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Research Article

Dynamic Mean-Variance Model with Borrowing Constraint under the Constant Elasticity of Variance Process

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This paper studies a continuous-time dynamic mean-variance portfolio selection problem with the constraint of a higher borrowing rate, in which stock price is governed by a constant elasticity of variance (CEV) process. Firstly, we apply Lagrange duality theorem to change an original mean-variance problem into an equivalent optimization one. Secondly, we use dynamic programming principle to get the Hamilton-Jacobi-Bellman (HJB) equation for the value function, which is a more sophisticated nonlinear second-order partial differential equation. Furthermore, we use Legendre transform and dual theory to transform the HJB equation into its dual one. Finally, the closed-form solutions to the optimal investment strategy and efficient frontier are derived by applying variable change technique.

1. Introduction

The main purpose of this paper is to focus on the portfolio selection problem under the constant elasticity of variance (CEV) model. The CEV model was originally proposed by Cox and Ross [1] as an alternative diffusion process for European option pricing. It is a natural extension of the geometric Brownian motion (GBM). The advantage of the CEV model is that it can explain the empirical bias exhibited by Black and Scholes [2] model, such as volatility smile. Therefore, the CEV model was often applied to analyze the option pricing formulas, for example, Schroder [3], Phelim and Yisong [4], Davydov and Linetsky [5], and so forth. Recently, the CEV model has been introduced into annuity contracts to study the optimal investment strategy in the defined contribution and defined benefit pension plan (referring to Xiao et al. [6], Gao [7, 8]), but those models were all considered in the utility function framework. In the existing literatures, as far as our knowledge, the CEV model in the mean-variance framework has not been reported.

In most of the real-world situations, different interest rates for borrowing and lending are often faced by the investors. It is clear that portfolio selection models with borrowing constraint will make it more practical. This attracts the attention of many authors, referring to Paxson [9], Fleming and Zariphopoulou [10], Vila and Zariphopoulou [11], Teplá [12], and Zariphopoulou [13]. However, those models were usually dealt with under expect utility criterion, and the risky asset price was usually supposed to be driven by a GBM. In addition, the risk and return relationship is implicit in the utility function approach and cannot be disentangled at the different level of the optimal strategy. As a matter of fact, the optimal investment strategy under the utility maximizing criterion is not necessarily mean-variance efficient.

This paper introduces a CEV model and borrowing constraint into the classical portfolio selection problem in a continuous-time mean-variance framework. For the mean-variance portfolio selection problem, stochastic linear-quadratic (LQ) control technique is an effective method (e.g., Zhou and Li [14], Li et al. [15], and Chiu and Li [16], Xie et al. [17]). But borrowing constraint forces this problem to become piecewise linear-quadratic and is hence no longer a LQ control problem (see [18]). In addition, the introduction of the CEV model gives rise to some new difficulties, which are not easily dealt with in solving the associated HJB equation.

In this paper, we firstly apply Lagrange duality theorem to change an original mean-variance problem into an equivalent optimization one. Secondly, we use dynamic programming principle to get the Hamilton-Jacobi-Bellman (HJB) equation for the value function, which is a more sophisticated nonlinear second-order partial differential equation. Further, we use Legendre transform and dual theory to transform the HJB equation into its dual one. Finally, the closed-form solutions to the optimal investment strategy and efficient frontier are derived by applying variable change technique. There are several innovations in this paper: (i) stock price is supposed to follow the CEV model, which is a natural extension of geometric Brownian motion; (ii) we consider a dynamic mean-variance portfolio selection problem with borrowing constraints under a CEV model and assume that the borrowing rate is larger than the risk-free interest rate; (iii) the closed-form solutions to the optimal investment strategy and the efficient frontier are obtained.

The paper is organized as follows. In Section 2, we introduce a CEV model and describe portfolio selection problem with borrowing constraint in a mean-variance framework. In Section 3, the closed-form solution to optimal investment strategy is derived by applying Legendre transform and dual theory. Section 4 gives the main results on the optimal strategy and the efficient frontier. Section 5 concludes this paper.

2. Mean-Variance Model

In this paper, we consider a financial market where two assets are traded continuously over $[0,T]$. One asset is a bond with price P_t at time t, whose price process P_t with borrowing constraint can be expressed in the following ordinary differential equation (see Fu et al. [18]):

$$dP_t = \begin{cases} rP_t dt, & \text{if } P_t \geq 0,\ t \in [0,T] \\ RP_t dt, & \text{if } P_t < 0,\ t \in [0,T] \end{cases}, \quad P_0 = p_0 > 0, \tag{1}$$

where the constant $r > 0$ is the interest rate of the bond and R is the borrowing rate being larger than r.

Letting $x^- = -\min(x,0)$, then (1) can be rewritten as

$$dP_t = \left(rP_t - (R-r)P_t^-\right)dt, \quad P_0 = p_0 > 0. \tag{2}$$

The another asset is a stock with prices S_t at time t, whose price process S_t is supposed to follow the constant elasticity of variance (CEV) model:

$$dS_t = S_t\left[\mu dt + kS_t^\beta dW_t\right], \quad S_0 = s_0 > 0, \tag{3}$$

where μ ($\mu > R > r$) is the instantaneous return rate of the stock. k and β are constant parameters, the elasticity parameter β satisfies the general condition: $\beta \leq 0$. kS_t^β is defined as the instantaneous volatility of the stock, and W_t is a one-dimensional standard and adapted Brownian motion defined on the filtered complete probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$.

