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Research Article

Applications of Soft Union Sets in the Ring Theory

Yongwei Yang, Xiaolong Xin, and Pengfei He

Department of Mathematics, Northwest University, Xi'an 710127, China

Correspondence should be addressed to Xiaolong Xin; xlxin@nwu.edu.cn

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The aim of the paper is to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties. In order to provide these soft algebraic structures, the notion of (λ, μ)-soft union rings which is a generalization of that of soft union rings is proposed. By introducing the notion of soft cosets, soft quotient rings based on (λ, μ)-soft union ideals are established. Moreover, through discussing quotient soft subsets, an approach for constructing quotient soft union rings is made. Finally, isomorphism theorems of (λ, μ)-soft union rings related to invariant soft sets are discussed.

1. Introduction

Fuzzy set theory [1], intuitionistic set theory [2], and probability theory are useful approaches to describe uncertainty, but each of these theories has its inherent difficulties. To overcome these problems, Molodtsov [3] initiated the concept of soft sets that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Maji et al. [4] gave the operations of soft sets and their properties; furthermore, they [5] introduced fuzzy soft sets which combine the strengths of both soft sets and fuzzy sets. As a generalization of the soft set theory, the fuzzy soft set theory makes description of the objective world more realistic, practical, and precise in some cases, making it very promising. Since its introduction, the concept of soft sets has gained considerable attention in many directions and has found applications in a wide variety of fields such as the theory of soft sets [6, 7] and soft decision making [8, 9].

Since the notion of soft groups was proposed by Aktaş and Çağman [10], then the soft set theory is used as a new tool to discuss algebraic structures. Acar et al. [11] initiated the concepts of soft rings similar to soft groups. Liu et al. further investigated isomorphism and fuzzy isomorphism theories of soft rings in [12, 13], respectively. Soft sets were also applied to other algebraic structures such as near-rings [14], Γ-hyperrings [15], Γ-modules [16, 17], and BCK/BCI-algebras [18]. The idea of quasicoincidence of a fuzzy point with a fuzzy set, which is mentioned in [19], has played a vital role in generating some different algebraic structures. By using the concepts of belongingness to (denoted by $\in$) and quasicoincidence (denoted by q) of a fuzzy point with a fuzzy subgroup, Bhakat and Das [20] proposed the concept of (α, β)-fuzzy subgroups. Inspired by the previous works, Zhan et al. [21] extended these results to BCI-algebras and obtained some important and useful generalizations of related algebraic structures. Moreover, they characterized filteristic soft BL-algebras [22] and filteristic soft MTL-algebras [23] based on $\in$-soft sets and q-soft sets.

Çağman et al. [24] studied on soft int-groups, which are different from the definition of soft groups [10]. The new approach is based on the inclusion relation and intersection of sets. It brings the soft set theory, the set theory, and the group theory together. On the basis of soft int-groups, Sezgin et al. [25] introduced the concept of soft intersection near-rings (soft int near-rings) by using intersection operation of sets and gave the applications of soft int near-rings to the near-ring theory. By introducing soft intersection-union products and soft characteristic functions, Sezer [26] made a new approach to the classical ring theory via the soft set theory, with the concepts of soft union rings, ideals, and bi-ideals. Jun et al. applied intersectional soft sets to BCK/BCI-algebras [27, 28] and obtained many results.

In the present paper, in order to further investigate the application of soft sets in the ring theory, we introduce the notions of (λ, μ)-soft union rings and (λ, μ)-soft union ideals as generalizations of that of soft union rings and soft union ideals, respectively. Then, we discuss the properties of images and inverse images of (λ, μ)-soft union ideals. Furthermore, we establish soft quotient rings based on (λ, μ)-soft union ideals by introducing the notion of soft cosets. Moreover, through discussing quotient soft subsets, we give an approach for constructing quotient soft union rings. Finally, we discuss isomorphism theorems of (λ, μ)-soft union rings related to invariant soft sets.

2. Preliminaries

In this section, we would like to recall some basic notions related to soft sets and soft union rings. An algebraic system $(R, +, \cdot)$ is called a ring if it satisfies the following conditions:

(1) $(R, +)$ forms an abelian group,

(2) $(R, \cdot)$ forms a semigroup,

(3) $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$, for all $a, b, c \in R$.

A subgroup S of $(R, +)$ with $SR \subseteq S$ and $RS \subseteq S$ is called an ideal of R.

Throughout the paper, R, R_1, and R_2 denote rings, and 0, 0_1, and 0_2 are the zero elements of R, R_1, and R_2, respectively. U is an initial universe and E is a set of parameters under consideration with respect to U. A and B are subsets of E. The set of all subsets of U is denoted by $\mathscr{P}(U)$. Molodtsov [3] defined the concept of soft sets in the following way.

Definition 1 (see [3]). A soft set f_A over U is defined as $f_A : E \to \mathscr{P}(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For all $\varepsilon \in A$, $f_A(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set f_A. A soft set f_A over U can be presented by the set of ordered pairs:

$$f_A = \{(x, f_A(x)) \mid x \in E, f_A(x) \in \mathscr{P}(U)\}. \quad (1)$$

Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [3].

If f_A is a soft set over U, then the image of f_A is defined by $\mathrm{Im}(f_A) = \{f_A(a) \mid a \in A\}$. The set of all soft sets over U will be denoted by $S(U)$. Some of the operations of soft sets are listed as follows.

Definition 2 (see [4]). Let $f_A, f_B \in S(U)$. If $f_A(x) \subseteq f_B(x)$, for all $x \in E$, then f_A is called a soft subset of f_B and denoted by $f_A \tilde{\subseteq} f_B$.

f_A and f_B are called soft equal, denoted by $f_A = f_B$, if and only if $f_A \tilde{\subseteq} f_B$ and $f_B \tilde{\subseteq} f_A$.

Definition 3 (see [24]). Let $f_A, f_B \in S(U)$ and let Ψ be a function from A to B. Then, the soft anti-image of f_A under Ψ, denoted by $\Psi(f_A)$, is a soft set over U defined by

$$\Psi(f_A)(b) = \begin{cases} \cap\{f_A(a) \mid a \in A, \Psi(a) = b\}, & \text{if } \Psi^{-1}(b) \neq \emptyset, \\ \emptyset, & \text{otherwise}, \end{cases} \quad (2)$$

for all $b \in B$. And the soft preimage of f_B under Ψ, denoted by $\Psi^{-1}(f_B)$, is a soft set over U defined by $\Psi^{-1}(f_B)(a) = f_B(\Psi(a))$, for all $a \in A$.

Note that the concept of level sets in the fuzzy set theory, Çağman et al. [24] initiated the concept of lower inclusions of soft sets which serves as a bridge between soft sets and crisp sets.

Definition 4 (see [26]). Let f_A be a soft set over U and $\alpha \subseteq U$. Then, lower α-inclusion of f_A, denoted by $L(f_A; \alpha)$, is defined as $L(f_A; \alpha) = \{x \in A \mid f_A(x) \subseteq \alpha\}$.

Inspired by the concept of soft int-groups [24], Sezer in [26] introduced the concept of soft union rings by the combination of the theories of soft sets and rings.

Definition 5 (see [26]). A soft set f_R over U is called a soft union ring of R if

(1) $f_R(x + y) \subseteq f_R(x) \cup f_R(y)$,

(2) $f_R(x) \subseteq f_R(-x)$,

(3) $f_R(xy) \subseteq f_R(x) \cup f_R(y)$,

for all $x, y \in R$.

Now, we proceed on to recall the notion of soft union ideals of rings.

