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Research Article

Upper and Lower Solution Method for Fractional Boundary Value Problems on the Half-Line

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We establish the existence of unbounded solutions for nonlinear fractional boundary value problems on the half-line. By the upper and lower solution method technique, sufficient conditions for the existence of solutions for the fractional boundary value problems are established. An example is presented to illustrate our main result.

1. Introduction

Boundary value problems on the half-line arise in the study of radially symmetric solutions of nonlinear elliptic equations and various physical phenomena, such as the theory of drain flows and plasma physics; see [1–10] and the references therein. In 2006, Lian and Ge in [11] investigated the following boundary value problem on the half-line for the second-order differential equation:

$$\begin{gathered} x''(t) + f\left(t, x(t), x'(t)\right) = 0, \quad t \in (0, +\infty), \\ x(0) = \alpha x(\eta), \qquad \lim_{t\to\infty} x'(t) = 0, \end{gathered} \tag{1}$$

where $\alpha \in \mathbb{R}$, $\alpha \neq 1$, and $\eta \in (0, \infty)$ are given. Based on Leray-Schauder continuation theorem, some suitable conditions for the existence of solutions to (1) are established.

On the other hand, fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary noninteger order. Fractional calculus is a wonderful technique to understand memory and hereditary properties of materials and processes. Some recent contributions to fractional differential equations are present in the monographs [12–19]. Very recently, Chen and Tang in [20] considered the following fractional differential boundary value problem on the half-line:

$$\begin{gathered} D_{0_+}^{\alpha} u(t) = f(t, u(t)), \quad t \in [0, \infty), \\ u(0) = u'(0) = u''(0) = 0, \qquad D_{0_+}^{\alpha-1} u(t) = \lim_{t\to\infty} D_{0_+}^{\alpha-1} u(t), \end{gathered} \tag{2}$$

where $3 < \alpha < 4$ and $D_{0_+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative. By the recent Leggett-Williams norm-type theorem, the existence of positive solutions is obtained. In 2011, [21] set up the global existence results of solutions of initial value problems on the half-axis as follows:

$$\begin{gathered} D_{0_+}^{\alpha} x(t) = f(t, x(t)), \quad t \in (0, \infty), \ 0 < \alpha \leq 1, \\ \lim_{t\to\infty} t^{1-\alpha} x(t) = u_0, \end{gathered} \tag{3}$$

where $D_{0_+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative. By constructing a special Banach space and employing fixed point theorems, some sufficient conditions for the existence of solutions are obtained. In [22], the authors studied the following boundary value problem of fractional order on the half-line:

$$\begin{gathered} D_{0_+}^{\alpha} u(t) + a(t) f\left(t, u(t), D_{0_+}^{\alpha-1} u(t)\right) = 0, \quad t \in [0, \infty), \\ u(0) = 0, \qquad \lim_{t\to\infty} D_{0_+}^{\alpha-1} u(t) = u_{\infty}, \end{gathered} \tag{4}$$

where $1 < \alpha \leq 2$, $f \in C([0,\infty) \times \mathbb{R}^2, \mathbb{R})$, $D_{0_+}^{\alpha-1}$ and $D_{0_+}^{\alpha}$ are the standard Riemann-Liouville fractional derivatives. By Schauder's fixed point theorem on an unbounded domain, they obtain the existence result for (4). Some papers have recently been done for fractional boundary value problem on the half-line or unbounded domain, see [22–31].

Inspired by the above-mentioned works, in this paper, we study the existence of solutions to the following fractional differential equations with boundary value problems:

$$\begin{gathered} D_{0_+}^{\alpha} u(t) + a(t) f\left(t, u(t), D_{0_+}^{\alpha-2} u(t), D_{0_+}^{\alpha-1} u(t)\right) = 0, \\ t \in (0,\infty), \quad 2 < \alpha \leq 3, \\ u(0) = D_{0_+}^{\alpha-2} u(0) = 0, \qquad \lim_{t\to\infty} D_{0_+}^{\alpha-1} u(t) = 0, \end{gathered} \tag{5}$$

where $a(t) : [0,\infty) \to [0,\infty)$, and $D_{0_+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative. By upper and lower solution method techniques, the sufficient conditions for solutions to (5) are obtained. Our main findings given in this paper have some new features. Firstly, the like Nagumo condition defined by us plays an important role in the nonlinear term involving the standard Riemann-Liouville derivatives. Secondly, to the best of our knowledge, no work has been done concerning fractional boundary value problems (5) and our method is different from that of [22, 26, 31]. Thirdly, the nonlinear term f may take negative values, and $f(t,x,y,z)$ depends on z allowed to be quadratic, referring to our example. The rest of this paper is organized as follows: in Section 2, we present some preliminaries and some lemmas that will be used in Section 3. The main result and proof will be given in Section 3. In addition, an example is given to demonstrate the application of our main result.

2. Preliminaries

We first present some basic definitions and preliminary results about fractional calculus; we refer the reader to [17, 18] for more details.

Definition 1 (see [18]). The integral

$$I_{0_+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \tag{6}$$

where $\alpha > 0$, is called the Riemann-Liouville fractional integral of order α and $\Gamma(\alpha)$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$, $\alpha > 0$.

Definition 2 (see [17]). A function $f(x)$ given in the interval $[0,\infty)$, the expression

$$D_{0_+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds, \tag{7}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, is called the Riemann-Liouville fractional derivative of order α.

Lemma 3 (see [17]). *Let $\alpha > 0$ and $u(t) \in C(0,\infty) \bigcap L(0,\infty)$. Then the fractional differential equation*

$$D_{0_+}^{\alpha} u(t) = 0 \tag{8}$$

has

$$\begin{gathered} u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad c_i \in R, \\ i = 0, 1, 2, 3, \ldots, n, \quad n = [\alpha] + 1 \end{gathered} \tag{9}$$

as a unique solution.

Lemma 4 (see [17]). *Assume that $u(t) \in C(0,\infty) \bigcap L(0,\infty)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,\infty) \bigcap L(0,\infty)$; then*

$$I_{0_+}^{\alpha} D_{0_+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{10}$$

for some $c_i \in R$, $i = 0, 1, 2, 3, \ldots, n$, $n = [\alpha] + 1$.

Denote by $C[0,\infty)$ the space of all continuous real functions defined on $[0,\infty)$ and $L_{\text{loc}}(0,\infty)$ the space of all real functions which are Lebesgue integrable on every bounded subinterval of $(0,\infty)$. Denote

$$\begin{gathered} E = \left\{ x \in C[0,\infty) : \sup_{0\leq t<\infty} \frac{|x(t)|}{(1+t)^{\alpha-1}} < \infty, \right. \\ \left. \sup_{0\leq t<\infty} \left|D_{0_+}^{\alpha-2} x(t)\right| < \infty, \lim_{t\to\infty} D_{0_+}^{\alpha-1} x(t) \text{ exists} \right\} \end{gathered} \tag{11}$$

with the norm $\|x\| = \max\{\|x\|_1, \|D_{0_+}^{\alpha-2} x\|_\infty, \|D_{0_+}^{\alpha-1} x\|_\infty\}$, where

$$\begin{gathered} \|x\|_1 = \sup_{t\in[0,\infty)} \left|\frac{x(t)}{(1+t)^{\alpha-1}}\right|, \\ \left\|D_{0_+}^{\alpha-2} x\right\|_\infty = \sup_{t\in[0,\infty)} \left|D_{0_+}^{\alpha-2} x(t)\right|, \\ \left\|D_{0_+}^{\alpha-1} x\right\|_\infty = \sup_{t\in[0,\infty)} \left|D_{0_+}^{\alpha-1} x(t)\right|. \end{gathered} \tag{12}$$

By standard arguments, it is easy to prove that $(E, \|\cdot\|)$ is a Banach space.