Remark 1. Note that there are four special interpretations for the elasticity parameter β:

(i) if $\beta = 0$, the CEV model is reduced to a geometric Brownian motion (GBM);

(ii) if $\beta = -1$, it is the Ornstein-Uhlenbeck process;

(iii) if $\beta = -1/2$, it is the model first presented by Cox and Ross [1] as an alternative diffusion process for valuation of options;

(iv) if $\beta < 0$, this means that the instantaneous volatility kS_t^β increases as the stock price decreases and can generate a distribution with a fatter left tail (referring to Gao [7]).

Suppose that short-selling of the stock is allowed and transaction cost and consumption are not considered. We denote by X_t the wealth of the investor at time $t \in [0,T]$ and by $N_i(t)$ the share of asset ith held by the investor at time t, $i = 1,2$. Let $\pi_t = N_2(t)S_t$ be the amount invested in the stock at time t, $t \in [0,T]$. Clearly, the amount invested in the bond is $\pi_t^0 = N_1(t)P_t = X_t - \pi_t$. The wealth process $X_t = N_1(t)P_t + N_2(t)S_t$ corresponding to trading strategy π_t is subject to the following stochastic differential equation:

$$\begin{aligned} dX_t &= N_1(t)\,dP_t + N_2(t)\,dS_t \\ &= N_1(t)\left(rP_t - (R-r)P_t^-\right)dt \\ &\quad + N_2(t)S_t\left(\mu dt + kS_t^\beta dW_t\right) \\ &= \left(rN_1(t)P_t - (R-r)N_1(t)P_t^-\right)dt \\ &\quad + \pi_t\left(\mu dt + kS_t^\beta dW_t\right). \end{aligned} \tag{4}$$

That is, we have

$$\begin{aligned} dX_t = &\left(rX_t + (\mu - r)\pi_t - (R-r)\left(X_t - \pi_t\right)^-\right)dt \\ &+ \pi_t kS_t^\beta dW_t, \quad X_0 = x_0 > 0. \end{aligned} \tag{5}$$

The investor's objective is to find an optimal portfolio π_t such that the expected terminal wealth satisfies $\mathbb{E}X_T = C$, for some constant $C \in \mathbb{R}$, while the risk measured by the variance of the terminal wealth

$$\operatorname{Var} X_T = \mathbb{E}\left[X_T - \mathbb{E}X_T\right]^2 = \mathbb{E}\left(X_T - C\right)^2 \tag{6}$$

is minimized. The problem of finding out such a portfolio π_t is referred to as the mean-variance portfolio choice problem.

In the modern portfolio selection theory, a portfolio π_t is said to be admissible if it is integrable and $\{\mathcal{F}_t\}_{t>0}$-adapted, and (5) has a unique solution corresponding to π_t. In this case, we refer to (X_t, π_t) as an admissible pair. Therefore, the mean-variance problem can be formulated as a linearly constrained stochastic optimization problem:

$$\begin{aligned} &\text{Minimize} \quad \operatorname{Var} X_T = \mathbb{E}\left(X_T - C\right)^2 \\ &\text{subject to} \quad \mathbb{E}X_T = C \\ &\left(X_t, \pi_t\right) \text{ satisfies } (5). \end{aligned} \tag{7}$$

Finally, an optimal investment strategy of the above problem is called an efficient portfolio corresponding to C, the corresponding $(C, \operatorname{Var} X_T)$ is called an efficient point, whereas the set of all the efficient points, when the parameter C runs over $[x_0 e^{rT}, +\infty)$, is called an efficient frontier.

Remark 2. If $\pi_t < 0$, this means that the investor is short-selling the stock. If $X_t - \pi_t < 0$, then the investor needs borrowing the money from the bank at interest rate R and the amount to borrow is $|X_t - \pi_t|$. Otherwise, we do not borrow the money to run the portfolio.

3. The Optimal Portfolio

To find out an optimal investment strategy for the problem (7) corresponding to the constraint $\mathbb{E}X_T = C$, we introduce a Lagrange multiplier $2\lambda \in R$ and arrive at a new objective function:

$$\begin{aligned}\widehat{J}(\pi_t, \lambda) &= \mathbb{E}\left[(X_T - C)^2 + 2\lambda (X_T - C)\right] \\ &= \mathbb{E}(X_T - (C - \lambda))^2 - \lambda^2.\end{aligned} \tag{8}$$

Letting $\gamma = C - \lambda$, we obtain the following stochastic control problem:

$$\begin{aligned}&\text{Minimize} \quad \overline{J}(\pi_t, \gamma) = \mathbb{E}(X_T - \gamma)^2 - (C - \gamma)^2 \\ &\text{subject to} \quad (X_t, \pi_t) \text{ satisfies } (5).\end{aligned} \tag{9}$$

The link between problem (7) and (9) is provided by Lagrange duality theorem (see Fu et al. [18] and Luenberger [19]):

$$\begin{aligned}\text{Minimize} \quad \operatorname{Var} X_T &= \operatorname*{Max}_{\lambda\in\mathbb{R}} \operatorname*{Min}_{\pi_t} \widehat{J}(\pi_t, \lambda) \\ &= \operatorname*{Max}_{\gamma\in\mathbb{R}} \operatorname*{Min}_{\pi_t} \overline{J}(\pi_t, \gamma).\end{aligned} \tag{10}$$

For a fixed constant γ, the problem (9) is clearly equivalent to

$$\begin{aligned}&\text{Minimize} \quad \mathbb{E}(X_T - \gamma)^2 \\ &\text{subject to} \quad (X_t, \pi_t) \text{ satisfies } (5).\end{aligned} \tag{11}$$

We define the value function $H(t, s, x)$ as

$$H(t, s, x) = \operatorname*{Min}_{\pi_t} \mathbb{E}\left((X_T - \gamma)^2 \mid S_t = s, X_t = x\right) \tag{12}$$

with boundary condition $H(T, s, x) = (x - \gamma)^2$.