Definition 6 (see [26]). A soft set f_R over U is called a soft union left (resp., right) ideal of R if

(1) $f_R(x - y) \subseteq f_R(x) \cup f_R(y)$,

(2) $f_R(xy) \subseteq f_R(y)$ (resp., $f_R(xy) \subseteq f_R(x)$),

for all $x, y \in R$.

A soft set f_R over U is called a soft union ideal of R if it is both a soft union left and a soft union right ideal of R over U.

3. (λ, μ)-Soft Union Rings and (λ, μ)-Soft Ideals of Rings

In this section, we introduce the notion of (λ, μ)-soft union rings (ideals) which is a generation of that of soft union rings (ideals) and investigate their basic properties. From now on, $\emptyset \subseteq \mu \subset \lambda \subseteq U$ unless otherwise specified.

Definition 7. Let f_R be a soft set over U. f_R is called a (λ, μ)-soft union ring of R if

(1) $f_R(x + y) \cap \lambda \subseteq f_R(x) \cup f_R(y) \cup \mu$,

(2) $f_R(x) \cap \lambda \subseteq f_R(-x) \cup \mu$,

(3) $f_R(xy) \cap \lambda \subseteq f_R(x) \cup f_R(y) \cup \mu$,

for all $x, y \in R$.

Remark 8. Let f_R be a soft union ring of R over U; then f_R is a $(U, \emptyset)$-soft union ring of R. Therefore, a soft union ring of R is a (λ, μ)-soft union ring, but the converse is not true in general.

We show this fact by the following example.

Example 9. Given a ring $R = \{\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] \mid a, b \in \mathbb{Z}_2\}$ with the operations addition and multiplication of matrices, $U = \mathbb{Z}_5$. We define a soft set f_R over U by

$$\begin{aligned} f_R\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) &= \{0, 2\}, \qquad f_R\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \{0, 1, 2, 4\}, \\ f_R\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) &= \{0, 1, 2\}, \qquad f_R\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = \{0, 1, 4\}. \end{aligned} \tag{3}$$

Then, one can easily show that f_R is a $(\{0, 1, 2, 4\}, \{0, 2\})$-soft union ring of R. But f_R is not a soft union ring of R because

$$\begin{aligned} f_R\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) &= f_R\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &= \{0, 2\} \not\subseteq f_R\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) \cup f_R\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) \\ &= \{0, 1, 4\}. \end{aligned} \tag{4}$$

In order to give some characterizations of (λ, μ)-soft union rings, we need the following lemmas.

Lemma 10. *Let f_R be a soft set over U. If f_R is a (λ, μ)-soft union ring of R, then*

(1) $(f_R(-x) \cap \lambda) \cup \mu \subseteq (f_R(x) \cap \lambda) \cup \mu$,

(2) $f_R(-x) \cap \lambda \subseteq f_R(x) \cup \mu$,

for all $x \in R$.

Proof. (1) Assume that f_R is a (λ, μ)-soft union ring of R; then for all $x \in R$, we get that $(f_R(-x) \cap \lambda) \cup \mu = (f_R(-x) \cap \lambda \cap \lambda) \cup \mu \subseteq ((f_R(-(-x)) \cup \mu) \cap \lambda) \cup \mu = (f_R(x) \cap \lambda) \cup \mu$.

(2) It is straightforward. □

Lemma 11. *Let f_R be a soft set over U. If f_R is a (λ, μ)-soft union ring of R, then*

(1) $(f_R(0) \cap \lambda) \cup \mu \subseteq (f_R(x) \cap \lambda) \cup \mu$,

(2) $f_R(0) \cap \lambda \subseteq f_R(x) \cup \mu$,

for all $x \in R$.

Proof. (1) Assume that f_R is a (λ, μ)-soft union ring of R; then for all $x \in R$, we have that $(f_R(0) \cap \lambda) \cup \mu = (f_R(x - x) \cap \lambda \cap \lambda) \cup \mu \subseteq ((f_R(x) \cup f_R(-x) \cup \mu) \cap \lambda) \cup \mu = ((f_R(x) \cap \lambda) \cup \mu) \cup (f_R(-x) \cap \lambda) \cup \mu \subseteq (f_R(x) \cap \lambda) \cup \mu$, by Lemma 10.

(2) It is straightforward. □

Combining Lemma 11 and Definition 7, we obtain the following characterization of (λ, μ)-soft union rings.

Theorem 12. *Let f_R be a soft set over U. Then, f_R is a (λ, μ)-soft union ring of R if and only if f_R satisfies the following conditions:*

(1) $f_R(x - y) \cap \lambda \subseteq f_R(x) \cup f_R(y) \cup \mu$,

(2) $f_R(xy) \cap \lambda \subseteq f_R(x) \cup f_R(y) \cup \mu$,

for all $x, y \in R$.

Proof. Assume that f_R is a (λ, μ)-soft union ring of R. By Definition 7 and Lemma 11, we have $f_R(xy) \cap \lambda \subseteq f_R(x) \cup f_R(y) \cup \mu$ and

$$\begin{aligned} f_R(x - y) \cap \lambda &= (f_R(x - y) \cap \lambda) \cap \lambda \\ &\subseteq (f_R(x) \cup f_R(-y) \cup \mu) \cap \lambda \\ &= (f_R(x) \cap \lambda) \cup (f_R(-y) \cap \lambda) \cup \mu \\ &\subseteq f_R(x) \cup f_R(y) \cup \mu, \end{aligned} \tag{5}$$

for all $x, y \in R$.

Conversely, since $f_R(x - y) \cap \lambda \subseteq (f_R(x) \cup f_R(y)) \cup \mu$, for all $x, y \in R$, by Lemma 11, we get

$$\begin{aligned} f_R(x) \cap \lambda &= (f_R(0 - (-x)) \cap \lambda) \cap \lambda \\ &\subseteq (f_R(0) \cup f_R(-x) \cup \mu) \cap \lambda \\ &= (f_R(0) \cap \lambda) \cup ((f_R(-x) \cup \mu) \cap \lambda) \\ &\subseteq (f_R(-x) \cup \mu) \cup (f_R(-x) \cup \mu) \\ &= f_R(-x) \cup \mu. \end{aligned} \tag{6}$$

Moreover, we have

$$\begin{aligned} f_R(x + y) \cap \lambda &= (f_R(x - (-y)) \cap \lambda) \cap \lambda \\ &\subseteq (f_R(x) \cup f_R(-y) \cup \mu) \cap \lambda \\ &= ((f_R(x) \cup \mu) \cap \lambda) \cup (f_R(-y) \cap \lambda) \\ &\subseteq f_R(x) \cup f_R(y) \cup \mu. \end{aligned} \tag{7}$$

And it follows from hypothesis that $f_R(xy) \cap \lambda \subseteq f_R(x) \cup f_R(y) \cup \mu$; therefore f_R is a (λ, μ)-soft union ring of R. □

Analogues to the notion of (λ, μ)-soft union rings, we can extend the concept of soft union ideals as follows.

Definition 13. A soft set f_R over U is called a (λ, μ)-soft union ideal of R if

(1) $f_R(x - y) \cap \lambda \subseteq f_R(x) \cup f_R(y) \cup \mu$,

(2) $f_R(xy) \cap \lambda \subseteq (f_R(x) \cap f_R(y)) \cup \mu$,

for all $x, y \in R$.

Remark 14. Let f_R be a soft union ideal of R over U; then f_R is a $(U, \emptyset)$-soft union ideal of R. Therefore, a soft union ideal of R is a (λ, μ)-soft union ideal; however it is important and interesting to note that the converse is not true in general.