Proposition 5 (see [32]). *Assume that f is in $C(0,\infty) \bigcap L_{\text{loc}}(0,\infty)$ with a fractional derivative of order $0 < \alpha < 1$ that belongs to $C[0,\infty) \bigcap L_{\text{loc}}[0,\infty)$. Then*

$$I^{\alpha} D^{\alpha} f(x) = f(x) + c x^{\alpha-1}, \tag{13}$$

for some $c \in \mathbb{R}$. When the function f is in $C[0,\infty)$, then $c = 0$.

Definition 6. A function $\gamma \in E \bigcap L_{\text{loc}}(0,\infty)$ is called a lower solution of (5) if

$$\begin{gathered} D_{0_+}^{\alpha} \gamma(t) + a(t) f\left(t, \gamma(t), D_{0_+}^{\alpha-2} \gamma(t), D_{0_+}^{\alpha-1} \gamma(t)\right) \geq 0, \\ t \in (0,\infty), \\ \gamma(0) \leq 0, \quad D_{0_+}^{\alpha-2} \gamma(0) \leq 0, \qquad \lim_{t\to\infty} D_{0_+}^{\alpha-1} \gamma(t) \leq 0. \end{gathered} \tag{14}$$

Similarly, we can define an upper solution $\beta \in E \bigcap L_{\text{loc}}(0,\infty)$ of (5) by reversing the above inequalities.

Remark 7. If

$$D_{0_+}^{\alpha-2}\gamma(t) \le D_{0_+}^{\alpha-2}\beta(t), \quad t \in [0,\infty), \tag{15}$$

and $\gamma(0) \le 0$, $\beta(0) \ge 0$, then we have $\gamma(t) \le \beta(t)$ for all $t \in [0,\infty)$.

We consider the following two cases.

Case 1. Consider $\alpha = 3$. The inequality (15) reduces to $\gamma'(t) \le \beta'(t)$. Then

$$\int_0^t \gamma'(s)\,ds \le \int_0^t \beta'(s)\,ds, \quad t \in (0,\infty); \tag{16}$$

that is,

$$\gamma(t) - \gamma(0) \le \beta(t) - \beta(0), \quad t \in (0,\infty). \tag{17}$$

From the boundary condition $\gamma(0) \le 0$ and $\beta(0) \ge 0$ we obtain $\gamma(0) \le \beta(0)$, and

$$\gamma(t) \le \beta(t) + \gamma(0) - \beta(0) \le \beta(t), \quad t \in (0,\infty). \tag{18}$$

Thus, $\gamma(t) \le \beta(t)$ for $t \in [0,\infty)$.

Case 2. Consider $2 < \alpha < 3$. If (15) holds, then

$$I_{0_+}^{\alpha-2}D_{0_+}^{\alpha-2}\gamma(t) \le I_{0_+}^{\alpha-2}D_{0_+}^{\alpha-2}\beta(t), \quad t \in [0,\infty). \tag{19}$$

Noting that $0 < \alpha-2 < 1$ and $\gamma, \beta \in C[0,\infty)$, from the above inequality and Proposition 5, we have that $\gamma(t) \le \beta(t)$ for $t \in [0,\infty)$.

For example, consider $\gamma(t) = t^{\alpha-2}$, $\beta(t) = 2t^{\alpha-2}$. Obviously, $D_{0_+}^{\alpha-2}\gamma(t) = \Gamma(\alpha+1)$, $D_{0_+}^{\alpha-2}\beta(t) = 2\Gamma(\alpha+1)$. Thus $\gamma(0) = \beta(0) = 0$ and

$$D_{0_+}^{\alpha-2}\gamma(t) \le D_{0_+}^{\alpha-2}\beta(t), \quad t \in [0,\infty). \tag{20}$$

We also get $\gamma(t) \le \beta(t)$ for each $t \in [0,\infty)$.

Definition 8. Let $\gamma, \beta \in E \bigcap L_{\mathrm{loc}}(0,\infty)$ be lower and upper solutions to (5) and suppose that (15) holds. A continuous function $f : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is said to satisfy the like Nagumo condition with respect to the pair of functions γ, β, if there exist a nonnegative function $\phi \in C[0,\infty)$ and a positive one $h \in C[0,\infty)$ such that

$$\left|f(t,x,y,z)\right| \le \phi(t)\,h(|z|), \tag{21}$$

for all $0 \le t < \infty$, $\gamma(t) \le x \le \beta(t)$, $D_{0_+}^{\alpha-2}\gamma(t) \le y \le D_{0_+}^{\alpha-2}\beta(t)$, $z \in \mathbb{R}$, and

$$\int_0^\infty \frac{s}{h(s)}\,ds = \infty. \tag{22}$$

We list some assumptions related to $a(t), \phi(t)$, and f as follows.

(H_1) Consider

$$\begin{gathered}\int_0^\infty \max\{s,1\}\,a(s)\,ds < \infty,\\ \int_0^\infty \max\{s,1\}\,a(s)\,\phi(s)\,ds < \infty.\end{gathered} \tag{23}$$

(H_2) Let $f : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function satisfying the like Nagumo condition with respect to the pair of functions γ, β, such that

$$\begin{gathered} f(t,\gamma(t),y,z) \le f(t,x,y,z) \le f(t,\beta(t),y,z),\\ \text{for } (t,x,y,z) \in [0,\infty) \times [\gamma(t),\beta(t)] \times \mathbb{R}^2. \end{gathered} \tag{24}$$

Lemma 9. *Let $\sigma \in C(0,\infty) \bigcap L(0,\infty)$, then the fractional boundary value problem*

$$\begin{gathered} D_{0_+}^{\alpha}u(t) + \sigma(t) = 0, \quad t \in (0,\infty),\ 2 < \alpha \le 3,\\ u(0) = D_{0_+}^{\alpha-2}u(0) = 0, \qquad \lim_{t\to\infty} D_{0_+}^{\alpha-1}u(t) = 0 \end{gathered} \tag{25}$$

has a unique solution $u(t) = \int_0^\infty G(t,s)\sigma(s)ds$, where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t < \infty,\\ t^{\alpha-1}, & 0 \le t \le s < \infty. \end{cases} \tag{26}$$

Proof. By Lemma 4, the solution of (25) has the following form:

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sigma(s)\,ds\\ &+ c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3}. \end{aligned} \tag{27}$$

