Research Article

Group Classification of a Generalized Lane-Emden System

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We perform the group classification of the generalized Lane-Emden system

\[ \frac{d^2 u}{dx^2} + \frac{n}{x} \frac{du}{dx} + H(v) = 0, \]
\[ \frac{d^2 v}{dx^2} + \frac{n}{x} \frac{dv}{dx} + G(u) = 0. \]

Such systems arise in the modeling of several physical phenomena, such as pattern formation, population evolution, and chemical reactions. We obtain four cases depending on the values of \( n \).

1. Introduction

The celebrated Lane-Emden equation

\[ \frac{d^2 y}{dx^2} + \frac{n}{x} \frac{dy}{dx} + f(y) = 0, \]  

where \( n \) is a real constant and \( f(y) \) is a real-valued function of the variable \( y \), has many applications in mathematical physics and astrophysics. Equation (1), for certain fixed values of \( n \) and \( f(y) \), models several phenomena such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gaseous sphere, and the theory of thermionic currents [1–3]. Several methods for the solution and many applications of the Lane-Emden Equation (1) can be found in the literature. The interested reader is referred to [4] and the references therein. It is worth mentioning that Wong [5], in his review paper of 1975, presented more than 140 references on this topic.

A natural extension of (1), called the generalized Lane-Emden system [6], is given by

\[ \frac{d^2 u}{dx^2} + \frac{n}{x} \frac{du}{dx} + H(v) = 0, \]
\[ \frac{d^2 v}{dx^2} + \frac{n}{x} \frac{dv}{dx} + G(u) = 0. \]

2. Equivalence Transformations

An equivalence transformation (see, e.g., [13]) of the system (2) is an invertible transformation involving the variables \( x, u, \) and \( v \) that map system (2) into itself, with possibly the form of the transformed functions being different from that
of the original functions $H(v)$ and $G(u)$. We write system (2) as
\[
\frac{d^2 u}{dx^2} + \frac{n}{x} \frac{du}{dx} + H(u) = 0, \\
\frac{d^2 v}{dx^2} + \frac{n}{x} \frac{dv}{dx} + G(u) = 0,
\]
where $u$ and $v$ are differential variables with independent variable $x$, and $H$ is a differential function of the independent variables $x$ and $v$, whereas $G$ is a differential function of the independent variables $x$ and $u$. We obtain the generators of the group of equivalence transformations as
\[
Y = \xi(x, u, v) \frac{\partial}{\partial x} + \eta^{\prime}(x, u, v) \frac{\partial}{\partial u} \\
+ \eta^2 (x, u, v) \frac{\partial}{\partial v} + \mu^1 (x, u, v, H, G) \frac{\partial}{\partial H} \\
+ \mu^2 (x, u, v, H, G) \frac{\partial}{\partial G}.
\]
We apply Lie's infinitesimal approach by using the prolongation of $Y$ to involve the derivatives in system (3) as, for example, in [14].

We summarize our results below.

Case 1 ($n \neq -1, 1, 3$). In this case system (3) has the nine-dimensional equivalence Lie algebra spanned by the equivalence generators
\[
X_1 = x \frac{\partial}{\partial x} - 2H \frac{\partial}{\partial H} - 2G \frac{\partial}{\partial G}, \\
X_2 = u \frac{\partial}{\partial u} + H \frac{\partial}{\partial H}, \\
X_3 = v \frac{\partial}{\partial v} + G \frac{\partial}{\partial G}, \\
X_4 = \frac{\partial}{\partial u}, \\
X_5 = \frac{\partial}{\partial v}, \\
X_6 = x^{1-n} \frac{\partial}{\partial u}, \\
X_7 = x^{1-n} \frac{\partial}{\partial v}, \\
X_8 = x^2 \frac{\partial}{\partial u} - 2(1 + n) \frac{\partial}{\partial H}, \\
X_9 = x^2 \frac{\partial}{\partial v} - 2(1 + n) \frac{\partial}{\partial G}.
\]

and hence the nine-parameter equivalence group is given by
\[
X_1 : \bar{x} = e^{a_1} x, \quad \bar{u} = u, \quad \bar{v} = v, \\
\bar{H} = e^{-2a_1} H, \quad \bar{G} = e^{-2a_1} G, \\
X_2 : \bar{x} = x, \quad \bar{u} = e^{a_1} u, \quad \bar{v} = v, \quad \bar{H} = e^{a_1} H, \quad \bar{G} = G, \\
X_3 : \bar{x} = x, \quad \bar{u} = u, \quad \bar{v} = v, \quad \bar{H} = H, \quad \bar{G} = e^{a_1} G, \\
X_4 : \bar{x} = x, \quad \bar{u} = u + a_4, \\
\bar{v} = v, \quad \bar{H} = H, \quad \bar{G} = G, \\
X_5 : \bar{x} = x, \quad \bar{u} = u, \quad \bar{v} = v + a_5, \\
\bar{H} = H, \quad \bar{G} = G, \\
X_6 : \bar{x} = x, \quad \bar{u} = u + a_6 x^{1-n}, \\
\bar{v} = v, \quad \bar{H} = H, \quad \bar{G} = G, \\
X_7 : \bar{x} = x, \quad \bar{u} = u, \quad \bar{v} = v + a_7 x^{1-n}, \\
\bar{H} = H, \quad \bar{G} = G, \\
X_8 : \bar{x} = x, \quad \bar{u} = u + a_8 x^2, \\
\bar{v} = v, \quad \bar{H} = H - 2(1 + n) a_8, \quad \bar{G} = G, \\
X_9 : \bar{x} = x, \quad \bar{u} = u, \quad \bar{v} = v + a_9 x^2, \\
\bar{H} = H, \quad \bar{G} = G - 2(1 + n) a_9.
\]