According to dynamic programming principle, $H(t, s, x)$ can be taken as the continuous solution to the following HJB equation:

$$\begin{aligned}&H_t + \mu s H_s + \frac{1}{2}k^2 s^{2\beta+2} H_{ss} \\ &\quad + \operatorname*{Min}_{\pi_t} \left\{ \left[rx + (\mu - r)\pi_t - (R - r)(x - \pi_t)^-\right] H_x \right. \\ &\quad \left. + \frac{1}{2}\left(\pi_t k s^\beta\right)^2 H_{xx} + k^2 s^{2\beta+1} \pi_t H_{xs} \right\} = 0,\end{aligned} \tag{13}$$

where H_t, H_s, H_{ss}, H_x, H_{xx}, and H_{xs} denote first-order and second-order partial derivatives with respect to time t, stock price S_t, and wealth process X_t.

Let us firstly describe borrowing situation. Not borrowing and investing in the bond means that $x - \pi_t \geq 0$ and borrowing to invest in the stock means that $x - \pi_t < 0$. We define the nonborrowing region Θ in the (t, x)-plane to be

$$\Theta = \{(t, x) \in [0, T] \times \mathbb{R} \mid x - \pi_t \geq 0\}. \tag{14}$$

Taking borrowing situation into consideration, we rewrite the HJB equation (13) as

$$\begin{aligned}&H_t + \mu s H_s + \frac{1}{2}k^2 s^{2\beta+2} H_{ss} \\ &\quad + \operatorname*{Min}_{\pi_t} \left\{ \left[rx + (\mu - r)\pi_t\right] H_x + \frac{1}{2}\left(\pi_t k s^\beta\right)^2 H_{xx} \right. \\ &\quad \left. + k^2 s^{2\beta+1} \pi_t H_{xs} \right\} = 0, \quad \text{if } (t, x) \in \Theta, \\ &H_t + \mu s H_s + \frac{1}{2}k^2 s^{2\beta+2} H_{ss} \\ &\quad + \operatorname*{Min}_{\pi_t} \left\{ \left[Rx + (\mu - R)\pi_t\right] H_x + \frac{1}{2}\left(\pi_t k s^\beta\right)^2 H_{xx} \right. \\ &\quad \left. + k^2 s^{2\beta+1} \pi_t H_{xs} \right\} = 0, \quad \text{if } (t, x) \notin \Theta.\end{aligned} \tag{15}$$

The optimal value π_t^* of (15) is given by

$$\pi_t^* = \begin{cases} \dfrac{-(\mu - r) H_x - k^2 s^{2\beta+1} H_{xs}}{k^2 s^{2\beta} H_{xx}}, & \text{if } (t, x) \in \Theta, \\ \dfrac{-(\mu - R) H_x - k^2 s^{2\beta+1} H_{xs}}{k^2 s^{2\beta} H_{xx}}, & \text{if } (t, x) \notin \Theta. \end{cases} \tag{16}$$

Putting (16) into (15), we have

$$\begin{aligned}&H_t + \mu s H_s + \frac{1}{2}k^2 s^{2\beta+2} H_{ss} + rx H_x - \frac{1}{2k^2 s^{2\beta} H_{xx}} \\ &\quad \times \left[(\mu - r) H_x + k^2 s^{2\beta+1} H_{xs}\right]^2 = 0, \quad \text{if } (t, x) \in \Theta, \\ &H_t + \mu s H_s + \frac{1}{2}k^2 s^{2\beta+2} H_{ss} + Rx H_x - \frac{1}{2k^2 s^{2\beta} H_{xx}} \\ &\quad \times \left[(\mu - R) H_x + k^2 s^{2\beta+1} H_{xs}\right]^2 = 0, \quad \text{if } (t, x) \notin \Theta.\end{aligned} \tag{17}$$

Letting $\tau = r$, if $(t, x) \in \Theta$; $\tau = R$, if $(t, x) \notin \Theta$, we get

$$\begin{aligned}&H_t + \mu s H_s + \frac{1}{2}k^2 s^{2\beta+2} H_{ss} + \tau x H_x \\ &\quad - \frac{1}{2k^2 s^{2\beta} H_{xx}} \left[(\mu - \tau) H_x + k^2 s^{2\beta+1} H_{xs}\right]^2 = 0.\end{aligned} \tag{18}$$

According to the convexity of the value function, we can define a Legendre transform:

$$\widehat{H}(t, s, z) = \sup_{x>0} \{H(t, s, x) - zx\}, \tag{19}$$

where $z > 0$ denotes the dual variable to x.