By boundary value condition $u(0) = 0$, one has $c_3 = 0$. Thus, we have by (27) that

$$\begin{aligned} D_{0_+}^{\alpha-2}u(t) &= -D_{0_+}^{\alpha-2}I_{0_+}^{\alpha}\sigma(t) + c_1D_{0_+}^{\alpha-2}t^{\alpha-1} + c_2D_{0_+}^{\alpha-2}t^{\alpha-2}\\ &= -\int_0^t (t-s)\,\sigma(s)\,ds + c_1\Gamma(\alpha)\,t + c_2\Gamma(\alpha-1). \end{aligned} \tag{28}$$

Substituting the condition $D_{0_+}^{\alpha-2}u(0) = 0$ into (28), we obtain $c_2 = 0$. Together with $\lim_{t\to\infty}D_{0_+}^{\alpha-1}u(t) = 0$, we get

$$c_1 = \frac{1}{\Gamma(\alpha)}\int_0^\infty \sigma(s)\,ds. \tag{29}$$

Hence, we have

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sigma(s)\,ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty \sigma(s)\,ds\\ &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sigma(s)\,ds\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t t^{\alpha-1}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_t^\infty t^{\alpha-1}\sigma(s)\,ds\\ &= \frac{1}{\Gamma(\alpha)}\int_0^t \left[t^{\alpha-1} - (t-s)^{\alpha-1}\right]\sigma(s)\,ds\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_t^\infty t^{\alpha-1}\sigma(s)\,ds\\ &= \int_0^\infty G(t,s)\,\sigma(s)\,ds. \end{aligned} \tag{30}$$

□

Lemma 10. *The function $G(t,s)$ defined by (26) satisfies the following properties:*

(1) *$G(t,s)$ is a continuous function and $G(t,s) \geq 0$ for $[0,\infty)\times[0,\infty)$;*

(2) *$G(t,s) \leq t^{\alpha-1}/\Gamma(\alpha)$, for $(t,s)\in[0,\infty)\times[0,\infty)$.*

The proof is easy, so we omit it here.

Set $C_l := \{y \in C[0,\infty) : \lim_{t\to\infty} y(t) \text{exists}\}$. For $y \in C_l$, define $\|y\| := \sup_{t\in[0,\infty)}|y(t)|$. Then C_l is a Banach space.

Lemma 11 (see [33]). *Let $M \subset C_l$. Then M is relatively compact if the following conditions hold:*

(a) *M is bounded in C_l;*

(b) *the functions belonging to M are locally equicontinuous on $[0,\infty)$;*

(c) *the functions from M are equiconvergent; that is given $\epsilon > 0$, there corresponds $T(\epsilon) > 0$ such that $|x(t) - x(\infty)| < \epsilon$, and $x \in M$.*

By Lemma 11, similar to the proof of Theorem 2.3 in [34], we can easily have the following lemma.

Lemma 12. *Let $M \subset E$. Then M is relatively compact if the following conditions hold:*

(1) *M is bounded in E;*

(2) *the functions belonging to $\{y : y = x/(1+t)^{\alpha-1}, x \in M\}$, $\{z : z = D_{0_+}^{\alpha-2}x(t), x \in M\}$, and $\{w : w = D_{0_+}^{\alpha-1}x(t),\ x \in M\}$, are locally equicontinuous on $[0,\infty)$;*

(3) *the functions from $\{y : y = x/(1+t)^{\alpha-1}, x \in M\}$, $\{z : z = D_{0_+}^{\alpha-2}x(t), x \in M\}$, and $\{w : w = D_{0_+}^{\alpha-1}x(t), x \in M\}$ are equiconvergent at ∞.*

Define the auxiliary functions

$$w(t,x) = \begin{cases} \gamma(t), & x < \gamma(t), \\ x(t), & \gamma(t) \leq x \leq \beta(t), \\ \beta(t), & x > \beta(t), \end{cases}$$
$$w_1(t,y) = \begin{cases} D_{0_+}^{\alpha-2}\gamma(t), & y < D_{0_+}^{\alpha-2}\gamma(t), \\ y(t), & D_{0_+}^{\alpha-2}\gamma(t) \leq y \leq D_{0_+}^{\alpha-2}\beta(t), \\ D_{0_+}^{\alpha-2}\beta(t), & y > D_{0_+}^{\alpha-2}\beta(t). \end{cases} \tag{31}$$

Consider the following boundary value problem:

$$\begin{gathered} D_{0_+}^{\alpha}u(t) + a(t) f^*\left(t, u(t), D_{0_+}^{\alpha-2}u(t), D_{0_+}^{\alpha-1}u(t)\right) = 0, \\ t \in (0,\infty), \quad 2 < \alpha \leq 3, \\ u(0) = D_{0_+}^{\alpha-2}u(0) = 0, \qquad \lim_{t\to\infty} D_{0_+}^{\alpha-1}u(t) = 0, \end{gathered} \tag{32}$$

where

$$f^*(t,x,y,z) = f(t, w_0(t,x), w_1(t,y), z) + \frac{w_1(t,y) - y}{1 + |w_1(t,y) - y|}. \tag{33}$$

For each $u \in E$, it follows from (21), (23), and (33) that

$$\begin{aligned} &\left|\int_0^\infty a(s) f^*\left(s, u(s), D_{0_+}^{\alpha-2}u(s), D_{0_+}^{\alpha-1}u(s)\right) ds\right| \\ &\quad \leq \int_0^\infty a(s)\left[\phi(s) h\left(\left|D_{0_+}^{\alpha-1}u(s)\right|\right) + 1\right] ds \\ &\quad \leq \int_0^\infty a(s)\left(H_0\phi(s) + 1\right) ds \\ &\quad \leq \int_0^\infty \max\{s,1\} a(s)\left(H_0\phi(s) + 1\right) ds < \infty, \end{aligned} \tag{34}$$

where $H_0 = \max_{0\leq t\leq \|D_{0_+}^{\alpha-1}u\|_\infty} h(t)$. From (34) and Lemma 9, we know that u is a solution of (32) if and only if u solves the operator equation $Tu = u$, and T is defined by

$$\begin{aligned} &Tu(t) \\ &\quad = \int_0^\infty G(t,s) a(s) f^*\left(s, u(s), D_{0_+}^{\alpha-2}u(s), D_{0_+}^{\alpha-1}u(s)\right) ds, \\ &\qquad u \in E, \quad t \in [0,\infty). \end{aligned} \tag{35}$$

Lemma 13. *Suppose that (H_1)-(H_2) hold; then $T : E \to E$ (the operator defined in (35)) is completely continuous.*

Proof. Consider the following.