Thus the composition of these transformations gives
\[
\bar{x} = e^{a_1} x, \\
\bar{u} = e^{a_1} \left( u + a_6 x^{1-n} + a_4 \right), \\
\bar{v} = e^{a_1} \left( v + a_7 x^{1-n} + a_5 \right), \\
\bar{H} = e^{a_1} e^{-2a_1} \left( H - 2(1 + n) a_8 \right), \\
\bar{G} = e^{a_1} e^{-2a_1} \left( G - 2(1 + n) a_9 \right).
\]

Case 2 ($n = -1$). In this case system (3) has the nine-dimensional equivalence Lie algebra spanned by the equivalence generators
\[
X_1 = x \frac{\partial}{\partial x} - 2H \frac{\partial}{\partial H} - 2G \frac{\partial}{\partial G}, \\
X_2 = u \frac{\partial}{\partial u} + H \frac{\partial}{\partial H}, \\
X_3 = v \frac{\partial}{\partial v} + G \frac{\partial}{\partial G}, \\
X_4 = \frac{\partial}{\partial u}, \\
X_5 = \frac{\partial}{\partial v}, \\
X_6 = x \frac{\partial}{\partial u}, \\
X_7 = x \frac{\partial}{\partial v}, \\
X_8 = x^2 \frac{\partial}{\partial u} - 2(1 + n) \frac{\partial}{\partial H}, \\
X_9 = x^2 \frac{\partial}{\partial v} - 2(1 + n) \frac{\partial}{\partial G}.
\]
\( X_3 = v \frac{\partial}{\partial v} + G \frac{\partial}{\partial G}, \)
\( X_4 = \frac{\partial}{\partial u}, \)
\( X_5 = \frac{\partial}{\partial v}, \)
\( X_6 = x^2 \ln x \frac{\partial}{\partial u} - 2 \frac{\partial}{\partial H}, \)
\( X_7 = x^2 \ln x \frac{\partial}{\partial v} - 2 \frac{\partial}{\partial G}, \)
\( X_8 = x^2 \frac{\partial}{\partial u}, \)
\( X_9 = x^2 \frac{\partial}{\partial v} \)

and hence the nine-parameter equivalence group is given by

\[
\begin{align*}
X_1 : \bar{x} &= e^{a_1} x, \quad \bar{u} = u, \quad \bar{v} = v, \\
\bar{H} &= e^{-2a_1} H, \quad \bar{G} = e^{-2a_1} G, \\
X_2 : \bar{x} &= x, \quad \bar{u} = e^{a_2} u, \quad \bar{v} = v, \\
\bar{H} &= e^{a_2} H, \quad \bar{G} = G, \\
X_3 : \bar{x} &= x, \quad \bar{u} = u, \quad \bar{v} = e^{a_3} v, \\
\bar{H} &= H, \quad \bar{G} = e^{a_3} G, \\
X_4 : \bar{x} &= x, \quad \bar{u} = u + a_4, \quad \bar{v} = v, \\
\bar{H} &= H, \quad \bar{G} = G, \\
X_5 : \bar{x} &= x, \quad \bar{u} = u, \quad \bar{v} = v + a_5, \\
\bar{H} &= H, \quad \bar{G} = G, \\
X_6 : \bar{x} &= x, \quad \bar{u} = u + a_6 x^2 \ln x, \quad \bar{v} = v, \\
\bar{H} &= H - 2a_6, \quad \bar{G} = G, \\
X_7 : \bar{x} &= x, \quad \bar{u} = u, \quad \bar{v} = v + a_7 x^2 \ln x, \\
\bar{H} &= H, \quad \bar{G} = G - 2a_7, \\
X_8 : \bar{x} &= x, \quad \bar{u} = u + a_8 x^2, \quad \bar{v} = v, \\
\bar{H} &= H, \quad \bar{G} = G, \\
X_9 : \bar{x} &= x, \quad \bar{u} = u, \quad \bar{v} = v + a_9 x^2, \\
\bar{H} &= H, \quad \bar{G} = G.
\end{align*}
\]

Hence the composition of these transformations gives

\[
\begin{align*}
\bar{x} &= e^{a_1} x, \\
\bar{u} &= e^{a_2} \left( u + a_4 x^2 \ln x + x^2 a_6 + a_4 \right), \\
\bar{v} &= e^{a_3} \left( v + a_7 x^2 \ln x + x^2 a_8 + a_5 \right), \\
\bar{H} &= e^{a_2} (H - 2a_6), \\
\bar{G} &= e^{a_3} (G - 2a_7).
\end{align*}
\]