Example 15. Let $R = \mathbb{Z}_4$ be a ring and $U = \mathbb{Z}^+$ a universal set. We construct a soft set f_R over U by $f_R(0) = \{1\}$, $f_R(1) = f_R(3) = \{2,3,4\}$, and $f_R(2) = \{3\}$. One can show that f_R is a $(\{1,2,4\},\{1,2\})$-soft union ideal. But f_R is not a soft union ideal of R, since $f_R(2-2) = f_R(0) = \{1\} \not\subseteq f_R(2) \cup f_R(2) = \{3\}$.

Proposition 16. *If f_R is a (λ,μ)-soft union ideal of R, then $f_R^* \triangleq \{x \in R \mid (f_R(x) \cap \lambda) \cup \mu = (f_R(0) \cap \lambda) \cup \mu\}$ is an ideal of R.*

Proof. We need to show that (i) $x - y \in f_R^*$, (ii) $rx \in f_R^*$, and (iii) $xr \in f_R^*$, for all $x, y \in f_R^*$ and $r \in R$. If $x, y \in f_R^*$, then $(f_R(x) \cap \lambda) \cup \mu = (f_R(y) \cap \lambda) \cup \mu = (f_R(0) \cap \lambda) \cup \mu$. By Lemma 11, we obtain that $(f_R(0) \cap \lambda) \cup \mu \subseteq (f_R(x-y) \cap \lambda) \cup \mu$, $(f_R(0) \cap \lambda) \cup \mu \subseteq (f_R(rx) \cap \lambda) \cup \mu$, and $(f_R(0) \cap \lambda) \cup \mu \subseteq (f_R(xr) \cap \lambda) \cup \mu$, for all $x, y \in f_R^*$ and $r \in R$. Since f_R is a (λ,μ)-soft union ideal, then for all $x, y \in f_R^*$ and $r \in R$, we have

$$\begin{aligned}
&(f_R(x-y) \cap \lambda) \cup \mu \\
&\quad = (f_R(x-y) \cap \lambda \cap \lambda) \cup \mu \\
&\quad \subseteq ((f_R(x) \cup f_R(y) \cup \mu) \cap \lambda) \cup \mu \\
&\quad = ((f_R(x) \cap \lambda) \cup \mu) \\
&\quad \cup ((f_R(y) \cap \lambda) \cup \mu) \\
&\quad = (f_R(0) \cap \lambda) \cup \mu, \\
&(f_R(rx) \cap \lambda) \cup \mu = (f_R(rx) \cap \lambda \cap \lambda) \cup \mu \\
&\quad \subseteq (((f_R(r) \cap f_R(x)) \cup \mu) \cap \lambda) \cup \mu \\
&\quad \subseteq (f_R(x) \cap \lambda) \cup \mu = (f_R(0) \cap \lambda) \cup \mu.
\end{aligned} \tag{8}$$

Similarly, we can prove that $(f_R(xr) \cap \lambda) \cup \mu \subseteq (f_R(0) \cap \lambda) \cup \mu$, for all $x \in f_R^*$ and $r \in R$. Hence, $(f_R(x-y) \cap \lambda) \cup \mu = (f_R(rx) \cap \lambda) \cup \mu = (f_R(xr) \cap \lambda) \cup \mu = (f_R(0) \cap \lambda) \cup \mu$. Therefore, (i) $x - y \in f_R^*$, (ii) $rx \in f_R^*$, and (iii) $xr \in f_R^*$, for all $x, y \in f_R^*$ and $r \in R$. And thus, f_R^* is an ideal of R. □

We will now display the relationship between (λ,μ)-soft union ideals and ideals. For this purpose, we require the following notion.

Definition 17. Let f_R be a (λ,μ)-soft union ideal of R; then $E\,\mathrm{Im}(f_R)$ is called the extended image set of f_R, where $E\,\mathrm{Im}(f_R) = \mathrm{Im}(f_R) \cup \{\lambda,\mu\}$.

Now, we characterize (λ,μ)-soft union ideals by lower inclusions.

Proposition 18. *Let f_R be a soft set over U and $E\,\mathrm{Im}(f_R)$ a totally ordered set by inclusion. Then, f_R is a (λ,μ)-soft union ideal of R if and only if $L(f_R;\alpha)$ is an ideal of R, whenever it is nonempty, for each $\alpha \subseteq U$ where $\mu \subseteq \alpha \subset \lambda$.*

Proof. Assume that f_R is a (λ,μ)-soft union ideal of R and $L(f_R;\alpha)$ is nonempty. It is sufficient to show that $x - y \in L(f_R;\alpha)$, $rx \in L(f_R;\alpha)$ and $xr \in L(f_R;\alpha)$, for all $x, y \in L(f_R;\alpha)$, and $r \in R$. Let $x, y \in L(f_R;\alpha)$. It follows that $f_R(x) \subseteq \alpha$ and $f_R(y) \subseteq \alpha$. Since f_R is a (λ,μ)-soft union ideal of R and $E\,\mathrm{Im}(f_R)$ is a totally ordered set, then $f_R(x-y) \cap \lambda \subseteq f_R(x) \cup f_R(y) \cup \mu \subseteq \alpha \cup \alpha \cup \alpha = \alpha \subset \lambda$, $f_R(rx) \cap \lambda \subseteq (f_R(r) \cap f_R(x)) \cup \mu \subseteq \alpha \cup \alpha = \alpha \subset \lambda$, and $f_R(xr) \cap \lambda \subseteq (f_R(x) \cap f_R(r)) \cup \mu \subseteq \alpha \cup \alpha = \alpha \subset \lambda$. And thus, $f_R(x-y) \subseteq \alpha$, $f_R(rx) \subseteq \alpha$, and $f_R(xr) \subseteq \alpha$; hence $x - y, rx, xr \in L(f_R;\alpha)$. Therefore, $L(f_R;\alpha)$ is an ideal of R.

Conversely, assume that $L(f_R;\alpha)$ is an ideal of R whenever it is nonempty, for each $\alpha \subseteq U$ where $\mu \subseteq \alpha \subset \lambda$. Suppose that $f_R(x-y) \cap \lambda \subseteq f_R(x) \cup f_R(y) \cup \mu$ does not hold for some $x, y \in R$; then there exist $x_0, y_0 \in R$ such that $f_R(x_0 - y_0) \cap \lambda \supset \alpha = (f_R(x_0) \cup f_R(y_0) \cup \mu)$. Therefore, $f_R(x_0) \cup f_R(y_0) \subseteq \alpha$ and $\mu \subseteq \alpha \subset \lambda$. It follows that $x_0, y_0 \in L(f_R;\alpha)$, but $f_R(x_0 - y_0) \supset \alpha$; that is, $x_0 - y_0 \notin L(f_R;\alpha)$, which is a contradiction. Hence, $f_R(x-y) \cap \lambda \subseteq f_R(x) \cup f_R(y) \cup \mu$, for all $x, y \in R$. Similarly, we can prove that $f_R(xy) \cap \lambda \subseteq (f_R(x) \cap f_R(y)) \cup \mu$, for all $x, y \in R$. Thus, f_R is a (λ,μ)-soft union ideal of R. □

In the rest of this section, we will show that the soft anti-image and soft preimage of a (λ,μ)-soft union ideal under a ring homomorphism are also (λ,μ)-soft union ideals.