Step 1. $T : E \to E$ is well defined. For $u \in E$, it follows from (34) that

$$\begin{aligned} &\int_0^\infty sa(s)\left(H_0\phi(s) + 1\right) ds \\ &\quad \leq \int_0^\infty \max\{s,1\} a(s)\left(H_0\phi(s) + 1\right) ds < \infty, \end{aligned} \tag{36}$$

which implies

$$\lim_{t\to\infty} ta(t)\left(H_0\phi(t) + 1\right) = 0. \tag{37}$$

We also have

$$\int_t^\infty a(s)\left(H_0\phi(s) + 1\right) ds \leq \int_t^\infty sa(s)\left(H_0\phi(s) + 1\right) ds, \quad t \geq 1. \tag{38}$$

Combining (36) with (38), one has

$$\lim_{t\to\infty} \int_t^\infty a(s)\left(H_0\phi(s) + 1\right) ds = 0. \tag{39}$$

If we apply Lebesgue dominated convergence theorem with (37) and (39), then

$$\begin{aligned} &\lim_{t\to\infty} \frac{|Tu(t)|}{(1+t)^{\alpha-1}} \\ &\quad \leq \lim_{t\to\infty} \int_0^\infty \frac{G(t,s)}{(1+t)^{\alpha-1}} a(s)\left(H_0\phi(s) + 1\right) ds = 0, \end{aligned} \tag{40}$$

which yields

$$\lim_{t\to\infty}\frac{Tu(t)}{(1+t)^{\alpha-1}}=0; \tag{41}$$

that is,

$$\sup_{0\le t<\infty}\frac{|Tu(t)|}{(1+t)^{\alpha-1}}<\infty. \tag{42}$$

By virtue of (34), we have

$$\begin{aligned}
&\sup_{0\le t<\infty}\left|D_{0^+}^{\alpha-2}Tu(t)\right| \\
&\quad=\sup_{0\le t<\infty}\left|D_{0^+}^{\alpha-2}\int_t^\infty G(t,s)\,a(s)\,f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),\right.\right. \\
&\qquad\qquad\left.\left.D_{0^+}^{\alpha-1}u(s)\right)ds\right| \\
&\quad=\sup_{0\le t<\infty}\left|\int_0^t sa(s)\,f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),D_{0^+}^{\alpha-1}u(s)\right)ds\right. \\
&\qquad+\int_t^\infty ta(s)\,f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),\right. \\
&\qquad\qquad\left.\left.D_{0^+}^{\alpha-1}u(s)\right)ds\right| \\
&\quad\le\sup_{0\le t<\infty}\int_0^t sa(s)\left(H_0\phi(s)+1\right)ds \\
&\qquad+\int_t^\infty ta(s)\left(H_0\phi(s)+1\right)ds \\
&\quad\le\int_0^\infty sa(s)\left(H_0\phi(s)+1\right)ds \\
&\quad\le\int_0^\infty \max\{s,1\}\,a(s)\left(H_0\phi(s)+1\right)ds<\infty.
\end{aligned} \tag{43}$$

It follows from (39) that

$$\begin{aligned}
&\left|\int_t^\infty a(s)\,f^*\left(s,u(s),D_{0_+}^{\alpha-2}u(s),D_{0_+}^{\alpha-1}u(s)\right)ds\right| \\
&\quad\le\int_t^\infty a(s)\left(H_0\phi(s)+1\right)ds\longrightarrow 0,\quad t\longrightarrow\infty.
\end{aligned} \tag{44}$$

Therefore, we have

$$\begin{aligned}
&\lim_{t\to\infty}D_{0_+}^{\alpha-1}Tu(t) \\
&\quad=\lim_{t\to\infty}\int_t^\infty a(s)\,f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),D_{0^+}^{\alpha-1}u(s)\right)ds \\
&\quad=0.
\end{aligned} \tag{45}$$

Thus, we conclude that $Tu\in E$.

Step 2. $T:E\to E$ is continuous. For any convergent sequence $u_n\to u$ in E, we find

$$\begin{aligned}
&u_n(t)\longrightarrow u(t),\qquad D_{0_+}^{\alpha-2}u_n\longrightarrow D_{0_+}^{\alpha-2}u, \\
&D_{0_+}^{\alpha-1}u_n\longrightarrow D_{0_+}^{\alpha-1}u,\quad \text{as } n\longrightarrow\infty,\ t\in[0,\infty).
\end{aligned} \tag{46}$$

From continuity of f^*, we obtain

$$\begin{aligned}
&\left|f^*\left(s,u_n(s),D_{0^+}^{\alpha-2}u_n(s),D_{0^+}^{\alpha-1}u_n(s)\right)\right. \\
&\qquad\left.-f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),D_{0^+}^{\alpha-1}u(s)\right)\right|\longrightarrow 0, \\
&\qquad\qquad n\longrightarrow\infty,\quad \text{for any } t\in[0,\infty).
\end{aligned} \tag{47}$$

Since $u_n\to u$, we have $\sup_{n\in N}\|D_{0_+}^{\alpha-1}u\|_\infty<\infty$. Set

$$H_p=\max_{0\le t\le\max\left\{\|D_{0_+}^{\alpha-1}u\|_\infty,\sup_{n\in N}\|D_{0_+}^{\alpha-1}u\|_\infty\right\}}h(t), \tag{48}$$

which follows that

$$\begin{aligned}
&\int_0^\infty sa(s)\left|f^*\left(s,u_n(s),D_{0^+}^{\alpha-2}u_n(s),D_{0^+}^{\alpha-1}u_n(s)\right)\right. \\
&\qquad\left.-f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),D_{0^+}^{\alpha-1}u(s)\right)\right|ds \\
&\quad\le 2\int_0^\infty sa(s)\left(H_p\phi(s)+1\right)ds<\infty.
\end{aligned} \tag{49}$$

Thus, applying the Lebesgue dominated convergence theorem and then using (49), we have

$$\begin{aligned}
&\|Tu_n-Tu\|_1 \\
&\quad=\sup_{0\le t<\infty}\frac{\left|Tu_n(t)-Tu(t)\right|}{(1+t)^{\alpha-1}} \\
&\quad=\sup_{0\le t<\infty}\left|\int_0^\infty\frac{G(t,s)}{(1+t)^{\alpha-1}}a(s)\right. \\
&\qquad\times\left(f^*\left(s,u_n(s),D_{0^+}^{\alpha-2}u_n(s),D_{0^+}^{\alpha-1}u_n(s)\right)\right. \\
&\qquad\qquad\left.\left.-f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),D_{0^+}^{\alpha-1}u(s)\right)\right)\right| \\
&\quad\le\frac{1}{\Gamma(\alpha)}\int_0^\infty a(s)\left|f^*\left(s,u_n(s),D_{0^+}^{\alpha-2}u_n(s),D_{0^+}^{\alpha-1}u_n(s)\right)\right. \\
&\qquad\qquad\left.-f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),D_{0^+}^{\alpha-1}u(s)\right)\right| \\
&\quad\longrightarrow 0,
\end{aligned} \tag{50}$$