**Case 3** \((n = 1)\). In this case system (3) has the nine-dimensional equivalence Lie algebra spanned by the equivalence generators

\[
\begin{align*}
X_1 &= x \frac{\partial}{\partial x} - 2H \frac{\partial}{\partial H} - 2G \frac{\partial}{\partial G}, \\
X_2 &= u \frac{\partial}{\partial u} + H \frac{\partial}{\partial H}, \\
X_3 &= v \frac{\partial}{\partial v} + G \frac{\partial}{\partial G}, \\
X_4 &= \frac{\partial}{\partial u}, \\
X_5 &= \frac{\partial}{\partial v}, \\
X_6 &= x^2 \ln x \frac{\partial}{\partial u} - 4 \frac{\partial}{\partial H}, \\
X_7 &= x^2 \ln x \frac{\partial}{\partial v} - 4 \frac{\partial}{\partial G}.
\end{align*}
\]

and hence the nine-parameter equivalence group is given by

\[
\begin{align*}
X_1 : \bar{x} &= e^{a_1} x, \quad \bar{u} = u, \quad \bar{v} = v, \\
\bar{H} &= e^{-2a_1} H, \quad \bar{G} = e^{-2a_1} G, \\
X_2 : \bar{x} &= x, \quad \bar{u} = e^{a_2} u, \quad \bar{v} = v, \\
\bar{H} &= e^{a_2} H, \quad \bar{G} = G, \\
X_3 : \bar{x} &= x, \quad \bar{u} = u, \quad \bar{v} = e^{a_3} v, \\
\bar{H} &= H, \quad \bar{G} = e^{a_3} G, \\
X_4 : \bar{x} &= x, \quad \bar{u} = u + a_4, \quad \bar{v} = v, \\
\bar{H} &= H, \quad \bar{G} = G, \\
X_5 : \bar{x} &= x, \quad \bar{u} = u + a_5 x^2, \quad \bar{v} = v, \\
\bar{H} &= H, \quad \bar{G} = G.
\end{align*}
\]
Table 1: Lie symmetries for $n \neq -1,1,3$, for various functions $H(v)$ and $G(u)$.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H(v)$ arbitrary, $G(u) = c, c$ a constant</td>
<td>$X_1 = \frac{\partial}{\partial u}, X_2 = x^{(1-n)} \frac{\partial}{\partial v}, X_3 = (cx^2 + 2nu + 2v) \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>$H(v)$ arbitrary, $G(u) = c, c$ a constant</td>
<td>$X_1 = \frac{\partial}{\partial v}, X_2 = x^{(1-n)} \frac{\partial}{\partial v}, X_3 = (dx^2 + 2nu + 2v) \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>$H(v)$ arbitrary, $G(u) = c, c$ a constant</td>
<td>$X_1 = \frac{\partial}{\partial u}, X_2 = x^{(1-n)} \frac{\partial}{\partial u}, X_3 = (cux^2 + 2nu + 2v) \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>$H(v)$ arbitrary, $G(u) = c, c$ a constant</td>
<td>$X_1 = \frac{\partial}{\partial v}, X_2 = x^{(1-n)} \frac{\partial}{\partial v}, X_3 = (dx^2 + 2nu + 2v) \frac{\partial}{\partial u}$</td>
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<tr>
<td>$H(v)$ arbitrary, $G(u) = c, c$ a constant</td>
<td>$X_1 = \frac{\partial}{\partial v}, X_2 = x^{(1-n)} \frac{\partial}{\partial v}, X_3 = (dx^2 + 2nu + 2v) \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>$H(v)$ arbitrary, $G(u) = c, c$ a constant</td>
<td>$X_1 = \frac{\partial}{\partial u}, X_2 = x^{(1-n)} \frac{\partial}{\partial u}, X_3 = (cux^2 + 2nu + 2v) \frac{\partial}{\partial u}$</td>
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<td>$H(v)$ arbitrary, $G(u) = c, c$ a constant</td>
<td>$X_1 = \frac{\partial}{\partial v}, X_2 = x^{(1-n)} \frac{\partial}{\partial v}, X_3 = (dx^2 + 2nu + 2v) \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>$H(v)$ arbitrary, $G(u) = c, c$ a constant</td>
<td>$X_1 = \frac{\partial}{\partial u}, X_2 = x^{(1-n)} \frac{\partial}{\partial u}, X_3 = (cux^2 + 2nu + 2v) \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>$H(v)$ arbitrary, $G(u) = c, c$ a constant</td>
<td>$X_1 = \frac{\partial}{\partial v}, X_2 = x^{(1-n)} \frac{\partial}{\partial v}, X_3 = (dx^2 + 2nu + 2v) \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>$H(v)$ arbitrary, $G(u) = c, c$ a constant</td>
<td>$X_1 = \frac{\partial}{\partial u}, X_2 = x^{(1-n)} \frac{\partial}{\partial u}, X_3 = (cux^2 + 2nu + 2v) \frac{\partial}{\partial u}$</td>
</tr>
</tbody>
</table>
Table 1: Continued.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H(v) = a + bv^m$, $a, m,$ and $b$ constants ($b, m \neq 0$), $G(u) = c, c$ a constant</td>
<td>$X_1 = \frac{\partial}{\partial u}, X_2 = x^{(1-n)} \frac{\partial}{\partial x}, X_3 = (c^2 + 2mv + 2v) \frac{\partial}{\partial u}$, $X_4 = (nx + x) \frac{\partial}{\partial x} + (2u - 2nmu + 2nu - amx^2 - 2mu) \frac{\partial}{\partial u} + (2nu + 2u) \frac{\partial}{\partial v}$</td>
</tr>
<tr>
<td>$H(u) = be^{-mu}, m$ and $b$ constants ($b, m \neq 0$), $G(u) = \alpha + \beta u, \alpha$ and $\beta$ constants ($\beta \neq 0$)</td>
<td>$X_1 = (\beta x + \beta mu) \frac{\partial}{\partial x} - (2\alpha + 2\alpha m + 2\beta mu - 2\beta u) \frac{\partial}{\partial u} + 4\beta v \frac{\partial}{\partial v}$</td>
</tr>
<tr>
<td>$H(u) = be^{-mu}, m$ and $b$ constants ($b, m \neq 0$), $G(u) = \alpha + \beta u, \alpha$ and $\beta$ constants ($\beta \neq 0$)</td>
<td>$X_1 = (\beta x + \beta mu) \frac{\partial}{\partial x} - (2\alpha m + 2\beta mu - 2\beta u) \frac{\partial}{\partial u} + 4\beta v \frac{\partial}{\partial v}$</td>
</tr>
<tr>
<td>$H(u) = a + b e^{mu}, a, m,$ and $b$ constants ($b, m \neq 0$), $G(u) = c, c$ constant</td>
<td>$X_1 = \frac{\partial}{\partial u}, X_2 = x^{(1-n)} \frac{\partial}{\partial x}, X_3 = (c^2 + 2mv + 2v) \frac{\partial}{\partial u}$, $X_4 = (nx + x) \frac{\partial}{\partial x} + (2u - 2nmu + 2nu - amx^2 - 2mu) \frac{\partial}{\partial u} + (2nu + 2u) \frac{\partial}{\partial v}$</td>
</tr>
</tbody>
</table>

