Research Article

Coincidence Points of Weaker Contractions in Partially Ordered Metric Spaces

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Received 6 June 2013; Accepted 4 August 2013

Academic Editor: E. Karapinar

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We prove new coincidence point theorems for the $(\varphi, \psi, \phi, \xi)$ -contractions and generalized Meir-Keeler-type α - ψ -contractions in partially ordered metric spaces. Our results generalize many recent coincidence point theorems in the literature.

1. Introduction and Preliminaries

Throughout this paper, by $\mathbb{R}^+$, we denote the set of all nonnegative real numbers, while $\mathbb{N}$ is the set of all natural numbers. Let (X, d) be a metric space, D a subset of X , and $f : D \rightarrow X$ a map. We say f is contractive if there exists $\alpha \in [0, 1)$ such that for all $x, y \in D$,

$$d(fx, fy) \leq \alpha \cdot d(x, y). \quad (1)$$

The well-known Banach's fixed point theorem asserts that if $D = X$, f is contractive and (X, d) is complete, then f has a unique fixed point in X . It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. Also, this principle has many generalizations. For instance, a mapping $f : X \rightarrow X$ is called a quasicontraction if there exists $k < 1$ such that

$$d(fx, fy) \leq k \cdot \max \{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}, \quad (2)$$

for any $x, y \in X$. In 1974, Ćirić [2] introduced these maps and proved an existence and uniqueness fixed point theorem.

Recently, Eslamian and Abkar proved the following theorem.

Theorem 1 (see [3]). *Let (X, d) be a complete metric space and $f : X \rightarrow X$ be such that*

$$\psi(d(fx, fy)) \leq \alpha(d(x, y)) - \beta(d(x, y)), \quad (3)$$

for each $x, y \in X$,

where $\psi, \alpha, \beta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are as follows: ψ is continuous and nondecreasing, α is continuous, β is lower semicontinuous, and

$$\begin{aligned} \psi(t) - \alpha(t) + \beta(t) &> 0 \quad \forall t > 0, \\ \psi(t) = 0 &\text{ iff } t = 0, \quad \alpha(0) = \beta(0) = 0. \end{aligned} \quad (4)$$

Then f has a fixed point in X .

Recently, fixed point theory has developed rapidly in partially ordered metric spaces (e.g., [4–8]).

In 2012, Choudhury and Kundu [9] proved the following coincidence theorem as a generalization of Theorem 1.

Theorem 2 (see [9]). *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space and $f, g : X \rightarrow X$ be such that $fX \subset gX$, f is g -nondecreasing, gX is closed, and*

$$\psi(d(fx, fy)) \leq \alpha(d(gx, gy)) - \beta(d(gx, gy)), \quad (5)$$

for each $x, y \in X$ such that $gx \sqsubseteq gy$,

where $\psi, \alpha, \beta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are such that ψ is continuous and nondecreasing, α is continuous, β is lower semicontinuous, and

$$\begin{aligned} \psi(t) - \alpha(t) + \beta(t) &> 0 \quad \forall t > 0, \\ \psi(t) &= 0 \quad \text{iff } t = 0, \quad \alpha(0) = \beta(0) = 0. \end{aligned} \quad (6)$$

Also, if any nondecreasing sequence $\{x_n\}$ in X converges to v , then we assume that

$$x_n \sqsubseteq v \quad \forall n \in \mathbb{N}. \quad (7)$$

If there exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$, then f and g have a coincidence point in X .

In this paper, we prove new coincidence point theorems for the $(\varphi, \psi, \phi, \xi)$ -contractions and generalized Meir-Keeler-type α - ψ -contractions in partially ordered metric spaces. Our results generalize many recent coincidence point theorems in the literature.

2. Main Results

We start with the following definition.

Definition 3 (g -nondecreasing mapping [4]). Let $(X, \sqsubseteq)$ be a partially ordered set and $f, g : X \rightarrow X$. Then f is said to be g -nondecreasing if, for $x, y \in X$,

$$gx \sqsubseteq gy \implies fx \sqsubseteq fy. \quad (8)$$

In the sequel, we denote by Ψ the class of functions $\psi : \mathbb{R}^{+5} \rightarrow \mathbb{R}^+$ satisfying the following conditions:

- (ψ_1) ψ is an increasing, continuous function in each coordinate,
- (ψ_2) for all $t \in \mathbb{R}^+$, $\psi(t, t, t, 0, 2t) \leq t$, $\psi(t, t, t, 2t, 0) \leq t$, $\psi(0, 0, t, t, 0) \leq t$, and $\psi(t, 0, 0, t, t) \leq t$,
- (ψ_3) $\psi(t_1, t_2, t_3, t_4, t_5) = 0$ if and only if $t_1 = t_2 = t_3 = t_4 = t_5 = 0$.