as $n \to \infty$. On the other hand, we have

$$\begin{aligned}
&\left\|D_{0_+}^{\alpha-2}Tu_n - D_{0_+}^{\alpha-2}Tu\right\|_\infty \\
&= \sup_{0\le t<\infty}\left|D_{0_+}^{\alpha-2}Tu_n(t) - D_{0_+}^{\alpha-2}Tu(t)\right| \\
&= \sup_{0\le t<\infty}\left|\int_0^t sa(s) \times \left(f^*\left(s,u_n(s),D_{0^+}^{\alpha-2}u_n(s),D_{0^+}^{\alpha-1}u_n(s)\right) - f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),D_{0^+}^{\alpha-1}u(s)\right)\right)ds \right. \\
&\qquad \left. + \int_t^\infty ta(s) \times \left(f^*\left(s,u_n(s),D_{0^+}^{\alpha-2}u_n(s),D_{0^+}^{\alpha-1}u_n(s)\right) - f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),D_{0^+}^{\alpha-1}u(s)\right)\right)ds\right| \\
&\le \sup_{0\le t<\infty}\left[\int_0^t sa(s) \times \left|f^*\left(s,u_n(s),D_{0^+}^{\alpha-2}u_n(s),D_{0^+}^{\alpha-1}u_n(s)\right)ds - f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),D_{0^+}^{\alpha-1}u(s)\right)\right|ds \right. \\
&\qquad \left. + \int_t^\infty sa(s) \times \left|f^*\left(s,u_n(s),D_{0^+}^{\alpha-2}u_n(s),D_{0^+}^{\alpha-1}u_n(s)\right) - f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),D_{0^+}^{\alpha-1}u(s)\right)\right|ds\right] \\
&= \int_0^\infty sa(s) \times \left|f^*\left(s,u_n(s),D_{0^+}^{\alpha-2}u_n(s),D_{0^+}^{\alpha-1}u_n(s)\right) - f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),D_{0^+}^{\alpha-1}u(s)\right)\right|ds \longrightarrow 0, \quad n\longrightarrow\infty.
\end{aligned} \tag{51}$$

From (49), it is clear that

$$\int_0^\infty a(s)\left|f^*\left(s,u_n(s),D_{0^+}^{\alpha-2}u_n(s),D_{0^+}^{\alpha-1}u_n(s)\right) - f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),D_{0^+}^{\alpha-1}u(s)\right)\right| < +\infty. \tag{52}$$

Moreover, by (52), we have

$$\begin{aligned}
&\left\|D_{0_+}^{\alpha-1}Tu_n - D_{0_+}^{\alpha-1}Tu\right\|_\infty \\
&= \sup_{0\le t<\infty}\left|D_{0_+}^{\alpha-1}Tu_n(t) - D_{0_+}^{\alpha-1}Tu(t)\right| \\
&= \sup_{0\le t<\infty}\left|\int_t^\infty a(s)\left(f^*\left(s,u_n(s),D_{0^+}^{\alpha-2}u_n(s),D_{0^+}^{\alpha-1}u_n(s)\right) - f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),D_{0^+}^{\alpha-1}u(s)\right)\right)\right| \\
&\le \int_0^\infty a(s)\left|f^*\left(s,u_n(s),D_{0^+}^{\alpha-2}u_n(s),D_{0^+}^{\alpha-1}u_n(s)\right) - f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),D_{0^+}^{\alpha-1}u(s)\right)\right| \longrightarrow 0, \quad n\longrightarrow\infty.
\end{aligned} \tag{53}$$

Therefore, combining (50), (51), and (53), we get $\|Tu_n - Tu\| \to 0$, as $n \to \infty$; then we claim that $T : E \to E$ is continuous.

Step 3. $T : E \to E$ is compact. Let A be any bounded subset of E; then, for $u \in A$, set

$$H_q = \sup_{0\le t\le\|D_{0_+}^{\alpha-1}\|_\infty, u\in A} h(t) < \infty, \tag{54}$$

in a similar manner as (50), (51), and (53); we have by (21) and (23) that

$$\begin{aligned}
&\|Tu\|_1 \\
&= \sup_{0\le t<\infty}\frac{|Tu(t)|}{(1+t)^{\alpha-1}} \\
&\le \sup_{0\le t<\infty}\int_0^\infty \frac{G(t,s)}{(1+t)^{\alpha-1}}a(s) \times \left|f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),D_{0^+}^{\alpha-1}u(s)\right)\right|ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^\infty a(s) \times \left|f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),D_{0^+}^{\alpha-1}u(s)\right)\right|ds \\
&< \infty,
\end{aligned}$$

$$\begin{aligned}
\left\|D_{0_+}^{\alpha-2}Tu\right\|_\infty &\le \int_0^\infty sa(s) \times \left|f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),D_{0^+}^{\alpha-1}u(s)\right)\right|ds \\
&\le \int_0^\infty sa(s)\left(H_q\phi(s)+1\right)ds < \infty,
\end{aligned}$$

$$\begin{aligned}
&\left\|D_{0_+}^{\alpha-1}Tu\right\|_\infty \\
&\quad\le \int_0^\infty a(s)\left|f^*\left(s,u(s),D_{0^+}^{\alpha-2}u(s),D_{0^+}^{\alpha-1}u(s)\right)\right|ds \\
&\quad\le \int_0^\infty a(s)\left(H_q\phi(s)+1\right)ds<\infty,
\end{aligned} \tag{55}$$

which implies that $\|Tu\| < \infty$. Hence, TA is uniformly bounded. Meanwhile, for any $B > 0$, for $t_1, t_2 \in [0, B]$, we have

$$\begin{aligned}
&\left|\frac{Tu(t_1)}{(1+t_1)^{\alpha-1}}-\frac{Tu(t_2)}{(1+t_2)^{\alpha-1}}\right| \\
&\quad=\left|\int_0^\infty\left(\frac{G(t_1,s)}{(1+t_1)^{\alpha-1}}-\frac{G(t_2,s)}{(1+t_2)^{\alpha-1}}\right)\right. \\
&\qquad\left.\times a(s)f^*\left(s,u(s),D_{0_+}^{\alpha-2}u(s),D_{0_+}^{\alpha-1}u(s)\right)\right| \\
&\quad\le\int_0^\infty\left|\frac{G(t_1,s)}{(1+t_1)^{\alpha-1}}-\frac{G(t_2,s)}{(1+t_2)^{\alpha-1}}\right| \\
&\qquad\times a(s)\left(H_q\phi(s)+1\right)ds\longrightarrow 0,\quad t_1\longrightarrow t_2.
\end{aligned} \tag{56}$$

We also have

$$\begin{aligned}
&\left|D_{0_+}^{\alpha-2}Tu(t_1)-D_{0_+}^{\alpha-2}Tu(t_2)\right| \\
&\quad=\left|\int_0^{t_1}sa(s)f^*\left(s,u(s),D_{0_+}^{\alpha-2}u(s),D_{0_+}^{\alpha-1}u(s)\right)ds\right. \\
&\qquad+\int_{t_1}^\infty t_1a(s)f^*\left(s,u(s),D_{0_+}^{\alpha-2}u(s),D_{0_+}^{\alpha-1}u(s)\right)ds \\
&\qquad-\int_0^{t_2}sa(s)f^*\left(s,u(s),D_{0_+}^{\alpha-2}u(s),D_{0_+}^{\alpha-1}u(s)\right)ds \\
&\qquad\left.-\int_{t_2}^\infty t_2a(s)f^*\left(s,u(s),D_{0_+}^{\alpha-2}u(s),D_{0_+}^{\alpha-1}u(s)\right)ds\right| \\
&\quad\le\left|\int_{t_1}^{t_2}sa(s)f^*\left(s,u(s),D_{0_+}^{\alpha-2}u(s),D_{0_+}^{\alpha-1}u(s)\right)ds\right| \\
&\qquad+\left|t_1\int_{t_1}^{t_2}a(s)f^*\left(s,u(s),D_{0_+}^{\alpha-2}u(s),D_{0_+}^{\alpha-1}u(s)\right)ds\right| \\
&\qquad+\left|(t_2-t_1)\int_{t_2}^\infty a(s)f^*\left(s,u(s),D_{0_+}^{\alpha-2}u(s),\right.\right. \\
&\qquad\qquad\left.\left.D_{0_+}^{\alpha-1}u(s)\right)ds\right| \\
&\quad\le\int_{t_1}^{t_2}sa(s)\left(H_q\phi(s+1)\right)ds \\
&\qquad+t_1\int_{t_1}^{t_2}a(s)\left(H_q\phi(s+1)\right)ds \\
&\qquad+|t_2-t_1|\int_{t_2}^\infty a(s)\left(H_q\phi(s+1)\right)ds,
\end{aligned} \tag{57}$$