The value of x which the maximum in the above equation will be attained at is denoted by $g(t,s,z)$, so we have

$$g(t,s,z) = \inf_{x>0}\left\{x \mid H(t,s,x) \geq zx + \widehat{H}(t,s,z)\right\}. \quad (20)$$

The functions $g(t,s,z)$ and $\widehat{H}(t,s,z)$ are closely related and we will refer to either one of them as the dual function of $H(t,s,x)$. In this paper, we will work mainly with the function $g(t,s,z)$.

The function $\widehat{H}(t,s,z)$ is related to $g(t,s,z)$ by $g(t,s,z) = -\widehat{H}_z(t,s,z)$.

Noting that $H(T,s,x) = (x-\gamma)^2$, we can define the following Legendre transform at terminal time:

$$\begin{aligned} \widehat{H}(T,s,z) &= \sup_{x>0}\left\{(x-\gamma)^2 - zx\right\}, \\ g(T,s,z) &= \inf_{x>0}\left\{(x \mid x-\gamma)^2 \geq zx + \widehat{H}(T,s,z)\right\}. \end{aligned} \quad (21)$$

So we have

$$g(T,s,z) = \frac{1}{2}z + \gamma. \quad (22)$$

According to (19), we have $H_x(t,s,x) = z$, and this leads to

$$g(t,s,z) = x, \qquad \widehat{H}(t,s,z) = H(t,s,g) - zg. \quad (23)$$

Referring to Jonsson and Sircar [20], Xiao et al. [6], and Gao [7], we get the following transformation rules:

$$\begin{aligned} &H_t = \widehat{H}_t, \qquad H_x = z, \qquad H_{xx} = -\frac{1}{\widehat{H}_{zz}}, \\ &H_s = \widehat{H}_s, \qquad H_{ss} = \widehat{H}_{ss} - \frac{\widehat{H}_{sz}^2}{\widehat{H}_{zz}}, \qquad H_{xs} = -\frac{\widehat{H}_{sz}}{\widehat{H}_{zz}}. \end{aligned} \quad (24)$$

Putting transformation rules (24) into (18), we get

$$\begin{aligned} &\widehat{H}_t + \mu s\widehat{H}_s + \frac{1}{2}k^2 s^{2\beta+2}\widehat{H}_{ss} + (\tau g)z \\ &\quad + \frac{(\mu-\tau)^2 z^2}{2k^2 s^{2\beta}}\widehat{H}_{zz} - (\mu-\tau)sz\widehat{H}_{sz} = 0. \end{aligned} \quad (25)$$

Differentiating $\widehat{H}$ with respect to z, we derive the following dual equation:

$$\begin{aligned} &g_t + \tau s g_s + \frac{1}{2}k^2 s^{2\beta+2} g_{ss} + \left(\frac{(\mu-\tau)^2}{k^2 s^{2\beta}} - \tau\right) z g_z \\ &\quad + \frac{(\mu-\tau)^2}{2k^2 s^{2\beta}} z^2 g_{zz} - (\mu-\tau)sz g_{sz} - \tau g = 0, \end{aligned} \quad (26)$$

where $\tau = r$, if $(t,x) \in \Theta$; $\tau = R$, if $(t,x) \notin \Theta$.

Taking (22) into consideration, we can fit a solution to (26) with the following structure:

$$g(t,s,z) = \begin{cases} f^r(t,y)z + h^r(t), \; y = s^{-2\beta}, & \text{if } (t,x) \in \Theta; \\ f^R(t,y)z + h^R(t), \; y = s^{-2\beta}, & \text{if } (t,x) \notin \Theta. \end{cases} \quad (27)$$

Considering $\tau = r$, if $(t,x) \in \Theta$; $\tau = R$, if $(t,x) \notin \Theta$, we can rewrite $g(t,s,z)$ as

$$g(t,s,z) = f^\tau(t,y)z + h^\tau(t), \qquad y = s^{-2\beta}, \quad (28)$$

with boundary conditions given by $f^\tau(T,y) = 1/2$ and $h^\tau(T) = \gamma$.

Further, we have

$$\begin{aligned} &g_t = f_t^\tau z + h_t^\tau, \qquad g_s = f_y^\tau(-2\beta)s^{-2\beta-1}z, \\ &g_{sz} = f_y^\tau(-2\beta)s^{-2\beta-1}, \\ &g_{ss} = f_{yy}^\tau\left((-2\beta)s^{-2\beta-1}\right)^2 \cdot z \\ &\quad + f_y^\tau(-2\beta)(-2\beta-1)s^{-2\beta-2}z, \\ &g_z = f^\tau, \qquad g_{zz} = 0. \end{aligned} \quad (29)$$

Putting the above partial derivatives into (26), we get

$$\begin{aligned} &\Big[f_t^\tau + \left(2\beta(\mu-2\tau)y + \beta(2\beta+1)k^2\right)f_y^\tau \\ &\quad + 2\beta^2 k^2 y f_{yy}^\tau + \left(\frac{(\mu-\tau)^2}{k^2}y - 2\tau\right)f^\tau\Big]z \\ &\quad + h_t^\tau - \tau h^\tau = 0. \end{aligned} \quad (30)$$