Theorem 19. *Let f_{R_1} be a soft set over U and Ψ a ring epimorphism from R_1 to R_2. If f_{R_1} is a (λ,μ)-soft union ideal of R_1, then $\Psi(f_{R_1})$ is a (λ,μ)-soft union ideal of R_2 and $(\Psi(f_{R_1})(0_2) \cap \lambda) \cup \mu = (f_{R_1}(0_1) \cap \lambda) \cup \mu$.*

Proof. Let $y_1, y_2 \in R_2$ and f_{R_1} a (λ,μ)-soft union ideal of R_1. Since Ψ is a ring epimorphism from R_1 to R_2, then $\Psi^{-1}(y_1) \neq \emptyset$ and $\Psi^{-1}(y_2) \neq \emptyset$. And thus, there exist $x_1, x_2 \in R_1$ such that $\Psi(x_1) = y_1$, $\Psi(x_2) = y_2$. Therefore, we have

$$\begin{aligned}
&\Psi(f_{R_1})(y_1 - y_2) \cap \lambda \\
&\quad = \cap\{f_{R_1}(x_1 - x_2) \mid \Psi(x_1 - x_2) = y_1 - y_2\} \cap \lambda \\
&\quad = \cap\{f_{R_1}(x_1 - x_2) \cap \lambda \mid \Psi(x_1 - x_2) = y_1 - y_2\} \\
&\quad \subseteq \cap\{f_{R_1}(x_1) \cup f_{R_1}(x_2) \cup \mu \mid \Psi(x_1) = y_1, \Psi(x_2) = y_2\} \\
&\quad = \cap\{f_{R_1}(x_1) \mid \Psi(x_1) = y_1\} \\
&\quad \cup \cap\{f_{R_1}(x_2) \mid \Psi(x_2) = y_2\} \cup \mu \\
&\quad = \Psi(f_{R_1})(y_1) \cup \Psi(f_{R_1})(y_2) \cup \mu, \\
&\Psi(f_{R_1})(y_1 y_2) \cap \lambda \\
&\quad = \cap\{f_{R_1}(x_1 x_2) \mid \Psi(x_1 x_2) = y_1 y_2\} \cap \lambda \\
&\quad = \cap\{f_{R_1}(x_1 x_2) \cap \lambda \mid \Psi(x_1 x_2) = y_1 y_2\} \\
&\quad \subseteq \cap\{(f_{R_1}(x_1) \cap f_{R_1}(x_2)) \cup \mu \mid \Psi(x_1) = y_1, \Psi(x_2) = y_2\} \\
&\quad = (\cap\{f_{R_1}(x_1) \mid \Psi(x_1) = y_1\}
\end{aligned}$$

$$\cap\cap\left\{f_{R_1}(x_2) \mid \Psi(x_2) = y_2\right\}\Big) \cup \mu$$
$$= \left(\Psi\left(f_{R_1}\right)(y_1) \cap \Psi\left(f_{R_1}\right)(y_1)\right) \cup \mu. \tag{9}$$

Therefore, $\Psi(f_{R_1})$ is a (λ, μ)-soft union ideal of R_2.

By Lemma 11, we have $(\Psi(f_{R_1})(0_2) \cap \lambda) \cup \mu = (\cap\{f_{R_1}(x) \mid x \in R_1, \Psi(x) = 0_2\} \cap \lambda) \cup \mu = \cap\{(f_{R_1}(x) \cap \lambda) \cup \mu \mid x \in R_1, \Psi(x) = 0_2\} = (f_{R_1}(0_1) \cap \lambda) \cup \mu$. □

Theorem 20. *Let f_{R_2} be a soft set over U and Ψ a ring homomorphism from R_1 to R_2. If f_{R_2} is a (λ, μ)-soft union ideal of R_2, then $\Psi^{-1}(f_{R_2})$ is a (λ, μ)-soft union ideal of R_1.*

Proof. Let $x_1, x_2 \in R_1$. Then,

$$\begin{aligned}
&\Psi^{-1}\left(f_{R_2}\right)(x_1 - x_2) \cap \lambda \\
&\quad = f_{R_2}\left(\Psi(x_1 - x_2)\right) \cap \lambda \\
&\quad = f_{R_2}\left(\Psi(x_1) - \Psi(x_2)\right) \cap \lambda \\
&\quad \subseteq f_{R_2}\left(\Psi(x_1)\right) \cup f_{R_2}\left(\Psi(x_2)\right) \cup \mu \\
&\quad = \Psi^{-1}\left(f_{R_2}\right)(x_1) \cup \Psi^{-1}\left(f_{R_2}\right)(x_1) \cup \mu.
\end{aligned} \tag{10}$$

Moreover, we have

$$\begin{aligned}
&\Psi^{-1}\left(f_{R_2}\right)(x_1 x_2) \cap \lambda \\
&\quad = f_{R_2}\left(\Psi(x_1 x_2)\right) \cap \lambda = f_{R_2}\left(\Psi(x_1) \cdot \Psi(x_2)\right) \cap \lambda \\
&\quad \subseteq \left(f_{R_2}\left(\Psi(x_1)\right) \cap f_{R_2}\left(\Psi(x_1)\right)\right) \cup \mu \\
&\quad = \left(\Psi^{-1}\left(f_{R_2}\right)(x_1) \cap \Psi^{-1}\left(f_{R_2}\right)(x_1)\right) \cup \mu.
\end{aligned} \tag{11}$$

Hence, $\Psi^{-1}(f_{R_2})$ is a (λ, μ)-soft union ideal of R_1. □

4. Soft Quotient Rings

The main purpose of this section is to give an approach for constructing soft quotient rings based on (λ, μ)-soft union ideals. Such approach involves the concept of soft cosets. In addition, some simple characterizations of soft cosets are presented.

Definition 21. Let f_R be a (λ, μ)-soft union ring of R over U and $r \in R$. Then, a soft coset $r \oplus f_R$ of f_R is defined by

$$(r \oplus f_R)(x) = \left(f_R(x - r) \cap \lambda\right) \cup \mu, \tag{12}$$

for all $x \in R$.

For the sake of simplicity and better understanding, we illustrate the above concept by the following example.

Example 22. Consider the $(\{0, 1, 2, 4\}, \{0, 2\})$-soft union ring f_R in Example 9. If $r = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, then we define a soft set $r \oplus f_R$ as

$$\begin{aligned}
(r \oplus f_R)\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) &= \{0, 1, 2, 4\}, \\
(r \oplus f_R)\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) &= \{0, 2\}, \\
(r \oplus f_R)\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) &= \{0, 1, 2, 4\}, \\
(r \oplus f_R)\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) &= \{0, 1, 2\}.
\end{aligned} \tag{13}$$

Then, it is easy to show that $r \oplus f_R$ is a soft coset of f_R.

Proposition 23. *Let f_R be a (λ, μ)-soft union ring over U and $a, b \in R$. Then, $(f_R(a - b) \cap \lambda) \cup \mu = (f_R(0) \cap \lambda) \cup \mu$ if and only if $(f_R(b - a) \cap \lambda) \cup \mu = (f_R(0) \cap \lambda) \cup \mu$.*

Proof. Suppose that $(f_R(b - a) \cap \lambda) \cup \mu = (f_R(0) \cap \lambda) \cup \mu$. Since f_R is a (λ, μ)-soft union ring, then $(f_R(a - b) \cap \lambda) \cup \mu = (f_R(a-b) \cap \lambda \cap \lambda) \cup \mu \subseteq ((f_R(b-a) \cup \mu) \cap \lambda) \cup \mu = (f_R(b-a) \cap \lambda) \cup \mu = (f_R(0) \cap \lambda) \cup \mu$. By Lemma 11, we have $(f_R(0) \cap \lambda) \cup \mu \subseteq (f_R(a - b) \cap \lambda) \cup \mu$. Thus, $(f_R(a - b) \cap \lambda) \cup \mu = (f_R(0) \cap \lambda) \cup \mu$.