which approaches 0, as $t_1 \to t_2$. Furthermore, we have

$$\begin{aligned}
&\left|D_{0_+}^{\alpha-1}Tu(t_1)-D_{0_+}^{\alpha-1}Tu(t_2)\right| \\
&\quad=\left|\int_{t_1}^{t_2}a(s)f^*\left(s,u(s),D_{0_+}^{\alpha-2}u(s),D_{0_+}^{\alpha-1}u(s)\right)ds\right| \\
&\quad\le\int_{t_1}^{t_2}a(s)\left(H_q\phi(s)+1\right)ds,
\end{aligned} \tag{58}$$

which approaches 0, as $t_1 \to t_2$. Hence, we get that TA is equicontinuous. From (41), we have

$$\begin{aligned}
&\left|\frac{Tu(t)}{(1+t)^{\alpha-1}}-\lim_{t\to\infty}\frac{Tu(t)}{(1+t)^{\alpha-1}}\right| \\
&\qquad=\left|\frac{Tu(t)}{(1+t)^{\alpha-1}}\right|\longrightarrow 0,\quad \text{as } t\longrightarrow\infty.
\end{aligned} \tag{59}$$

Since

$$\begin{aligned}
&\left|\int_t^\infty ta(s)f^*\left(s,u(s),D_{0_+}^{\alpha-2}u(s),D_{0_+}^{\alpha-1}u(s)\right)\right| \\
&\quad\le\int_t^\infty ta(s)\left(H_0\phi(s)+1\right)ds \\
&\quad\le\int_t^\infty sa(s)\left(H_0\phi(s)+1\right)ds\longrightarrow 0,\quad t\longrightarrow\infty, \\
&\lim_{t\to\infty}D_{0_+}^{\alpha-2}Tu(t) \\
&\quad=\lim_{t\to\infty}\left[\int_0^t sa(s)f^*\left(s,u(s),D_{0_+}^{\alpha-2}u(s),D_{0_+}^{\alpha-1}u(s)\right)ds\right. \\
&\qquad\left.+\int_t^\infty ta(s)f^*\left(s,u(s),D_{0_+}^{\alpha-2}u(s),\right.\right. \\
&\qquad\qquad\left.\left.D_{0_+}^{\alpha-1}u(s)\right)ds\right] \\
&\quad=\int_0^\infty sa(s)f^*\left(s,u(s),D_{0_+}^{\alpha-2}u(s),D_{0_+}^{\alpha-1}u(s)\right)ds.
\end{aligned} \tag{60}$$

Thus, we have

$$\begin{aligned}
&\left|D_{0_+}^{\alpha-2}Tu(t)-\lim_{t\to\infty}D_{0_+}^{\alpha-2}Tu(t)\right| \\
&\quad=\left|\int_t^\infty ta(s)f^*\left(s,u(s),D_{0_+}^{\alpha-2}u(s),D_{0_+}^{\alpha-1}u(s)\right)ds\right. \\
&\qquad\left.-\int_t^\infty sa(s)f^*\left(s,u(s),D_{0_+}^{\alpha-2}u(s),D_{0_+}^{\alpha-1}u(s)\right)ds\right|,
\end{aligned} \tag{61}$$

which approaches 0, as $t \to \infty$. We also have by (45) that

$$\begin{aligned}&\left|D_{0_+}^{\alpha-1}Tu(t)-\lim_{t\to\infty}D_{0_+}^{\alpha-1}Tu(t)\right|\\&=\left|D_{0_+}^{\alpha-1}Tu(t)\right|\\&=\int_t^{\infty}a(s)f^*\left(s,u(s),D_{0_+}^{\alpha-1}u(s),D_{0_+}^{\alpha-2}u(s)\right)ds,\end{aligned}\tag{62}$$

which approaches 0 as $t \to \infty$. Thus, TA is equiconvergent at ∞. Then TA is relatively compact. Therefore, $T : E \to E$ is completely continuous. The proof is complete. □

3. Main Result

We are in the position to state the main existence result.

Theorem 14. *Let $\gamma, \beta \in E \bigcap L_{\text{loc}}(0,\infty)$ be lower and upper solutions to (5), and suppose that (15) holds. Moreover, (H_1)-(H_2) hold. Then fractional boundary value problem (5) has at least one solution $u(t) \in E \bigcap L_{\text{loc}}(0,\infty)$ satisfying*

$$\begin{gathered}\gamma(t)\le u(t)\le\beta(t),\\D_{0_+}^{\alpha-2}\gamma(t)\le D_{0_+}^{\alpha-2}u(t)\le D_{0_+}^{\alpha-2}\beta(t),\\\left|D_{0_+}^{\alpha-1}u(t)\right|\le N,\quad\forall t\in[0,\infty),\end{gathered}\tag{63}$$

where N is a constant dependent only on γ, β, and ϕ.

Proof. By Lemma 13, we know that $T : E \to E$ is completely continuous. By the Schauder fixed point theorem, we can easily obtain that T has at least one fixed point $u \in E$. Thus, u is a solution of (32). Next, we will show that u satisfies the inequalities

$$\begin{gathered}\gamma(t)\le u(t)\le\beta(t),\\D_{0_+}^{\alpha-2}\gamma(t)\le D_{0_+}^{\alpha-2}u(t)\le D_{0_+}^{\alpha-2}\beta(t),\quad t\in[0,\infty),\end{gathered}\tag{64}$$

which implies that u is a solution of (5). First of all, we will show that $D_{0_+}^{\alpha-2}u(t) \le D_{0_+}^{\alpha-2}\beta(t)$, for all $t \in [0,\infty)$.

If not, then

$$\sup_{0\le t<\infty}\left(D_{0_+}^{\alpha-2}u(t)-D_{0_+}^{\alpha-2}\beta(t)\right)>0.\tag{65}$$

Note that $\lim_{t\to\infty}(D_{0_+}^{\alpha-1}u(t) - D_{0_+}^{\alpha-1}\beta(t)) < 0$; then there are two cases.