The composition of these transformations gives

$$X_5 : \bar{x} = x, \quad \bar{u} = u, \quad \bar{v} = v + a_5,$$

$$\bar{H} = H, \quad \bar{G} = G,$$

and so the composition of these transformations gives

$$\bar{x} = e^{a_1} x,$$

$$\bar{u} = e^{a_1} \left( u + a_6 \ln x + a_7 x^2 + a_8 \right),$$

$$\bar{v} = e^{a_1} \left( v + a_7 \ln x + a_9 x^2 + a_8 \right),$$

(13)

$$\bar{H} = e^{a_1} (H - 4a_8),$$

$$\bar{G} = e^{a_1} (G - 4a_9).$$

Case 4 ($n = 3$). In this case system (3) has the ten-dimensional equivalence Lie algebra spanned by the equivalence generators

$$X_1 = x \frac{\partial}{\partial x} - 2H \frac{\partial}{\partial H} - 2G \frac{\partial}{\partial G},$$

$$X_2 = x^{-1} \frac{\partial}{\partial x} - 2x^{-2} u \frac{\partial}{\partial u} - 2x^{-2} v \frac{\partial}{\partial v},$$

(12)
Table 2: Lie symmetries for $n = -1$, for various functions $H(v)$ and $G(u)$.

<table>
<thead>
<tr>
<th>$H(v)$ arbitrary, $G(u) = c, c$ a constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = \frac{\partial}{\partial u}, X_2 = x^2 \frac{\partial}{\partial u}, X_3 = (cx^2 \ln x + 2v) \frac{\partial}{\partial u}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$H(v) = d, d$ a constant, $G(u)$ arbitrary</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = \frac{\partial}{\partial u}, X_2 = x^2 \frac{\partial}{\partial u}, X_3 = (dx^2 \ln x + 2u) \frac{\partial}{\partial u}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$H(v) = d, d$ a constant, $G(u) = \alpha + \beta u, \alpha$ and $\beta$ constants ($\beta \neq 0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = \frac{\partial}{\partial u}, X_2 = x^2 \frac{\partial}{\partial u}, X_3 = (dx^2 \ln x + 2u) \frac{\partial}{\partial u}, X_4 = \beta x^4 \frac{\partial}{\partial u} - 8x^2 \frac{\partial}{\partial u}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$H(v) = d, d$ a constant, $G(u) = \alpha + \beta e^{-ku}, \alpha, k$, and $\beta$ constants ($\beta, k \neq 0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = \frac{\partial}{\partial u}, X_2 = x^2 \frac{\partial}{\partial u}, X_3 = (dx^2 \ln x + 2u) \frac{\partial}{\partial u}, X_4 = 2 \frac{\partial}{\partial u} - (ax^2 \ln x + 2uv) \frac{\partial}{\partial u}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$H(v) = a + bu, a$ and $b$ constants ($b \neq 0$), $G(u) = c, c$ constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = \frac{\partial}{\partial u}, X_2 = x^2 \frac{\partial}{\partial u}, X_3 = (cx^2 \ln x + 2v) \frac{\partial}{\partial u}, X_4 = cx^2 \ln x \frac{\partial}{\partial u} - 2x \frac{\partial}{\partial u}$</td>
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</table>

<table>
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<tr>
<th>$H(v) = a + bu, a$ and $b$ constants ($b \neq 0$), $G(u) = \alpha + \beta u, \alpha$ and $\beta$ constants ($\beta \neq 0$)</th>
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</thead>
<tbody>
<tr>
<td>$X_1 = F(x) \frac{\partial}{\partial u}, X_2 = W(x) \frac{\partial}{\partial u}, X_3 = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial u}, X_4 = bu \frac{\partial}{\partial u} + \beta u \frac{\partial}{\partial u}$</td>
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</tbody>
</table>