Eliminating the dependence on z, we obtain

$$h_t^\tau - \tau h^\tau = 0, \qquad h(T) = \gamma; \quad (31)$$

$$\begin{aligned} &f_t^\tau + \left(2\beta(\mu-2\tau)y + \beta(2\beta+1)k^2\right)f_y^\tau + 2\beta^2 k^2 y f_{yy}^\tau \\ &\quad + \left(\frac{(\mu-\tau)^2}{k^2}y - 2\tau\right)f^\tau = 0, \qquad f^\tau(T,y) = \frac{1}{2}. \end{aligned} \quad (32)$$

The solution to (31) is

$$h(t) = \gamma e^{-\tau(T-t)}. \quad (33)$$

Lemma 3. *Assume that the structure of the solution to* (32) *is* $f^\tau(t,y) = A^\tau(t)e^{B^\tau(t)y}$, *with the boundary conditions given by* $A^\tau(T) = 1/2$ *and* $B^\tau(T) = 0$; *then* $A^\tau(t)$ *and* $B^\tau(t)$ *are given by* (43) *and* (42), *respectively.*

Proof. Putting $f^\tau(t,y) = A^\tau(t)e^{B^\tau(t)y}$ into (32), we have

$$\begin{aligned} &\Big[A^\tau(t)\frac{dB^\tau(t)}{dt} + 2\beta(\mu-2\tau)A^\tau(t)B^\tau(t) \\ &\quad + 2\beta^2 k^2 A^\tau(t)B^{\tau 2}(t) + \frac{(\mu-\tau)^2}{k^2}A^\tau(t)\Big]y \\ &\quad + \frac{dA^\tau(t)}{dt} + \beta(2\beta+1)k^2 A^\tau(t)B^\tau(t) - 2\tau A^\tau(t) = 0. \end{aligned} \quad (34)$$

By matching the coefficients, we get

$$\frac{dA^{\tau}(t)}{dt}+\beta(2\beta+1)k^2A^{\tau}(t)B^{\tau}(t)-2\tau A^{\tau}(t)=0,\qquad A^{\tau}(T)=\frac{1}{2},\tag{35}$$

$$\frac{dB^{\tau}(t)}{dt}+2\beta(\mu-2\tau)B^{\tau}(t)+2\beta^2k^2B^{\tau 2}(t)+\frac{(\mu-\tau)^2}{k^2}=0,\qquad B^{\tau}(T)=0.\tag{36}$$

Equation (36) can be reduced to

$$\frac{dB^{\tau}(t)}{dt}=-2\beta^2k^2B^{\tau 2}(t)-2\beta(\mu-2\tau)B^{\tau}(t)-\frac{(\mu-\tau)^2}{k^2},\qquad B^{\tau}(T)=0.\tag{37}$$

Let Δ denote the discriminant of the quadratic equation $-2\beta^2k^2B^{\tau 2}(t)-2\beta(\mu-2\tau)B^{\tau}(t)-((\mu-\tau)^2/k^2)=0$.

Easy calculation leads to $\Delta=4\beta^2(2\tau^2-\mu^2)$. Assume that $\Delta>0$, that is, $-\sqrt{2}\tau<\mu<\sqrt{2}\tau$, then the quadratic equation has two real roots:

$$m_1^{\tau}=\frac{-(\mu-2\tau)+\sqrt{2\tau^2-\mu^2}}{2\beta k^2},\qquad m_2^{\tau}=\frac{-(\mu-2\tau)-\sqrt{2\tau^2-\mu^2}}{2\beta k^2}.\tag{38}$$

So (37) can be rewritten as

$$\frac{1}{m_1^{\tau}-m_2^{\tau}}\int_t^T\left(\frac{1}{B^{\tau}(t)-m_1^{\tau}}-\frac{1}{B^{\tau}(t)-m_2^{\tau}}\right)dB^{\tau}(t)=-2\beta^2k^2(T-t).\tag{39}$$

Further, we obtain

$$B^{\tau}(t)=\frac{m_1^{\tau}m_2^{\tau}\left(1-e^{-2\beta^2k^2(m_1^{\tau}-m_2^{\tau})(T-t)}\right)}{m_1^{\tau}-m_2^{\tau}e^{-2\beta^2k^2(m_1^{\tau}-m_2^{\tau})(T-t)}}.\tag{40}$$

Letting

$$\lambda_1^{\tau}=\frac{-(\mu-2\tau)+\sqrt{2\tau^2-\mu^2}}{2\beta},\qquad \lambda_2^{\tau}=\frac{-(\mu-2\tau)-\sqrt{2\tau^2-\mu^2}}{2\beta}.\tag{41}$$

We get

$$B^{\tau}(t)=k^{-2}I^{\tau}(t),\qquad I^{\tau}(t)=\frac{\lambda_1^{\tau}\lambda_2^{\tau}\left(1-e^{-2\beta^2(\lambda_1^{\tau}-\lambda_2^{\tau})(T-t)}\right)}{\lambda_1^{\tau}-\lambda_2^{\tau}e^{-2\beta^2(\lambda_1^{\tau}-\lambda_2^{\tau})(T-t)}}.\tag{42}$$

Plugging (42) into (35) yields

$$A^{\tau}(t)=\frac{1}{2}e^{-\int_t^T(2\tau-\beta(2\beta+1)I^{\tau}(t))dt}.\tag{43}$$

Therefore, Lemma 3 is completed. □

Finally, summarizing the above results, we obtain the optimal trading strategy for the problem (11).