Case 1. There exists a $t_0 \in [0,\infty)$ such that

$$\begin{aligned}&D_{0_+}^{\alpha-2}u(t_0)-D_{0_+}^{\alpha-2}\beta(t_0)\\&\quad=\sup_{t\in[0,\infty)}\left(D_{0_+}^{\alpha-2}u(t)-D_{0_+}^{\alpha-2}\beta(t)\right)>0.\end{aligned}\tag{66}$$

Then $D_{0_+}^{\alpha-1}u(t_0) = D_{0_+}^{\alpha-1}\beta(t_0)$ and

$$D_{0_+}^{\alpha}u(t_0)\le D_{0_+}^{\alpha}\beta(t_0).\tag{67}$$

On the other hand, in view of (32), (33), and (H_2), we have

$$\begin{aligned}&D_{0_+}^{\alpha}u(t_0)\\&=-a(t_0)\Bigg[f\left(t_0,w_0(t_0,u),w_1\left(t_0,D_{0_+}^{\alpha-2}u\right),D_{0_+}^{\alpha-1}u\right)\\&\qquad+\frac{w_1\left(t_0,D_{0_+}^{\alpha-2}u\right)-D_{0_+}^{\alpha-2}u(t_0)}{1+\left|w_1\left(t_0,D_{0_+}^{\alpha-2}u\right)-D_{0_+}^{\alpha-2}u(t_0)\right|}\Bigg]\\&=-a(t_0)\Bigg[f\left(t_0,w_0(t_0,u),\right.\\&\qquad\left.w_1\left(t_0,D_{0_+}^{\alpha-2}\beta(t_0)\right),D_{0_+}^{\alpha-1}\beta(t_0)\right)\\&\qquad+\frac{D_{0_+}^{\alpha-2}\beta(t_0)-D_{0_+}^{\alpha-2}u(t_0)}{1+\left|D_{0_+}^{\alpha-2}\beta(t_0)-D_{0_+}^{\alpha-2}u(t_0)\right|}\Bigg]\\&\ge-a(t_0)f\left(t_0,\beta(t_0),D_{0_+}^{\alpha-2}\beta(t_0),D_{0_+}^{\alpha-1}\beta(t_0)\right)\\&\quad+a(t_0)\frac{D_{0_+}^{\alpha-2}u(t_0)-D_{0_+}^{\alpha-2}\beta(t_0)}{1+\left|D_{0_+}^{\alpha-2}u(t_0)-D_{0_+}^{\alpha-2}\beta(t_0)\right|}\\&\ge-a(t_0)f\left(t_0,\beta(t_0),D_{0_+}^{\alpha-2}\beta(t_0),D_{0_+}^{\alpha-1}\beta(t_0)\right)\\&\ge D_{0_+}^{\alpha-1}\beta(t_0),\end{aligned}\tag{68}$$

which contradicts (67).

Case 2. Consider $D_{0_+}^{\alpha-2}u(0) - D_{0_+}^{\alpha-2}\beta(0) = \lim_{t\to 0^+}(D_{0_+}^{\alpha-2}u(t) - D_{0_+}^{\alpha-2}\beta(t)) = \sup_{t\in[0,\infty)}(D_{0_+}^{\alpha-2}u(t) - D_{0_+}^{\alpha-2}\beta(t)) > 0$. By the boundary condition, we have the contradiction $D_{0_+}^{\alpha-2}u(0) - D_{0_+}^{\alpha-2}\beta(0) \le 0$. Consequently, $D_{0_+}^{\alpha-2}u(t) \le D_{0_+}^{\alpha-2}\beta(t)$ holds for all $t \in [0,\infty)$. By using the similar method, we also can prove that $D_{0_+}^{\alpha-2}\gamma(t) \le D_{0_+}^{\alpha-2}u(t)$ for all $t \in [0,\infty)$. By Remark 7, we have

$$\gamma(t)\le u(t)\le\beta(t),\quad\text{for any } t\in[0,\infty).\tag{69}$$

If $\sigma > 0$, we choose

$$\begin{aligned}r\ge\max\Bigg\{&\sup_{t\in[\sigma,\infty)}\frac{D_{0_+}^{\alpha-2}\beta(t)-D_{0_+}^{\alpha-2}\gamma(0)}{t},\\&\sup_{t\in[\sigma,\infty)}\frac{D_{0_+}^{\alpha-2}\beta(0)-D_{0_+}^{\alpha-2}\gamma(t)}{t}\Bigg\},\end{aligned}\tag{70}$$

and $N > r$ such that

$$\int_r^N\frac{s}{h(s)}ds\ge m\left(\sup_{0\le t<\infty}D_{0_+}^{\alpha-2}\beta(t)-\inf_{0\le t<\infty}D_{0_+}^{\alpha-2}\gamma(t)\right),\tag{71}$$

where $m = \sup_{t\in[0,\infty)}a(t)\phi(t)$. From (23), it is clear that $\int_0^\infty a(s)\phi(s)ds < \infty$. Hence, $m < \infty$.

Finally, we will check that $|D_{0_+}^{\alpha-1}u(t)| < N$ for $t \in [0,\infty)$. If $|D_{0_+}^{\alpha-1}u(t)| \le r$, for every $t \in [0,\infty)$, then $|D_{0_+}^{\alpha-1}u(t)| < N$. If $D_{0_+}^{\alpha-1}u(t) > r$, for all $t \in [0,\infty)$, then for any $R \ge \sigma$; using (70), we obtain

$$\begin{aligned}\frac{D_{0_+}^{\alpha-2}\beta(R) - D_{0_+}^{\alpha-2}\gamma(0)}{R} &\ge \frac{D_{0_+}^{\alpha-2}u(R) - D_{0_+}^{\alpha-2}u(0)}{R} \\ &= \frac{\int_0^R (d/ds)\left(D_{0_+}^{\alpha-2}u(s)\right)ds}{R} \\ &= \frac{\int_0^R D_{0_+}^{\alpha-1}u(s)\,ds}{R} \\ &> r \ge \frac{D_{0_+}^{\alpha-2}\beta(R) - D_{0_+}^{\alpha-2}\gamma(0)}{R},\end{aligned} \tag{72}$$

which is a contradiction. If $D_{0_+}^{\alpha-1}u(t) < -r$, for every $t \in [0,\infty)$, we can also have a similar contradiction. There exists $t_0 \in [0,\infty)$ such that $|D_{0_+}^{\alpha-2}u(t_0)| \le r$. Thus, there exists $[t_1,t_2] \subset [0,\infty)$ such that $|D_{0_+}^{\alpha-1}u(t_1)| = r$, $|D_{0_+}^{\alpha-1}u(t)| > r$, $t \in (t_1,t_2]$. Otherwise, $|D_{0_+}^{\alpha-1}u(t_2)| = r$, $|D_{0_+}^{\alpha-1}u(t)| > r$, $t \in [t_1,t_2)$. Without loss of generality, we assume that $D_{0_+}^{\alpha-1}u(t_1) = r$, $D_{0_+}^{\alpha-1}u(t) > r$, $t \in (t_1,t_2]$. Therefore, by a convenient change of variable and using (21) and (71), we get