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<tr>
<th>$H(v) = a + bu, a$ and $b$ constants ($b \neq 0$), $G(u) = \beta u^p, \beta$ and $p$ constants ($\beta, p \neq 0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = (bx + bpx) \frac{\partial}{\partial x} + 4bu \frac{\partial}{\partial u} + (2a + 2bv - 2ap - 2bpv) \frac{\partial}{\partial u}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$H(v) = a + bu, a$ and $b$ constants ($b \neq 0$), $G(u) = \beta e^{-ku}, \beta$ and $k$ constants ($\beta, k \neq 0$)</th>
</tr>
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<tbody>
<tr>
<td>$X_1 = (b + bpu) \frac{\partial}{\partial x} + 4bu \frac{\partial}{\partial u} - (2ak + 2bkv) \frac{\partial}{\partial u}$</td>
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<th>$H(v) = a + bu, a$ and $b$ constants ($b, m \neq 0$), $G(u) = c, c$ a constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = \frac{\partial}{\partial u}, X_2 = x^2 \frac{\partial}{\partial u}, X_3 = (cx^2 \ln x + 2v) \frac{\partial}{\partial u}, X_4 = x^2 \frac{\partial}{\partial x} + (2u - amx^2 \ln x - 2mu) \frac{\partial}{\partial u} + 2u \frac{\partial}{\partial v}$</td>
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Table 2: Continued.

<table>
<thead>
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<tbody>
<tr>
<td>$H(v) = bv - m$ and $b$ constants $(b, m \neq 0)$, $G(u) = \alpha + \beta u$, $\alpha$ and $\beta$ constants $(\beta \neq 0)$</td>
<td>$X_1 = (\beta x + \beta m x) \frac{\partial}{\partial x} + (2\alpha - 2\alpha m + 2\beta u - 2\beta m u) \frac{\partial}{\partial u} + 4\beta v \frac{\partial}{\partial v}$</td>
</tr>
<tr>
<td>$H(v) = bv - m$ and $b$ constants $(b, m \neq 0)$, $G(u) = \beta e^{-\gamma u}$, $k$ and $\beta$ constants $(\beta, k \neq 0)$</td>
<td>$X_1 = k m x \frac{\partial}{\partial x} + (2m - 2) \frac{\partial}{\partial u} + 2k v \frac{\partial}{\partial v}$</td>
</tr>
<tr>
<td>$H(v) = a + b \ln v$, $a$, $m$, and $b$ constants $(b \neq 0)$, $G(u) = c$</td>
<td>$X_1 = \frac{\partial}{\partial u}$, $X_2 = x^2 \frac{\partial}{\partial u}$, $X_3 = (c x^2 \ln x + 2v) \frac{\partial}{\partial u}$, $X_4 = x^2 \frac{\partial}{\partial x} + (2u - b x^2 \ln x) \frac{\partial}{\partial u} + 2v \frac{\partial}{\partial v}$</td>
</tr>
</tbody>
</table>

and hence the ten-parameter equivalence group is given by

$X_1: \bar{x} = x$, $\bar{u} = e^{a}u$, $\bar{v} = v$,

$\bar{H} = e^{b}H$, $\bar{G} = G$,

$X_2: \bar{x} = (x^2 + 2a_2)^{1/2}$, $\bar{u} = ux^2 (x^2 + 2a_2)^{-1}$, $\bar{v} = ux^2 (x^2 + 2a_2)^{-1}$, $\bar{H} = H$, $\bar{G} = G$,

$X_3: \bar{x} = x$, $\bar{u} = u + a_5 x^{-2}$, $\bar{v} = v + a_5$, $\bar{H} = H - 8a_5$, $\bar{G} = G - 8a_5$.

(14)
Table 3: Lie symmetries for $n = 1$, for various functions $H(v)$ and $G(u)$.

| $H(v)$ arbitrary, $G(u) = c, c$ a constant |
| $X_1 = \frac{\partial}{\partial u}, X_2 = \ln x \frac{\partial}{\partial u}, X_3 = (cx^2 + 4v) \frac{\partial}{\partial v}$ |

| $H(v) = d, d$ a constant, $G(u)$ arbitrary |
| $X_1 = \frac{\partial}{\partial v}, X_2 = \ln x \frac{\partial}{\partial v}, X_3 = (dx^2 \ln x + 4u) \frac{\partial}{\partial v}$ |

| $H(v) = d, d$ a constant, $G(u) = \alpha + \beta u, \alpha$ and $\beta$ constants ($\beta \neq 0$) |
| $X_1 = \frac{\partial}{\partial v}, X_2 = \ln x \frac{\partial}{\partial v}, X_3 = (dx^2 \ln x + 4u) \frac{\partial}{\partial v}, X_4 = \beta x^2 \frac{\partial}{\partial v} - 4 \frac{\partial}{\partial u}$, |
| $X_5 = (16dx^2 + 64u) \frac{\partial}{\partial u} + (16x^2 + 64v - \beta dx^4) \frac{\partial}{\partial v}$, |
| $X_7 = 16x \frac{\partial}{\partial x} - (8ax^2 - \beta dx^4) \frac{\partial}{\partial x} - (16dx^2 + 32u) \frac{\partial}{\partial u}$, |

| $H(v) = a + bv, a$ and $b$ constants ($b \neq 0$), $G(u) = c, c$ constant |
| $X_1 = \frac{\partial}{\partial u}, X_2 = \ln x \frac{\partial}{\partial u}, X_3 = (cx^2 + 4v) \frac{\partial}{\partial u}, X_4 = 4 \ln x \frac{\partial}{\partial v} - (bx^2 \ln x - bx^2) \frac{\partial}{\partial v}$, |