Theorem 4. *For a given λ, T and $C\geqslant x_0e^{rT}$, the optimal investment strategy with borrowing constraint under a mean-variance criterion corresponding to the problem* (11) *is*

$$\pi_t^*=\begin{cases}-\dfrac{(\mu-r)}{k^2S_t^{2\beta}}\left(X_t-\gamma e^{-r(T-t)}\right)\widetilde{K}^r(t),\\ \quad \text{if } X_t\geqslant\gamma\rho(t),\quad r<\mu<\sqrt{2}r,\\ -\dfrac{(\mu-R)}{k^2S_t^{2\beta}}\left(X_t-\gamma e^{-R(T-t)}\right)\widetilde{K}^R(t),\\ \quad \text{if } X_t<\gamma\rho(t),\quad R<\mu<\sqrt{2}R,\end{cases}\tag{44}$$

where $K^{\tau}(t)$ and $\rho(t)$ are given by (48) *and* (50)*, respectively.*

Proof. Under the transformation rules, the optimal strategy (16) is derived as follows:

$$\begin{aligned}\pi_t^*&=\frac{-(\mu-\tau)H_x-k^2s^{2\beta+1}H_{xs}}{k^2s^{2\beta}H_{xx}}\\&=\frac{-(\mu-\tau)H_x/H_{xx}-k^2s^{2\beta+1}H_{xs}/H_{xx}}{k^2s^{2\beta}}\\&=\frac{(\mu-\tau)z\widehat{H}_{zz}-k^2s^{2\beta+1}\widehat{H}_{sz}}{k^2s^{2\beta}}\\&=\frac{-(\mu-\tau)zg_z+k^2s^{2\beta+1}g_s}{k^2s^{2\beta}}.\end{aligned}\tag{45}$$

Taking (33),(35), and Lemma 3 into consideration, we have

$$\begin{aligned}\pi_t^*&=\frac{-(\mu-\tau)zg_z+k^2s^{2\beta+1}g_s}{k^2s^{2\beta}}\\&=\frac{-(\mu-\tau)zf^{\tau}+k^2s^{2\beta+1}zf_y^{\tau}(-2\beta)s^{-2\beta-1}}{k^2s^{2\beta}}\\&=\frac{-(\mu-\tau)(g-h^{\tau}(t))+k^2zA^{\tau}(t)B^{\tau}(t)e^{B^{\tau}(t)y}(-2\beta)}{k^2s^{2\beta}}\\&=\frac{-(\mu-\tau)(x-h^{\tau}(t))+k^2B^{\tau}(t)(-2\beta)(x-h^{\tau}(t))}{k^2s^{2\beta}}\\&=\frac{-(\mu-\tau)(x-h^{\tau}(t))+I^{\tau}(t)(-2\beta)(x-h^{\tau}(t))}{k^2s^{2\beta}}\\&=-\frac{(\mu-\tau)+2\beta I^{\tau}(t)}{k^2s^{2\beta}}(x-h^{\tau}(t)).\end{aligned}\tag{46}$$

Therefore, the optimal strategy is reduced to

$$\pi_t^* = \begin{cases} -\dfrac{(\mu - r)}{k^2 S_t^{2\beta}} \left(X_t - \gamma e^{-r(T-t)}\right) \widetilde{K}^r(t), \\ \quad \text{if } (t,x) \in \Theta,\ r < \mu < \sqrt{2}r, \\ -\dfrac{(\mu - R)}{k^2 S_t^{2\beta}} \left(X_t - \gamma e^{-R(T-t)}\right) \widetilde{K}^R(t), \\ \quad \text{if } (t,x) \notin \Theta,\ \ R < \mu < \sqrt{2}R, \end{cases} \tag{47}$$

where

$$\widetilde{K}^\tau(t) = \left[1 + \frac{2\beta I^\tau(t)}{\mu - \tau}\right], \quad \tau = r, \text{ if } (t,x) \in \Theta, \qquad \tau = R, \text{ if } (t,x) \notin \Theta. \tag{48}$$

As the boundary condition of borrowing the money from the bank is $X_t - \pi_t^* = 0$; that is,

$$X_t + \frac{(\mu - R)}{k^2 S_t^{2\beta}} \left(X_t - \gamma e^{-R(T-t)}\right) \widetilde{K}^R(t) = 0. \tag{49}$$

Denoting by $\gamma\rho(t)$ the borrowing curve, we yield

$$\rho(t) = \frac{\left((\mu - R)/k^2 S_t^{2\beta}\right) e^{-R(T-t)} \widetilde{K}^R(t)}{1 + \left((\mu - R)/k^2 S_t^{2\beta}\right) \widetilde{K}^R(t)}. \tag{50}$$

Therefore, nonborrowing region Θ in the (t,x)-plane can be rewritten as

$$\Theta = \left\{(t,x) \in [0,T] \times \mathbb{R} \mid X_t \geqslant \gamma\rho(t)\right\}. \tag{51}$$

Hence, the proof of the Theorem 4 is completed. □

Remark 5. We can draw some conclusions from (44).

(i) If $X_t \geqslant \gamma\rho(t)$ and $r < \mu < \sqrt{2}r$, the investor need not borrowing the money from the bank and the optimal amount invested in the stock can be calculated by the first equation of (44), while the amount invested in the bond is $X_t - \pi_t^*$.

(ii) If $X_t < \gamma\rho(t)$ and $R < \mu < \sqrt{2}R$, the optimal amount invested in the stock is given by the second equation of (44). In addition, investors need to borrow the money to invest the stock and the amount to borrow is $|X_t - \pi_t^*|$, while the bond need not be invested.