Conversely, assume that $(f_R(a-b) \cap \lambda) \cup \mu = (f_R(0) \cap \lambda) \cup \mu$. We can prove that $(f_R(b - a) \cap \lambda) \cup \mu = (f_R(0) \cap \lambda) \cup \mu$ in a similar way. □

Proposition 24. *Let f_R be a (λ, μ)-soft union ring over U and $a, b \in R$. Then, $a \oplus f_R = b \oplus f_R$ if and only if $(f_R(a-b) \cap \lambda) \cup \mu = (f_R(0) \cap \lambda) \cup \mu$.*

Proof. Suppose that $(f_R(a - b) \cap \lambda) \cup \mu = (f_R(0) \cap \lambda) \cup \mu$; then

$$\begin{aligned}
(b \oplus f_R)(x) &= \left(f_R(x - b) \cap \lambda\right) \cup \mu \\
&= \left(f_R(x - a + a - b) \cap \lambda\right) \cup \mu \\
&= \left(f_R(x - a + a - b) \cap \lambda \cap \lambda\right) \cup \mu \\
&\subseteq \left(\left(f_R(x - a) \cup f_R(a - b) \cup \mu\right) \cap \lambda\right) \cup \mu \\
&= \left(\left(f_R(x - a) \cap \lambda\right) \cup \mu\right) \cup \left(\left(f_R(a - b) \cap \lambda\right) \cup \mu\right) \\
&= \left(\left(f_R(x - a) \cap \lambda\right) \cup \mu\right) \cup \left(\left(f_R(0) \cap \lambda\right) \cup \mu\right) \\
&\subseteq \left(f_R(x - a) \cap \lambda\right) \cup \mu = (a \oplus f_R)(x),
\end{aligned} \tag{14}$$

for all $x \in R$. Therefore, $b \oplus f_R \tilde{\subseteq} a \oplus f_R$. Similarly, we can show that $a \oplus f_R \tilde{\subseteq} b \oplus f_R$. Hence, $a \oplus f_R = b \oplus f_R$.

Conversely, assume that $a \oplus f_R = b \oplus f_R$. It follows that $(f_R(a - b) \cap \lambda) \cup \mu = (b \oplus f_R)(a) = (a \oplus f_R)(a) = (f_R(0) \cap \lambda) \cup \mu$. □

Based on the above proposition, we give a property related to soft cosets as follows.

Proposition 25. *Let f_R be a (λ,μ)-soft union ideal over U and $a,b,x,y\in R$. If*

$$\begin{aligned} x\oplus f_R &= a\oplus f_R,\\ y\oplus f_R &= b\oplus f_R,\\ \text{then}\ \ (x+y)\oplus f_R &= (a+b)\oplus f_R,\\ xy\oplus f_R &= ab\oplus f_R. \end{aligned} \tag{15}$$

Proof. Suppose that $x\oplus f_R = a\oplus f_R$, $y\oplus f_R = b\oplus f_R$. Then, $(f_R(x-a)\cap\lambda)\cup\mu = (f_R(0)\cap\lambda)\cup\mu$ and $(f_R(y-b)\cap\lambda)\cup\mu = (f_R(0)\cap\lambda)\cup\mu$, by Proposition 24. Since f_R is a (λ,μ)-soft union ideal, then

$$\begin{aligned} &(f_R((x+y)-(a+b))\cap\lambda)\cup\mu\\ &\quad= (f_R((x-a)+(y-b))\cap\lambda)\cup\mu\\ &\quad= (f_R((x-a)+(y-b))\cap\lambda\cap\lambda)\cup\mu\\ &\quad\subseteq ((f_R(x-a)\cup f_R(y-b)\cup\mu)\cap\lambda)\cup\mu\\ &\quad= (f_R(x-a)\cap\lambda)\cup(f_R(y-b)\cap\lambda)\cup\mu\\ &\quad= (f_R(0)\cap\lambda)\cup\mu. \end{aligned} \tag{16}$$

On the other hand, it follows from Lemma 11 that $(f_R(0)\cap\lambda)\cup\mu\subseteq(f_R((x+y)-(a+b))\cap\lambda)\cup\mu$. Hence, $(f_R((x+y)-(a+b))\cap\lambda)\cup\mu = (f_R(0)\cap\lambda)\cup\mu$, and so $(x+y)\oplus f_R = (a+b)\oplus f_R$.

Moreover,

$$\begin{aligned} &(f_R(xy-ab)\cap\lambda)\cup\mu\\ &\quad= (f_R((x-a)y+a(y-b))\cap\lambda)\cup\mu\\ &\quad= (f_R((x-a)y+a(y-b))\cap\lambda\cap\lambda)\cup\mu\\ &\quad\subseteq ((f_R((x-a)y)\cup f_R(a(y-b))\cup\mu)\cap\lambda)\cup\mu\\ &\quad= (f_R((x-a)y)\cap\lambda\cap\lambda)\\ &\quad\cup(f_R(a(y-b))\cap\lambda\cap\lambda)\cup\mu\\ &\quad\subseteq ((f_R(x-a)\cup\mu)\cap\lambda)\\ &\quad\cup((f_R(y-b)\cup\mu)\cap\lambda)\cup\mu\\ &\quad= ((f_R(x-a)\cap\lambda)\cup\mu)\cup((f_R(y-b)\cap\lambda)\cup\mu)\\ &\quad= (f_R(0)\cap\lambda)\cup\mu. \end{aligned} \tag{17}$$

According to Lemma 11, we get that $(f_R(0)\cap\lambda)\cup\mu\subseteq(f_R(xy-ab)\cap\lambda)\cup\mu$. Therefore, $(f_R(xy-ab)\cap\lambda)\cup\mu = (f_R(0)\cap\lambda)\cup\mu$; that is, $(xy)\oplus f_R = (ab)\oplus f_R$. □

In view of Proposition 25, we have the following result.

Proposition 26. *Let f_R be a (λ,μ)-soft union ideal over U. Then, $(R/f_R,+,\cdot)$ is a ring, where $R/f_R \triangleq \{a\oplus f_R \mid a\in R\}$, $(x\oplus f_R)+(y\oplus f_R)\triangleq(x+y)\oplus f_R$, and $(x\oplus f_R)\cdot(y\oplus f_R)\triangleq(xy)\oplus f_R$, for all $x,y\in R$.*

Proof. It is straightforward. □

Definition 27. Let f_R be a (λ,μ)-soft union ideal over U. Then, $(R/f_R,+,\cdot)$ is called a soft quotient ring.

Theorem 28. *Let f_R be a (λ,μ)-soft union ideal over U. Then, $R/f_R^*\cong R/f_R$.*

Proof. Assume that $g: R\to R/f_R$ such that $g(x)=x\oplus f_R$, for all $x\in R$. It is easy to see that g is a surjective homomorphism from R to R/f_R. Since $\mathrm{Ker}(g)=\{x\in R\mid g(x)=0\oplus f_R\}=\{x\in R\mid x\oplus f_R=0\oplus f_R\}=\{x\in R\mid (f_R(x)\cap\lambda)\cup\mu=(f_R(0)\cap\lambda)\cup\mu\}=f_R^*$, therefore $R/f_R^*\cong R/f_R$. □

5. Quotient Soft Union Rings

In this section, quotient soft union rings are constructed by introducing the notion of quotient soft subsets and some isomorphism theorems of (λ,μ)-soft union rings related to invariant soft sets are discussed.

