $(1, t_1)$ and strictly increasing on (t_1, ∞).

Equations (25)-(26) and the monotonicity of h lead to the conclusion that there exists $t_2 \in (1, \infty)$ such that $h(t) < 0$ for $t \in (1, t_2)$ and $h(t) > 0$ for $t \in (t_2, \infty)$. Therefore, g'' is strictly decreasing on $(1, t_2)$ and strictly increasing on (t_2, ∞).

From (22) and (23) together with the monotonicity of g'', we clearly see that there exists $t_3 \in (1, \infty)$ such that g' is strictly decreasing on $(1, t_3)$ and strictly increasing on (t_3, ∞).

Equations (19)-(20) and the monotonicity of g' imply that there exists $t_4 \in (1, \infty)$ such that g is strictly decreasing on $(1, t_4)$ and strictly increasing on (t_4, ∞).

Therefore, Lemma 3 follows from (16) and (17) together with the monotonicity of g. □

3. Main Results

Theorem 4. *Let E be as in Lemma 1 and let p be the unique solution of the equation $E(x) = 1/\pi$. Then, for all $a, b > 0$, $a \neq b$, the inequality*

$$L_p(a, b) < P(a, b) \tag{37}$$

holds, and $L_p(a, b)$ is the best possible lower generalized logarithmic mean bound for the Seiffert mean $P(a, b)$.

Proof. From Remark 2, it follows that

$$p \in (-0.242, -0.241). \tag{38}$$

We first prove that the inequality (37) holds. Without loss of generality, we assume that $a > b$. If $t = a/b > 1$, then, from (1) and (2), we have

$$\begin{aligned} &\log L_p(a, b) - \log P(a, b) \\ &\quad = \frac{1}{p} \log \frac{t^{p+1} - 1}{(p+1)(t-1)} \\ &\qquad + \log\left(4 \arctan \sqrt{t} - \pi\right) - \log(t-1), \end{aligned} \tag{39}$$

If

$$\begin{aligned} f(t) &= \frac{1}{p} \log \frac{t^{p+1} - 1}{(p+1)(t-1)} \\ &\quad + \log\left(4 \arctan \sqrt{t} - \pi\right) - \log(t-1), \end{aligned} \tag{40}$$

then simple computations lead to

$$\lim_{t \to 1} f(t) = \lim_{t \to +\infty} f(t) = 0, \tag{41}$$

$$f'(t) = \frac{(p+1)(t^p - 1)}{p\left(t^{p+1} - 1\right)(t-1)\left(4 \arctan \sqrt{t} - \pi\right)} f_1(t), \tag{42}$$

where

$$\begin{gathered} f_1(t) = -4 \arctan \sqrt{t} + \pi + \frac{2p\left(t^{p+1} - 1\right)(t-1)}{(p+1)\sqrt{t}(t+1)(t^p - 1)}, \\ \lim_{t \to 1} f_1(t) = 0, \\ \lim_{t \to +\infty} f_1(t) = +\infty, \end{gathered} \tag{43}$$

$$f_1'(t) = \frac{f_2(t)}{(1+p)(t+1)^2 (t^p - 1)^2 t^{3/2}}, \tag{44}$$

where

$$\begin{aligned} f_2(t) = {} & p t^{2p+3} + 2(p-1) t^{2p+2} - (3p+2) t^{2p+1} \\ & - p(2p+1) t^{p+3} + \left(2p^2 + p + 4\right) t^{p+2} \\ & + \left(2p^2 + p + 4\right) t^{p+1} - p(2p+1) t^p \\ & - (3p+2) t^2 + 2(p-1) t + p. \end{aligned} \tag{45}$$

From (38) and (44)-(45) together with Lemma 3, we clearly see that there exists $\lambda \in (1, +\infty)$ such that f_1 is strictly decreasing on $(1, \lambda)$ and strictly increasing on (λ, ∞). Then, (38) and (42)-(43) together with the monotonicity of f_1 imply that there exists $\mu \in (1, +\infty)$ such that f is strictly decreasing on $(1, \mu)$ and strictly increasing on $(\mu, +\infty)$.

Therefore, $L_p(a, b) < P(a, b)$ follows from (39)–(41) and the monotonicity of f.

Next, we prove that $L_p(a, b)$ is the best possible lower generalized logarithmic mean bound for the Seiffert mean $P(a, b)$.

For any $0 < \varepsilon < -p$, from (1) and (2), we get

$$\lim_{t \to +\infty} \frac{L_{p+\varepsilon}(t, 1)}{P(t, 1)} = \pi(p + 1 + \varepsilon)^{-1/(p+\varepsilon)}. \tag{46}$$

Lemma 1 and (46) lead to

$$\lim_{t \to +\infty} \frac{L_{p+\varepsilon}(t, 1)}{P(t, 1)} > 1. \tag{47}$$

Inequality (47) implies that, for $0 < \varepsilon < -p$, there exists $T = T(\varepsilon) > 1$ such that $L_{p+\varepsilon}(t, 1) > P(t, 1)$ for $t \in (T, \infty)$. □

Theorem 5. *$L_0(a, b)$ is the best possible upper generalized logarithmic mean bound for the Seiffert mean $P(a, b)$.*

Proof. For any $0 < \varepsilon < 1$ and $x > 0$, from (1) and (2), we have

$$\begin{aligned} &[P(1+x, 1)]^\varepsilon - \left[L_{-\varepsilon}(1+x, 1)\right]^\varepsilon \\ &\quad = \frac{f(x)}{\left[(1+x)^{1-\varepsilon} - 1\right]\left[4 \arctan \sqrt{1+x} - \pi\right]^\varepsilon}, \end{aligned} \tag{48}$$

where $f(x) = [(1+x)^{1-\varepsilon} - 1] x^\varepsilon - (1-\varepsilon)(4 \arctan \sqrt{1+x} - \pi)^\varepsilon x$.

When $x \to 0$, then making use of the Taylor expansion, we get

$$\begin{aligned} f(x) &= \left[(1-\varepsilon)x - \frac{1}{2}\varepsilon(1-\varepsilon)x^2 \right. \\ &\quad \left. + \frac{1}{6}\varepsilon(1-\varepsilon)(1+\varepsilon)x^3 + o\left(x^3\right)\right]x^{\varepsilon} \\ &\quad - (1-\varepsilon)\left[x - \frac{1}{2}x^2 + \frac{7}{24}x^3 + o\left(x^3\right)\right]^{\varepsilon} x \\ &= \frac{1}{24}\varepsilon^2(1-\varepsilon)x^{3+\varepsilon} + o\left(x^{3+\varepsilon}\right). \end{aligned} \tag{49}$$

Equations (48) and (49) imply that for any $0 < \varepsilon < 1$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$P(1+x,1) > L_{-\varepsilon}(1+x,1) \tag{50}$$

for $x \in (0,\delta)$.

Therefore, Theorem 5 follows from inequalities (4) and (50). □

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