4. The Efficient Frontier

In this section, we apply Lagrange duality theorem to derive the efficient frontier for the mean-variance portfolio selection problem (7). To simplify the presentation, we denote by τ either the interest rate r or the borrowing rate R, and letting

$$\theta^\tau = \frac{\mu - \tau}{k S_t^\beta}, \tag{52}$$

where $\tau = r$, if $X_t \geqslant \gamma\rho(t)$ and $\tau = R$, if $X_t < \gamma\rho(t)$.

In both cases above, the wealth equation (5) is reduced to

$$dX_t = \left(\tau X_t + (\mu - \tau)\pi_t\right) dt + \pi_t k S_t^\beta dW_t, \quad X_0 = x_0 > 0. \tag{53}$$

For any fixed γ, under the efficient strategy in the Theorem 4, the dynamics of the wealth equation (5) are

$$\begin{aligned} dX_t = {} & \left(\left(\tau - (\theta^\tau)^2 \widetilde{K}^\tau(t)\right) X_t + (\theta^\tau)^2 \gamma e^{-\tau(T-t)} \widetilde{K}^\tau(t)\right) dt \\ & - \theta^\tau \left(X_t - \gamma e^{-\tau(T-t)}\right) \widetilde{K}^\tau(t)\, dW_t, \quad X_0 = x_0 > 0. \end{aligned} \tag{54}$$

Applying *Itô's* lemma to the wealth process (54), we yield

$$\begin{aligned} dX_t^2 = {} & \Big(\left[2\tau + (\theta^\tau)^2 \left(\widetilde{K}^{2\tau}(t) - 2\widetilde{K}^\tau(t)\right)\right] X_t^2 \\ & - (\theta^\tau)^2 2\gamma e^{-\tau(T-t)} \left(\widetilde{K}^{2\tau}(t) - \widetilde{K}^\tau(t)\right) X_t \\ & + (\theta^\tau)^2 \left(\gamma e^{-\tau(T-t)} \widetilde{K}^\tau(t)\right)^2 \Big) dt \\ & - \theta^\tau \left(X_t - \gamma e^{-\tau(T-t)}\right) \widetilde{K}^\tau(t)\, 2X_t dW_t, \quad X_0^2 = x_0^2 > 0. \end{aligned} \tag{55}$$

Taking expectations on both sides of (54) and (55), respectively, one has

$$d\mathbb{E}X_t = \left(\left(\tau - (\theta^\tau)^2 \widetilde{K}^\tau(t)\right) \mathbb{E}X_t + (\theta^\tau)^2 \gamma e^{-\tau(T-t)} \widetilde{K}^\tau(t)\right) dt, \qquad \mathbb{E}X_0 = x_0 > 0, \tag{56}$$

$$\begin{aligned} d\mathbb{E}X_t^2 = {} & \Big(\left[2\tau + (\theta^\tau)^2 \left(\widetilde{K}^{2\tau}(t) - 2\widetilde{K}^\tau(t)\right)\right] \mathbb{E}X_t^2 \\ & - (\theta^\tau)^2 2\gamma e^{-\tau(T-t)} \left(\widetilde{K}^{2\tau}(t) - \widetilde{K}^\tau(t)\right) \mathbb{E}X_t \\ & + (\theta^\tau)^2 \left(\gamma e^{-\tau(T-t)} \widetilde{K}^\tau(t)\right)^2 \Big) dt, \quad \mathbb{E}X_0^2 = x_0^2 > 0. \end{aligned} \tag{57}$$

The solution of the linear ordinary differential equation (56) is

$$\mathbb{E}X_t = x_0 e^{\int_0^t (\tau - (\theta^\tau)^2 \widetilde{K}^\tau(t))dt} + \gamma e^{-\tau(T-t)} \left[1 - e^{-\int_0^t (\theta^\tau)^2 \widetilde{K}^\tau(t)dt}\right], \tag{58}$$

and it results in

$$\mathbb{E}X_T = x_0 e^{\int_0^T (\tau - (\theta^\tau)^2 \widetilde{K}^\tau(t))dt} + \gamma \left[1 - e^{-\int_0^T (\theta^\tau)^2 \widetilde{K}^\tau(t)dt}\right]. \tag{59}$$

Similarly, by solving (57), one has

$$\begin{aligned} \mathbb{E}X_T^2 = {} & e^{\int_0^T (\theta^\tau)^2 (\widetilde{K}^{2\tau}(t) - 2\widetilde{K}^\tau(t))dt} \left(x_0 e^{\tau T} - \gamma\right)^2 \\ & + 2\gamma e^{-\int_0^T (\theta^\tau)^2 \widetilde{K}^\tau(t)dt} \left(x_0 e^{\tau T} - \gamma\right) + \gamma^2. \end{aligned} \tag{60}$$