Definition 29. Let f_R be a (λ,μ)-soft union ideal over U and g_R a soft set. Then, a quotient soft subset g_R/f_R of R/f_R is defined by

$$g_R/f_R(r\oplus f_R)=\cap\{g_R(x)\mid x\in R, x\oplus f_R=r\oplus f_R\}, \tag{18}$$

for all $r\in R$.

We will illustrate the above concept by the following example.

Example 30. Consider the $(\{1,2,4\},\{1,2\})$-soft union ideal f_R over $\mathbb{Z}^+$ in Example 15. We construct a soft set g_R over U by $g_R(0)=\{1,2,3,5,6\}$, $g_R(1)=\{1,3,5,6\}$, $g_R(2)=\{1,2,4,5\}$, and $g_R(3)=\{3,6\}$. We define a soft set g_R/f_R such that $g_R/f_R(0\oplus f_R)=g_R/f_R(2\oplus f_R)=\{1,2,5\}$, $g_R/f_R(1\oplus f_R)=g_R/f_R(3\oplus f_R)=\{3,6\}$. Then, one can show that g_R/f_R is a quotient soft subset of R/f_R.

Proposition 31. *Let f_R be a (λ,μ)-soft union ideal of R over U. Then, $\widehat{f}_R$ is a (λ,μ)-soft union ideal of R/f_R, where $\widehat{f}_R(x\oplus f_R)\triangleq f_R(x)$, for all $x\in R$.*

Proof. It is easy to see that $\widehat{f}_R$ is a soft set. For all $x,y\in R$, we have

$$\begin{aligned} &\widehat{f}_R((x\oplus f_R)-(y\oplus f_R))\cap\lambda\\ &\quad= \widehat{f}_R((x-y)\oplus f_R)\cap\lambda\\ &\quad= f_R(x-y)\cap\lambda\subseteq f_R(x)\cup f_R(y)\cup\mu\\ &\quad= \widehat{f}_R(x\oplus f_R)\cup\widehat{f}_R(y\oplus f_R)\cup\mu. \end{aligned} \tag{19}$$

Moreover, since $\widehat{f}_R((x\oplus f_R)\cdot(y\oplus f_R))\cap\lambda=\widehat{f}_R((xy)\oplus f_R)\cap\lambda=f_R(xy)\cap\lambda\subseteq(f_R(x)\cap f_R(y))\cup\mu=(\widehat{f}_R(x\oplus f_R)\cap\widehat{f}_R(y\oplus f_R))\cup\mu$, for all $x,y\in R$, then $\widehat{f}_R$ is a (λ,μ)-soft union ideal of R/f_R. □

Proposition 32. *Let f_R be a (λ,μ)-soft union ideal and g_R a soft union ring. Then, g_R/f_R is a soft union ring of R/f_R.*

Proof. For all $x, y \in R$, we have

$$\begin{aligned} & g_R/f_R\left((x \oplus f_R) - (y \oplus f_R)\right) \\ & \quad = g_R/f_R\left((x - y) \oplus f_R\right) \\ & \quad = \cap \ \{g_R(u - v) \mid u, v \in R, (u - v) \\ & \qquad \oplus f_R = (x - y) \oplus f_R\} \\ & \quad \subseteq \cap \{g_R(u) \cup g_R(v) \mid u, v \in R, \\ & \qquad u \oplus f_R = x \oplus f_R, v \oplus f_R = y \oplus f_R\} \\ & \quad = \cap \{g_R(u) \mid u \in R, u \oplus f_R = x \oplus f_R\} \\ & \quad \cup \cap \{g_R(v) \mid v \in R, v \oplus f_R = y \oplus f_R\} \\ & \quad = g_R/f_R(x \oplus f_R) \cup g_R/f_R(y \oplus f_R). \end{aligned} \tag{20}$$

Similarly, we can show that $g_R/f_R((x \oplus f_R) \cdot (y \oplus f_R)) \subseteq g_R/f_R(x \oplus f_R) \cup g_R/f_R(y \oplus f_R)$, for all $x, y \in R$. Thus, g_R/f_R is a soft union ring of R/f_R. □

Definition 33. Let f_R be a (λ, μ)-soft union ideal and g_R a soft union ring. Then, $(g_R/f_R, +, \cdot)$ is called a quotient soft union ring of R/f_R.

In order to investigate isomorphism theorems of (λ, μ)-soft union rings, we need a concept of invariant soft sets which will play an important role in the sequel.

Definition 34. Let $\Psi : R_1 \to R_2$ be a ring homomorphism. A soft set f_{R_1} over U is called an invariant soft set with respect to Ψ if $\Psi(x_1) = \Psi(x_2)$ implies $f_{R_1}(x_1) = f_{R_1}(x_2)$, for all $x_1, x_2 \in R_1$.

Proposition 35. *Let $\Psi : R_1 \to R_2$ be a ring homomorphism and f_{R_2} a soft set of R_2 over U. Then, $\Psi^{-1}(f_{R_2})$ is an invariant soft set with respect to Ψ.*

Proof. Let $x_1, x_2 \in R_1$ such that $\Psi(x_1) = \Psi(x_2)$. Then, $\Psi^{-1}(f_{R_2})(x_1) = f_{R_2}(\Psi(x_1)) = f_{R_2}(\Psi(x_2)) = \Psi^{-1}(f_{R_2})(x_2)$. Hence, $\Psi^{-1}(f_{R_2})$ is an invariant soft set with respect to Ψ. □

Next, we establish isomorphism theorems of (λ, μ)-soft union rings.

Theorem 36 (first isomorphism theorem). *Let $\Psi : R_1 \to R_2$ be an epimorphism and let (λ, μ)-soft union ideal f_{R_1} be a invariant soft set with respect to Ψ. Then, $R_1/f_{R_1} \cong R_2/\Psi(f_{R_1})$.*

Proof. Let $\psi : R_1 \to R_2/\Psi(f_{R_1})$ be a mapping such that $\psi(x) = \Psi(x) \oplus \Psi(f_{R_1})$, for all $x \in R_1$. Obviously, ψ is an epimorphism. Since f_{R_1} is an invariant soft set with respect to Ψ, then $\mathrm{Ker}(\psi) = \{x \in R_1 \mid \psi(x) = 0_2 \oplus \Psi(f_{R_1})\} = \{x \in R_1 \mid \Psi(x) \oplus \Psi(f_{R_1}) = 0_2 \oplus \Psi(f_{R_1})\} = \{x \in R_1 \mid (\Psi(f_{R_1})(\Psi(x)) \cap \lambda) \cup \mu = (\Psi(f_{R_1})(0_2) \cap \lambda) \cup \mu\} = \{x \in R_1 \mid x \oplus f_{R_1} = 0_1 \oplus f_{R_1}\} = f^*_{R_1}$. Therefore, $R_1/f^*_{R_1} \cong R_2/\Psi(f_{R_1})$. By Theorem 28, we have $R_1/f^*_{R_1} \cong R_1/f_{R_1}$. Hence, $R_1/f_{R_1} \cong R_2/\Psi(f_{R_1})$. □

Proposition 37. *Let $\Psi : R_1 \to R_2$ be an epimorphism and (λ, μ)-soft union ideal and f_{R_2} a invariant soft set with respect to Ψ. Then, $R_1/\Psi^{-1}(f_{R_2}) \cong R_2/f_{R_2}$.*