$$\begin{aligned}&\int_{D_{0_+}^{\alpha-1}u(t_1)}^{D_{0_+}^{\alpha-1}u(t_2)} \frac{s}{h(s)}ds \\ &= \int_{t_1}^{t_2} \frac{D_{0_+}^{\alpha-1}u(t)}{h\left(D_{0_+}^{\alpha-1}u(t)\right)} D_{0_+}^{\alpha}u(t)\,dt \\ &= \int_{t_1}^{t_2} -a(t)\,f\left(t,u(t),D_{0_+}^{\alpha-2}u(t),D_{0_+}^{\alpha-1}u(t)\right) \\ &\qquad \times \frac{D_{0_+}^{\alpha-1}u(t)}{h\left(D_{0_+}^{\alpha-1}u(t)\right)}dt \\ &\le \int_{t_1}^{t_2} a(t)\,\phi(t)\,D_{0_+}^{\alpha-1}u(t)\,dt \\ &\le m\int_{t_1}^{t_2} D_{0_+}^{\alpha-1}u(t)\,dt \\ &= m\left(D_{0_+}^{\alpha-2}u(t_2) - D_{0_+}^{\alpha-2}u(t_1)\right) \\ &\le m\left(\sup_{t\in[0,\infty)} D_{0_+}^{\alpha-2}\beta(t) - \inf_{t\in[0,\infty)} D_{0_+}^{\alpha-2}\gamma(t)\right) \\ &\le \int_r^N \frac{s}{h(s)}ds,\end{aligned} \tag{73}$$

which implies that $D_{0_+}^{\alpha-1}u(t_2) \le N$, since t_2 can be arbitrarily as long as $D_{0_+}^{\alpha-1}u(t) > r$; we have $D_{0_+}^{\alpha-1}u(t) > r$, for any $t \in [0,\infty)$, which follows that $D_{0_+}^{\alpha-1}u(t) \le N$. By a similar analysis, we can also obtain that if $D_{0_+}^{\alpha-1}u(t_1) = -r$, $D_{0_+}^{\alpha-1}u(t) < -r$, $t \in (t_1,t_2]$, then $D_{0_+}^{\alpha-1}u(t) > -N$, $t \in [0,\infty)$. Therefore,

$$\begin{aligned}D_{0_+}^{\alpha}u(t) &= -a(t)\,f^*\left(t,u(t),D_{0_+}^{\alpha-2}u(t),D_{0_+}^{\alpha-1}u(t)\right) \\ &= -a(t)\,f\left(t,u(t),D_{0_+}^{\alpha-2}u(t),D_{0_+}^{\alpha-1}u(t)\right);\end{aligned} \tag{74}$$

that is, u is a solution of (5), which completes the proof. □

Example 15. Consider the boundary value problem of the fractional differential equation on the half-line

$$\begin{aligned}&D_{0_+}^{8/3}u(t) + e^{-t}\left(t^2+u^2(t)\right)\left(\Gamma\left(\frac{5}{3}\right) - D_{0_+}^{2/3}u(t)\right) \\ &\qquad \times\left(1+\arctan\left(D_{0_+}^{5/3}u(t)\right)^2\right) = 0, \quad t\in(0,\infty), \\ &u(0) = D_{0_+}^{2/3}u(0) = 0, \qquad D_{0_+}^{5/3}u(\infty) = 0.\end{aligned} \tag{75}$$

In this case, $\alpha = 8/3, a(t) = e^{-t}$, $f(t,x,y,z) = (t^2 + x^2)(\Gamma(5/3) - y)(1+\arctan(z^2))$. Obviously, $\gamma(t) = -t^{2/3}$ and $\beta(t) = t^{2/3}$ are a pair of lower and upper solutions of (75). Furthermore, we have $\gamma,\beta \in E$, $\gamma(t) \le \beta(t)$, for $t \in [0,\infty)$. It is clear that f is continuous on $[0,\infty)\times\mathbb{R}^3$. If $0 \le t < \infty$, $-t^{2/3} \le x \le t^{2/3}$, $-\Gamma(5/3) \le y \le \Gamma(5/3)$, we have

$$\left|f(t,x,y,z)\right| \le \phi(t)\,h(|z|), \tag{76}$$

where $\phi(t) = 4\Gamma(5/3)(1+t^2)$ and $h(z) = 1+z^2$; we have

$$\int_0^\infty \frac{s}{h(s)}ds = \int_0^\infty \frac{s}{1+s^2}ds = \infty, \tag{77}$$

and f satisfies the like Nagumo condition with respect to $-t^{2/3}, t^{2/3}$. Furthermore, we have

$$\begin{aligned}&\int_0^\infty \max\{s,1\}\,a(s)\,ds = \int_0^1 e^{-s}ds + \int_1^\infty se^{-s}ds < \infty, \\ &\int_0^\infty \max\{s,1\}\,a(s)\,\phi(s)\,ds \\ &\qquad = 4\Gamma\left(\frac{5}{3}\right)\int_0^1\left(1+s^2\right)e^{-s}ds \\ &\qquad\quad + 4\Gamma\left(\frac{5}{3}\right)\int_1^\infty s\left(1+s^2\right)e^{-s}ds < \infty.\end{aligned} \tag{78}$$

Thus, we conclude by Theorem 14 that there exists at least one solution $u(t)$ to boundary value problem (75) such that

$$\begin{aligned}&-t^{2/3} \le u(t) \le t^{2/3}, \\ &-\Gamma\left(\frac{5}{3}\right) \le D_{0_+}^{\alpha-2}u(t) \le \Gamma\left(\frac{5}{3}\right), \quad t\in[0,\infty).\end{aligned} \tag{79}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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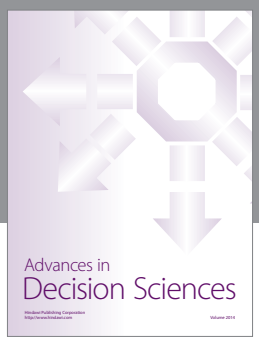

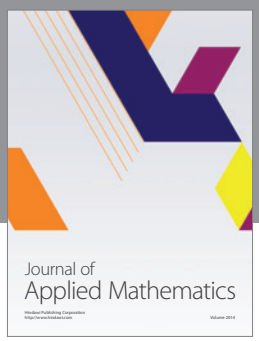

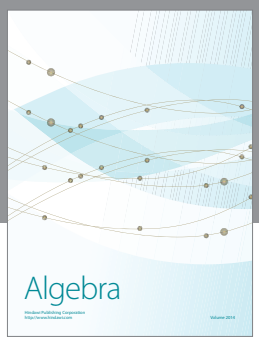

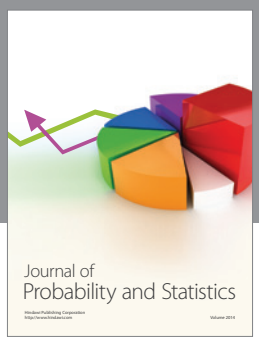

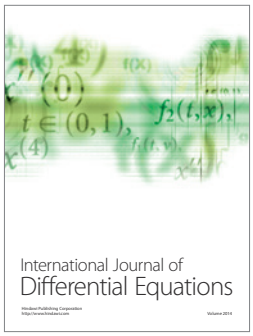

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