Therefore, the objective function of the problem (9), as a explicit function of parameter γ, is given by

$$\begin{aligned}\overline{J}_{\min}(\pi_t^*,\gamma) &= \mathbb{E}(X_T-\gamma)^2-(C-\gamma)^2\\ &= \mathbb{E}X_T^2-2\gamma\mathbb{E}X_T+2\gamma C-C^2\\ &= \gamma^2\left(e^{\int_0^T(\theta^\tau)^2(\widetilde{K}^{2\tau}(t)-2\widetilde{K}^\tau(t))dt}-1\right)\\ &\quad+2\gamma\left(C-x_0e^{\tau T}e^{\int_0^T(\theta^\tau)^2(\widetilde{K}^{2\tau}(t)-2\widetilde{K}^\tau(t))dt}\right)\\ &\quad+x_0^2e^{2\tau T}e^{\int_0^T(\theta^\tau)^2(\widetilde{K}^{2\tau}(t)-2\widetilde{K}^\tau(t))dt}-C^2.\end{aligned} \tag{61}$$

Using (10) obtained by Lagrange duality theorem, the minimum variance $\operatorname{Var}X_T$ is achieved for

$$\gamma_\tau^*=\frac{C-x_0e^{\tau T}e^{\int_0^T(\theta^\tau)^2(\widetilde{K}^{2\tau}(t)-2\widetilde{K}^\tau(t))dt}}{1-e^{\int_0^T(\theta^\tau)^2(\widetilde{K}^{2\tau}(t)-2\widetilde{K}^\tau(t))dt}}. \tag{62}$$

In addition, we obtain

$$\overline{J}_{\max\min}(\pi_t^*,\gamma_\tau^*)=\frac{\left(C-x_0e^{\tau T}\right)^2}{e^{\int_0^T(\theta^\tau)^2(\widetilde{K}^{2\tau}(t)-2\widetilde{K}^\tau(t))dt}-1}. \tag{63}$$

Letting $\operatorname{Var}^\tau X_T=\overline{J}_{\max\min}(\pi_t^*,\gamma_\tau^*)$. The optimal value of γ and the minimum variance $\operatorname{Var}X_T$ are

$$\begin{aligned}\operatorname{Var}X_T &= \operatorname{Max}\left\{\operatorname{Var}^rX_T,\operatorname{Var}^RX_T\right\}\\ &= \operatorname{Max}\left\{\frac{\left(C-x_0e^{rT}\right)^2}{e^{\int_0^T(\theta^r)^2(\widetilde{K}^{2r}(t)-2\widetilde{K}^r(t))dt}-1},\right.\\ &\qquad\left.\frac{\left(C-x_0e^{RT}\right)^2}{e^{\int_0^T(\theta^R)^2(\widetilde{K}^{2R}(t)-2\widetilde{K}^R(t))dt}-1}\right\},\end{aligned} \tag{64}$$

$$\gamma^*=\begin{cases}\gamma_r^*, & \text{if } \operatorname{Var}^rX_T=\operatorname{Var}X_T,\\ \gamma_R^*, & \text{if } \operatorname{Var}^RX_T=\operatorname{Var}X_T.\end{cases} \tag{65}$$

Putting (65) into the optimal strategy (44) in Theorem 4, we can summarize our main results in the following theorem.

Theorem 6. *The optimal investment strategy for the mean-variance portfolio selection problem* (7) *with borrowing constraint under a CEV process is*

$$\pi_t^*=\begin{cases}-\dfrac{(\mu-r)}{k^2S_t^{2\beta}}\left(X_t-\gamma^*e^{-r(T-t)}\right)\widetilde{K}^r(t),\\ \quad\text{if } X_t\geqslant\gamma^*\rho(t),\quad r<\mu<\sqrt{2}r,\\ -\dfrac{(\mu-R)}{k^2S_t^{2\beta}}\left(X_t-\gamma^*e^{-R(T-t)}\right)\widetilde{K}^R(t),\\ \quad\text{if } X_t<\gamma^*\rho(t),\quad R<\mu<\sqrt{2}R.\end{cases} \tag{66}$$

Moreover, the efficient frontier is given by

$$\begin{aligned}\operatorname{Var}X_T &= \operatorname{Max}\left\{\frac{\left(C-x_0e^{rT}\right)^2}{e^{\int_0^T(\theta^r)^2(\widetilde{K}^{2r}(t)-2\widetilde{K}^r(t))dt}-1},\right.\\ &\qquad\left.\frac{\left(C-x_0e^{RT}\right)^2}{e^{\int_0^T(\theta^R)^2(\widetilde{K}^{2R}(t)-2\widetilde{K}^R(t))dt}-1}\right\},\end{aligned} \tag{67}$$

where $K^\tau(t)$ and $\rho(t)$ are given by (48) *and* (50)*, respectively.*

Remark 7. When $\beta=0$, the results in the Theorem 6 are reduced to the ones under a geometric Brownian motion model, which is obtained by [18]. When $\beta=-1/2$ and $\beta=-1$, the corresponding results are all given by (66) and (67). Therefore, extending a geometric Brownian motion to a CEV model is the most important innovation in our paper.

5. Conclusions

This paper is concerned with a continuous-time dynamic portfolio selection problem in a mean-variance framework, in which the constraint of the borrowing rate higher than the lending rate is allowed and stock price process is supposed to follow the constant elasticity of variance (CEV) model. The closed-form solution to the optimal investment strategy is derived by applying Legendre transform and dual theory. In addition, the efficient strategy and efficient frontier are derived by using Lagrange duality theorem.

In future research, we will continue to concentrate on continuous-time portfolio selection problems under a CEV model. It would be interesting to extend our model to those with more sophisticated cases, such as introducing consumption and transaction cost, short-selling constraint, and liability process. We leave these problems and corresponding verification theorem for future research.

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