Proof. It follows from Theorem 28 and Proposition 35 that $\Psi^{-1}(f_{R_2})$ is a (λ, μ)-soft union ideal of R_1 and $\Psi^{-1}(f_{R_2})$ is an invariant soft set with respect to Ψ. Since Ψ is an epimorphism, then $\Psi(\Psi^{-1}(f_{R_2})) = f_{R_2}$. By Theorem 36, we get that $R_1/\Psi^{-1}(f_{R_2}) \cong R_2/f_{R_2}$. □

Theorem 38. *Let $\Psi : R_1 \to R_2$ be an epimorphism, g_{R_1} a soft union ring of R_1, and f_{R_1} a (λ, μ)-soft union ideal of R_1. If f_{R_1} is an invariant soft set with respect to Ψ, then $g_{R_1}/f_{R_1} \cong \Psi(g_{R_1})/\Psi(f_{R_1})$.*

Proof. Let $\varphi : R_1/f_{R_1} \to R_2/\Psi(f_{R_1})$ be a mapping such that $\varphi(x \oplus f_{R_1}) = \Psi(x) \oplus \Psi(f_{R_1})$, for all $x \in R_1$. By Theorem 36, we have that φ is a ring isomorphism. Let $y \in R_2$. Since Ψ is a subjective homomorphism, then there exists $x \in R_1$ such that $\Psi(x) = y$. Considering that f_{R_1} is an invariant soft set with respect to Ψ, we obtain

$$\begin{aligned} & \Psi\left(g_{R_1}\right)/\Psi\left(f_{R_1}\right)\left(y \oplus \Psi\left(f_{R_1}\right)\right) \\ & \quad = \Psi\left(g_{R_1}\right)/\Psi\left(f_{R_1}\right)\left(\Psi(x) \oplus \Psi\left(f_{R_1}\right)\right) \\ & \quad = \cap \left\{\Psi\left(g_{R_1}\right)(v) \mid v \in R_2, v \oplus \Psi\left(f_{R_1}\right)\right. \\ & \qquad \left. = \Psi(x) \oplus \Psi\left(f_{R_1}\right)\right\} \\ & \quad = \cap \left\{\cap \left\{g_{R_1}(u) \mid u \in R_1, \Psi(u) = v\right\} \mid v \in R_2,\right. \\ & \qquad \left. v \oplus \Psi\left(f_{R_1}\right) = \Psi(x) \oplus \Psi\left(f_{R_1}\right)\right\} \\ & \quad = \cap \left\{g_{R_1}(u) \mid u \in R_1, \Psi(u) \oplus \Psi\left(f_{R_1}\right)\right. \\ & \qquad \left. = \Psi(x) \oplus \Psi\left(f_{R_1}\right)\right\} \\ & \quad = \cap \left\{g_{R_1}(u) \mid u \in R_1, \left(\Psi\left(f_{R_1}\right)(\Psi(u) - \Psi(x)) \cap \lambda\right) \cup \mu\right. \\ & \qquad \left. = \left(\Psi\left(f_{R_1}\right)(0_2) \cap \lambda\right) \cup \mu\right\} \\ & \quad = \cap \left\{g_{R_1}(u) \mid u \in R_1, \left(f_{R_1}(u - x) \cap \lambda\right) \cup \mu\right. \\ & \quad \left. = \left(f_{R_1}(0_1) \cap \lambda\right) \cup \mu\right\} \\ & \quad = \cap \left\{g_{R_1}(u) \mid u \in R_1, u \oplus f_{R_1} = x \oplus f_{R_1}\right\} \\ & \quad = g_{R_1}/f_{R_1}\left(x \oplus f_{R_1}\right) \\ & \quad = \varphi^{-1}\left(\varphi\left(g_{R_1}/f_{R_1}\right)\right)\left(x \oplus f_{R_1}\right) \\ & \quad = \varphi\left(g_{R_1}/f_{R_1}\right)\left(\varphi\left(x \oplus f_{R_1}\right)\right) \\ & \quad = \varphi\left(g_{R_1}/f_{R_1}\right)\left(\Psi(x) \oplus \Psi\left(f_{R_1}\right)\right) \\ & \quad = \varphi\left(g_{R_1}/f_{R_1}\right)\left(y \oplus \Psi\left(f_{R_1}\right)\right). \end{aligned} \tag{21}$$

Hence, $\varphi(g_{R_1}/f_{R_1}) = \Psi(g_{R_1})/\Psi(f_{R_1})$, and so $g_{R_1}/f_{R_1} \cong \Psi(g_{R_1})/\Psi(f_{R_1})$. □

Proposition 39. *Let $\Psi : R_1 \to R_2$ be an epimorphism. If g_{R_2} is a soft union ring of R_2 and f_{R_2} is a (λ, μ)-soft union ideal of R_2, then $\Psi^{-1}(g_{R_2})/\Psi^{-1}(f_{R_2}) \cong g_{R_2}/f_{R_2}$.*

Proof. Since Ψ is a subjective homomorphism, then $\Psi(\Psi^{-1}(f_{R_2})) = f_{R_2}$ and $\Psi(\Psi^{-1}(g_{R_2})) = g_{R_2}$. By Proposition 35, we get that $\Psi^{-1}(f_{R_1})$ is an invariant soft set with respect to Ψ. It follows from Theorem 38 that $\Psi^{-1}(g_{R_2})/\Psi^{-1}(f_{R_2}) \cong g_{R_2}/f_{R_2}$. □

Lemma 40. *Let f_R and g_R be (λ, μ)-soft union ideals of R such that $g_R \widetilde{\subset} f_R$ and $(f_R(0) \cap \lambda) \cup \mu = (g_R(0) \cap \lambda) \cup \mu$. Then, $x \oplus f_R \oplus g_R/f_R = y \oplus f_R \oplus g_R/f_R$ if and only if $x \oplus g_R = y \oplus g_R$, for all $x, y \in R$.*

Proof. Assume that $x \oplus f_R \oplus g_R/f_R = y \oplus f_R \oplus g_R/f_R$, for all $x, y \in R$. Then, we have that $((g_R/f_R((x-y) \oplus f_R)) \cap \lambda) \cup \mu = ((g_R/f_R(0 \oplus f_R)) \cap \lambda) \cup \mu = (g_R(0) \cap \lambda) \cup \mu$, and so

$$\begin{aligned}
&(\cap \{g_R(t) \mid t \in R, t \oplus f_R = (x-y) \oplus f_R\} \cap \lambda) \cup \mu \\
&\quad = \cap \{(g_R(t) \cap \lambda) \cup \mu \mid t \in R, t \oplus f_R = (x-y) \oplus f_R\} \\
&\quad = (g_R(0) \cap \lambda) \cup \mu.
\end{aligned} \tag{22}$$

If $t \oplus f_R = (x-y) \oplus f_R$, then $(f_R(x-y-t) \cap \lambda) \cup \mu = (f_R(0) \cap \lambda) \cup \mu$. Therefore,

$$\begin{aligned}
&(g_R(x-y) \cap \lambda) \cup \mu \\
&\quad = (g_R(x-y-t+t) \cap \lambda) \cup \mu \\
&\quad = (g_R(x-y-t+t) \cap \lambda \cap \lambda) \cup \mu \\
&\quad \subseteq ((g_R(x-y-t) \cup g_R(t) \cup \mu) \cap \lambda) \cup \mu \\
&\quad = ((g_R(x-y-t) \cap \lambda) \cup \mu) \cup ((g_R(t) \cap \lambda) \cup \mu) \\
&\quad \subseteq ((f_R(x-y-t) \cap \lambda) \cup \mu) \cup ((g_R(t) \cap \lambda) \cup \mu) \\
&\quad = ((f_R(0) \cap \lambda) \cup \mu) \cup ((g_R(t) \cap \lambda) \cup \mu) \\
&\quad \subseteq (g_R(t) \cap \lambda) \cup \mu.
\end{aligned} \tag{23}$$

It follows that $(g_R(x-y) \cap \lambda) \cup \mu \subseteq \cap\{(g_R(t) \cap \lambda) \cup \mu \mid t \in R, t \oplus f_R = (x-y) \oplus f_R\} = (g_R(0) \cap \lambda) \cup \mu$. On the other hand, we have $(g_R(0) \cap \lambda) \cup \mu \subseteq (g_R(x-y) \cap \lambda) \cup \mu$, by Lemma 11. Hence, $(g_R(x-y) \cap \lambda) \cup \mu = (g_R(0) \cap \lambda) \cup \mu$. And thus, $x \oplus g_R = y \oplus g_R$.