| $H(v) = a + bv, a$ and $b$ constants ($b \neq 0$), $G(u) = \alpha + \beta u, \alpha$ and $\beta$ constants ($\beta \neq 0$) |
| $X_1 = F(x) \frac{\partial}{\partial u}, X_2 = W(x) \frac{\partial}{\partial u}, X_3 = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}$, |
| $X_4 = \frac{\partial}{\partial v} - \beta u \frac{\partial}{\partial v}$ |

| $H(v) = a + bv, a$ and $b$ constants ($b \neq 0$), $G(u) = \beta u, \beta$ and $p$ constants ($\beta, p \neq 0$) |
| $X_1 = (bx + bpx) \frac{\partial}{\partial x} + 4bu \frac{\partial}{\partial u} + (2a + 2bv - 2ap - 2bpv) \frac{\partial}{\partial x}$ |

| $H(v) = a + bv, a$ and $b$ constants ($b \neq 0$), $G(u) = \beta e^{-kx}, \beta$ and $k$ constants ($\beta, k \neq 0$) |
| $X_1 = bx \frac{\partial}{\partial x} + 4b \frac{\partial}{\partial u} - (2ak + 2bk) \frac{\partial}{\partial x}$ |

| $H(v) = a + bv, a, m, and b$ constants ($b, m \neq 0$), $G(u) = c, c$ constant |
| $X_1 = \frac{\partial}{\partial u}, X_2 = \ln x \frac{\partial}{\partial u}, X_3 = (cx^2 + 4v) \frac{\partial}{\partial u}$, |
| $X_4 = 2x \frac{\partial}{\partial x} + (4u - amx^2 - 4mu) \frac{\partial}{\partial u} + 4v \frac{\partial}{\partial v}$ |
Table 3: Continued.

- \( H(v) = b v^m \) and \( b \) constants \( (b, m \neq 0) \), \( G(u) = \alpha + \beta u \) and \( \beta \) constants \( (\beta \neq 0) \)
  \[ X_1 = (b x + b m x) \frac{\partial}{\partial x} + (2(2a + 2b u + 2b m u)) \frac{\partial}{\partial u} + 4 b v \frac{\partial}{\partial v} \]
  \[ X_2 = (a m x^2 + 4 m u) \frac{\partial}{\partial u} - 4 \frac{\partial}{\partial v} \]

- \( H(v) = b v^m \) and \( b \) constants \( (b, m \neq 0) \), \( G(u) = \alpha + \beta u \) and \( \beta \) constants \( (\beta \neq 0) \)
  \[ X_1 = (b x + b m x) \frac{\partial}{\partial x} + (2(2a + 2b u + 2b m u)) \frac{\partial}{\partial u} + 4 b v \frac{\partial}{\partial v} \]

Therefore the composition of these transformations gives

\[ \overline{x} = e^{\theta}(x^2 + 2a_2)^{1/2}, \]
\[ \overline{u} = e^{5\theta}(x^2 + 2a_2)^{-1/2} (x^2 u + a_2 x^2 + a_2 x^2), \]
\[ \overline{v} = e^{5\theta}(x^2 + 2a_2)^{-1/2} (x^2 v + a_1 x^2 + a_2 x^2 + a_3 x^2), \]
\[ \overline{H} = e^{-\theta} (H - 8a_2), \]
\[ \overline{G} = e^{-\theta} (G - 8a_2). \]  

3. Principal Lie Algebra and Lie Group Classification

The generalized Lane-Emden system (2) admits a Lie point symmetry

\[ X = \xi (x, u, v) \frac{\partial}{\partial x} + \eta^1 (x, u, v) \frac{\partial}{\partial u} + \eta^2 (x, u, v) \frac{\partial}{\partial v} \]  

Consequently, we conclude that the principal Lie algebra of (2) is trivial and the classifying relations are

\[ (au + \beta) G'(u) + \gamma G(u) + \delta = 0, \]
\[ (\theta v + \lambda) H'(v) + \phi H(v) + \omega = 0, \]  

where \( \alpha, \beta, \gamma, \delta, \theta, \lambda, \psi, \) and \( \omega \) are constants.
Table 4: Lie symmetries for $n=3$, for various functions $H(v)$ and $G(u)$.

<table>
<thead>
<tr>
<th>$H(v)$ arbitrary, $G(u) = c, c$ a constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = \frac{\partial}{\partial u}, X_2 = x^\alpha \frac{\partial}{\partial v}, X_3 = (cx^2 + 8u) \frac{\partial}{\partial v}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$H(v) = d, d$ a constant, $G(u) = \alpha + \beta u, \alpha$ and $\beta$ constants ($\beta \neq 0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = \frac{\partial}{\partial v}, X_2 = x^\alpha \frac{\partial}{\partial v}, X_3 = (dx^2 + 8u) \frac{\partial}{\partial v}, X_4 = \beta \ln x \frac{\partial}{\partial u} - 2x^{-2} \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>$X_3 = bx^2 \frac{\partial}{\partial v} - 8 \frac{\partial}{\partial v}$,</td>
</tr>
<tr>
<td>$X_4 = 96x \frac{\partial}{\partial x} - 24dx^2 \frac{\partial}{\partial u} + (\beta dx^4 + 192u) \frac{\partial}{\partial v}$,</td>
</tr>
<tr>
<td>$X_5 = (192u + 4dx^2) \frac{\partial}{\partial u} + (192v - \beta dx^2 + 24ax^2) \frac{\partial}{\partial v}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$H(v) = d, d$ a constant, $G(u) = \alpha + \beta e^{-ku}, \alpha, k$ and $\beta$ constants ($\beta, k \neq 0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = \frac{\partial}{\partial v}, X_2 = x^\alpha \frac{\partial}{\partial v}, X_3 = (dx^2 + 8u) \frac{\partial}{\partial v}, X_4 = \beta kx \frac{\partial}{\partial u} - 2x^{-2} \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>$X_3 = bx^2 \frac{\partial}{\partial v} - 8 \frac{\partial}{\partial v}$,</td>
</tr>
<tr>
<td>$X_4 = 8x \frac{\partial}{\partial u} - (ax^2 + 8ku) \frac{\partial}{\partial v}$</td>
</tr>
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</table>

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<tr>
<th>$H(v) = d, d$ a constant, $G(u) = \alpha + \beta \ln u, \alpha$ and $\beta$ constants ($\beta \neq 0$)</th>
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</thead>
<tbody>
<tr>
<td>$X_1 = \frac{\partial}{\partial v}, X_2 = x^\alpha \frac{\partial}{\partial v}, X_3 = (dx^2 + 8u) \frac{\partial}{\partial v}, X_4 = 8x \frac{\partial}{\partial u} - (ax^2 + 8ku) \frac{\partial}{\partial v}$</td>
</tr>
<tr>
<td>$X_3 = x^\alpha \frac{\partial}{\partial u} - 2x^{-2} u \frac{\partial}{\partial u} - (2x^{-2} u - \beta \ln x) \frac{\partial}{\partial v}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$H(v) = a + bv, a$ and $b$ constants ($b \neq 0$), $G(u) = c, c$ constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = \frac{\partial}{\partial u}, X_2 = x^\alpha \frac{\partial}{\partial v}, X_3 = (cx^2 + 8u) \frac{\partial}{\partial v}, X_4 = b \ln x \frac{\partial}{\partial u} - 2x^{-2} \frac{\partial}{\partial v}$</td>
</tr>
<tr>
<td>$X_3 = bx^2 \frac{\partial}{\partial v} - 8 \frac{\partial}{\partial v}$,</td>
</tr>
<tr>
<td>$X_4 = 96x \frac{\partial}{\partial x} + (bcx^4 + 192u) \frac{\partial}{\partial v} - 24cx^2 \frac{\partial}{\partial v}$,</td>
</tr>
<tr>
<td>$X_5 = (24ax^2 + 192u - bcx^4) \frac{\partial}{\partial u} + (24ax^2 + 192u) \frac{\partial}{\partial v}$</td>
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<th>$H(v) = a + bv, a$ and $b$ constants ($b \neq 0$), $G(u) = \alpha + \beta u, \alpha$ and $\beta$ constants ($\beta \neq 0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = F(x) \frac{\partial}{\partial v}, X_2 = W(x) \frac{\partial}{\partial u}, X_3 = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}$</td>
</tr>
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<td>$X_4 = bu \frac{\partial}{\partial u} + \beta u \frac{\partial}{\partial v}$</td>
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<th>$H(v) = a + bv, a$ and $b$ constants ($b \neq 0$), $G(u) = \beta u^p, \beta$ and $p$ constants ($\beta, p \neq 0$)</th>
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<td>$X_1 = (bx + bpx) \frac{\partial}{\partial x} + 4bu \frac{\partial}{\partial u} + (2a + 2bv - 2ap - 2bpv) \frac{\partial}{\partial v}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$H(v) = a + bv, a$ and $b$ constants ($b \neq 0$) $G(u) = \beta e^{-ku}, \beta$ and $k$ constants ($\beta, k \neq 0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = bkx \frac{\partial}{\partial x} + 4b \frac{\partial}{\partial u} - (2ak + 2bkv) \frac{\partial}{\partial v}$</td>
</tr>
</tbody>
</table>
\[ H(v) = a + bv^m, \text{ and } b \text{ constants } (b, m \neq 0), G(u) = c, c \text{ a constant} \]
\[ X_1 = \frac{\partial}{\partial u}, X_2 = x^2 \frac{\partial}{\partial u}, X_3 = (cx^2 + 8u) \frac{\partial}{\partial u}, \]
\[ X_4 = 4x \frac{\partial}{\partial x} + (8u - amx^2 - 8mu) \frac{\partial}{\partial u} + 8v \frac{\partial}{\partial v} \]

\[ H(u) = bv^m, m \text{ and } b \text{ constants } (b, m \neq 0), G(u) = \alpha + \beta u, \alpha \text{ and } \beta \text{ constants } (\beta \neq 0) \]
\[ X_1 = (\beta + \beta mx) \frac{\partial}{\partial x} + (2\alpha - 2\alpha m + 2\beta u - 2\beta mu) \frac{\partial}{\partial u} + 4\beta v \frac{\partial}{\partial v} \]

\[ H(u) = bv^m, m \text{ and } b \text{ constants } (b, m \neq 0), G(u) = \alpha + \beta u, \alpha \text{ and } \beta \text{ constants } (\beta \neq 0) \]
\[ X_1 = \frac{\partial}{\partial x}, X_2 = x^2 \frac{\partial}{\partial u}, X_3 = (cx^2 + 8u) \frac{\partial}{\partial u}, X_4 = 4x \frac{\partial}{\partial x} + (8u - amx^2 - 8mu) \frac{\partial}{\partial u} + 8v \frac{\partial}{\partial v} \]