Conversely, suppose that $x \oplus g_R = y \oplus g_R$, for all $x, y \in R$. It follows that $(g_R(x-y) \cap \lambda) \cup \mu = (g_R(0) \cap \lambda) \cup \mu = (g_R/f_R(0 \oplus f_R) \cap \lambda) \cup \mu$. Therefore,

$$\begin{aligned}
&(g_R((x-y) \oplus f_R) \cap \lambda) \cup \mu \\
&\quad = (\cap \{g_R(t) \mid t \in R, \\
&\qquad\qquad t \oplus f_R = (x-y) \oplus f_R\} \cap \lambda) \cup \mu \\
&\quad = \cap \{(g_R(t) \cap \lambda) \cup \mu \mid t \in R, \\
&\qquad\qquad t \oplus f_R = (x-y) \oplus f_R\} = (g_R(0) \cap \lambda) \cup \mu \\
&\quad = ((g_R/f_R(0 \oplus f_R)) \cap \lambda) \cup \mu;
\end{aligned} \tag{24}$$

that is, $(g_R/f_R((x \oplus f_R) - (y \oplus f_R)) \cap \lambda) \cup \mu = (g_R/f_R(0 \oplus f_R) \cap \lambda) \cup \mu$. It follows from Proposition 24 that $x \oplus f_R \oplus g_R/f_R = y \oplus f_R \oplus g_R/f_R$. □

Theorem 41 (third isomorphism theorem). *Let f_R and g_R be (λ, μ)-soft union ideals of R such that $g_R \widetilde{\subseteq} f_R$ and $(f_R(0) \cap \lambda) \cup \mu = (g_R(0) \cap \lambda) \cup \mu$. If h_R is a soft union ring, then $(h_R/f_R)/(g_R/f_R) \cong h_R/g_R$.*

Proof. Let $\Psi : (R/f_R)/(R/f_R) \to R/g_R$ be a mapping such that $\Psi(r \oplus f_R \oplus g_R/f_R) = r \oplus g_R$, for all $r \in R$. According to Lemma 40, we obtain that Ψ is an injective mapping. Obviously, Ψ is subjective. For all $x, y \in R$, we have

$$\begin{aligned}
&\Psi((x \oplus f_R \oplus g_R/f_R) + (y \oplus f_R \oplus g_R/f_R)) \\
&\quad = \Psi((x+y) \oplus f_R \oplus g_R/f_R) \\
&\quad = (x+y) \oplus g_R = (x \oplus g_R) + (y \oplus g_R), \\
&\Psi((x \oplus f_R \oplus g_R/f_R) \cdot (y \oplus f_R \oplus g_R/f_R)) \\
&\quad = \Psi((xy) \oplus f_R \oplus g_R/f_R) \\
&\quad = (xy) \oplus g_R = (x \oplus g_R) \cdot (y \oplus g_R).
\end{aligned} \tag{25}$$

Thus, Ψ is a ring isomorphism.

Moreover, since

$$\begin{aligned}
&((h_R/f_R)/(g_R/f_R))(x \oplus f_R \oplus g_R/f_R) \\
&\quad = \cap \{h_R/f_R(u \oplus f_R) \mid u \in R, u \oplus f_R \oplus g_R/f_R \\
&\qquad\qquad = x \oplus f_R \oplus g_R/f_R\} \\
&\quad = \cap \{\cap \{h_R(v) \mid v \in R, v \oplus f_R \\
&\qquad\qquad = u \oplus f_R\} \mid u \in R, u \oplus f_R \oplus g_R/f_R \\
&\qquad\qquad = x \oplus f_R \oplus g_R/f_R\} \\
&\quad = \cap \{h_R(v) \mid v \in R, v \oplus f_R \oplus g_R/f_R \\
&\qquad\qquad = x \oplus f_R \oplus g_R/f_R\} \\
&\quad = \cap \{h_R(v) \mid v \in R, v \oplus g_R = x \oplus g_R\} \\
&\quad = (h_R/g_R)(x \oplus g_R) \\
&\quad = (h_R/g_R)(\Psi(x \oplus f_R \oplus g_R/f_R)) \\
&\quad = \Psi^{-1}((h_R/g_R)(x \oplus f_R \oplus g_R/f_R)),
\end{aligned} \tag{26}$$

that is, $(h_R/f_R)/(g_R/f_R) = \Psi^{-1}(h_R/g_R)$ therefore $\Psi((h_R/f_R)/(g_R/f_R)) = h_R/g_R$, and so $(h_R/f_R)/(g_R/f_R) \cong h_R/g_R$. □

6. Conclusions

We introduced the notions of (λ, μ)-soft union rings and (λ, μ)-soft union ideals which are generations of that of soft union rings and soft union ideals in the paper. Then, we established soft quotient rings based on (λ, μ)-soft union ideals by using soft cosets and gave an approach for constructing quotient soft union rings. Finally, we considered isomorphism theorems of (λ, μ)-soft union rings related to invariant soft sets. Motivated by the notion of soft intersection near-rings in [25], we will investigate the properties of (λ, μ)-soft near-rings. In addition, it is interesting to apply the theory of soft union (intersection) sets to other algebraic structures such as modules.

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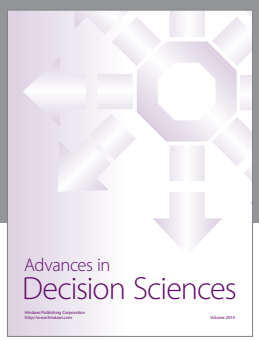

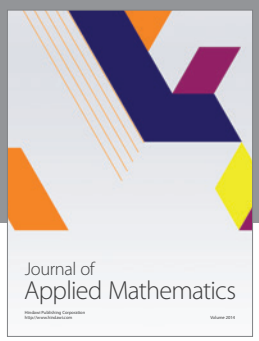

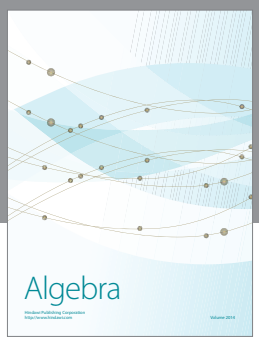

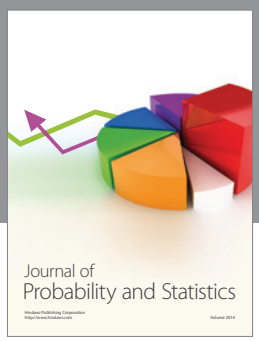

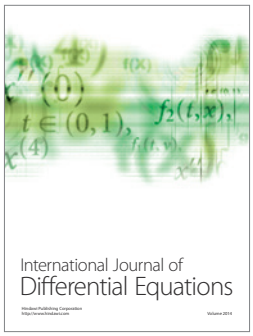

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