\[ H(u) = bv^m, m \text{ and } b \text{ constants } (b, m \neq 0), G(u) = \alpha + \beta u, \alpha \text{ and } \beta \text{ constants } (\beta \neq 0) \]
\[ X_1 = \frac{\partial}{\partial x}, X_2 = x^2 \frac{\partial}{\partial u}, X_3 = (cx^2 + 8u) \frac{\partial}{\partial u}, X_4 = 4x \frac{\partial}{\partial x} + (8u - amx^2 - 8mu) \frac{\partial}{\partial u} + 8v \frac{\partial}{\partial v} \]

\[ H(u) = a + \beta e^{mu}, a, m, \text{ and } b \text{ constants } (b, m \neq 0), G(u) = \alpha + \beta u, \alpha \text{ and } \beta \text{ constants } (\beta \neq 0) \]
\[ X_1 = \frac{\partial}{\partial x}, X_2 = x^2 \frac{\partial}{\partial u}, X_3 = (cx^2 + 8u) \frac{\partial}{\partial u}, X_4 = 4x \frac{\partial}{\partial x} + (8u - amx^2 - 8mu) \frac{\partial}{\partial u} + 8v \frac{\partial}{\partial v} \]

\[ H(u) = a + \beta e^{mu}, a, m, \text{ and } b \text{ constants } (b, m \neq 0), G(u) = \alpha + \beta u, \alpha \text{ and } \beta \text{ constants } (\beta \neq 0) \]
\[ X_1 = \frac{\partial}{\partial x}, X_2 = x^2 \frac{\partial}{\partial u}, X_3 = (cx^2 + 8u) \frac{\partial}{\partial u}, X_4 = 4x \frac{\partial}{\partial x} + (8u - amx^2 - 8mu) \frac{\partial}{\partial u} + 8v \frac{\partial}{\partial v} \]

\[ H(u) = a + \beta e^{mu}, a, m, \text{ and } b \text{ constants } (b, m \neq 0), G(u) = \alpha + \beta u, \alpha \text{ and } \beta \text{ constants } (\beta \neq 0) \]
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\[ H(u) = a + \beta e^{mu}, a, m, \text{ and } b \text{ constants } (b, m \neq 0), G(u) = \alpha + \beta u, \alpha \text{ and } \beta \text{ constants } (\beta \neq 0) \]
\[ X_1 = \frac{\partial}{\partial x}, X_2 = x^2 \frac{\partial}{\partial u}, X_3 = (cx^2 + 8u) \frac{\partial}{\partial u}, X_4 = 4x \frac{\partial}{\partial x} + (8u - amx^2 - 8mu) \frac{\partial}{\partial u} + 8v \frac{\partial}{\partial v} \]

These classifying relations are invariant under the equivalence transformation (7) if

\[ \bar{\alpha} = \alpha, \]
\[ \bar{\beta} = \beta \left( a_0 x^2 + a_0 x^{1-n} + a_4 \right) + \beta e^{-\alpha_1}, \]
\[ \bar{\gamma} = \gamma, \]
\[ \bar{\delta} = \delta e^{2n-\alpha_1} - 2\gamma (1 + n) a_2, \]
\[ \bar{\theta} = \theta, \]
\[ \bar{\lambda} = \theta \left( a_0 x^2 + a_0 x^{1-n} + a_4 \right) + \lambda e^{-\alpha_1}, \]
\[ \bar{\varphi} = \varphi, \]
\[ \bar{\omega} = \omega e^{2n-\alpha_1} - 2\varphi (1 + n) a_6. \]
It is also noted that the classifying relations (20) are invariant under the equivalence transformation (13) if
\[ \alpha = \alpha, \]
\[ \beta = e (a_1 \ln x + a_2 x^2 + a_3) + \beta e^{-\omega}, \]
\[ \gamma = \gamma, \]
\[ \delta = \delta e^{2a_1 x} - 4\gamma a_3, \]
\[ \theta = \theta, \]
\[ \lambda = \theta (a_3 \ln x + a_4 x^2 + a_5) + \lambda e^{a}, \]
\[ \phi = \phi, \]
\[ \omega = \omega e^{2a_1 x} - 4\phi a_3. \]

The classifying relations (20) are also invariant under the equivalence transformation (16) if
\[ \alpha = \alpha, \]
\[ \beta = e (a_1 x^2 + a_2 x^2 + a_3) + \beta (1 + 2a_2 x^2) e^{-a}, \]
\[ \gamma = \gamma, \]
\[ \delta = \delta e^{2a_1 x} - 8\gamma a_3, \]
\[ \theta = \theta, \]
\[ \lambda = \theta (a_4 x^2 + a_5 x^2 + a_6) + \lambda (1 + 2a_2 x^2) e^{-a}, \]
\[ \phi = \phi, \]
\[ \omega = \omega e^{2a_1 x} - 8\phi a_3. \]

The above relations are now used to find the nonequivalence forms of \( H \) and \( G \) and their corresponding Lie point symmetry. Several cases arise and are presented in Tables 1, 2, 3, and 4.

The Noether symmetries given in [6] from (25) to (44) always form a proper subalgebra of the Lie algebra that is obtained above. This can be seen from Tables 1, 2, 3, and 4. However, in [6] the first integrals were also presented.

4. Concluding Remarks
We have studied a generalized coupled Lane-Emden system from the algebraic viewpoint. A complete group classification of the underlying system was performed. We showed that the generalized coupled Lane-Emden system admits a nine- or ten-dimensional equivalence Lie algebra. The principal Lie algebra, which was found to be trivial, had several possible extensions. We deduced the results for all possible cases of the values of \( n \). There were in fact four cases that arose.

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