Next, we denote by Φ the class of functions $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying the following conditions:

- (ϕ_1) ϕ is a continuous function and monotone nondecreasing;
- (ϕ_2) $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$;
- (ϕ_3) ϕ is subadditive, that is, $\phi(t_1 + t_2) \leq \phi(t_1) + \phi(t_2)$ for all $t_1, t_2 > 0$.

And, we denote the following sets of functions:

$$\Theta = \{\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ such that } \varphi \text{ is continuous}\},$$

$$\Xi = \{\xi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ such that } \xi \text{ is lower continuous}\}. \quad (9)$$

Let X be a nonempty set and $(X, \sqsubseteq)$ be a partially ordered set endowed with a metric d . Then, the triple $(X, \sqsubseteq, d)$ is called a partially ordered metric space.

We now state the $(\varphi, \psi, \phi, \xi)$ -contraction and the main fixed point theorem for the $(\varphi, \psi, \phi, \xi)$ -contraction in partially ordered metric spaces, as follows.

Definition 4. Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, and let $f, g : X \rightarrow X$. Then the pair (f, g) is called a $(\varphi, \psi, \phi, \xi)$ -contraction if the following inequality holds:

$$\begin{aligned} &\varphi(d(fx, fy)) \\ &\leq \psi(\phi(d(gx, gy)), \phi(d(gx, fx)), \\ &\quad \phi(d(gy, fy)), \phi(d(gx, fy)), \phi(d(gy, fx))) \\ &\quad - \xi(\max\{d(gx, gy), d(gx, fx), d(gy, fy)\}), \end{aligned} \quad (10)$$

for all $x, y \in X$ with $gx \sqsubseteq gy$, where $\varphi \in \Theta$, $\psi \in \Psi$, $\phi \in \Phi$ and $\xi \in \Xi$.

We now state the main fixed point theorem for the $(\varphi, \psi, \phi, \xi)$ -contraction in partially ordered metric spaces, as follows.

Theorem 5. Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, and let $f, g : X \rightarrow X$ be such that $fX \subset gX$, f is g -nondecreasing and gX is closed. Suppose the pair (f, g) is a $(\varphi, \psi, \phi, \xi)$ -contraction, and

$$\begin{aligned} \varphi(t) - \phi(t) + \xi(t) &> 0 \quad \forall t > 0, \\ \varphi(t) &= 0 \quad \text{iff } t = 0, \quad \phi(0) = \xi(0) = 0. \end{aligned} \quad (11)$$

Also, if any nondecreasing sequence $\{x_n\}$ in X converges to v , then we assume that

$$x_n \sqsubseteq v \quad \forall n \in \mathbb{N}. \quad (12)$$

If there exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$, then f and g have a coincidence point in X .

Proof. Since $fX \subset gX$ and there exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$, we can choose $x_1 \in X$ such that $gx_1 = fx_0$. Since f is g -nondecreasing, we have $fx_0 \sqsubseteq fx_1$. In this process, we construct the sequence $\{x_n\}$ recursively as

$$fx_n = gx_{n+1} \quad \forall n \in \mathbb{N}. \quad (13)$$

Thus, we also conclude that

$$\begin{aligned} gx_0 \sqsubseteq fx_0 &= gx_1 \sqsubseteq fx_1 = gx_2 \sqsubseteq \cdots \sqsubseteq fx_{n-1} \\ &= gx_n \sqsubseteq fx_n = gx_{n+1} \sqsubseteq \cdots \end{aligned} \quad (14)$$

If any two consecutive terms in (14) are equal, then the conclusion of the theorem follows. So we may assume that

$$d(fx_{n-1}, fx_n) \neq 0, \quad \forall n \in \mathbb{N}. \quad (15)$$

Now, we claim that $d(fx_n, fx_{n+1}) \leq d(fx_{n-1}, fx_n)$ for all $n \in \mathbb{N}$. If not, we assume that $d(fx_{n-1}, fx_n) < d(fx_n, fx_{n+1})$ for

some $n \in \mathbb{N}$, substituting $x = x_n$ and $y = x_{n+1}$ in (10) and using the definition of the function ψ , we have

$$\begin{aligned}
 & \psi(\phi(d(gx_n, gx_{n+1})), \phi(d(gx_n, fx_n)), \\
 & \quad \phi(d(gx_{n+1}, fx_{n+1})), \phi(d(gx_n, fx_{n+1})), \\
 & \quad \phi(d(gx_{n+1}, fx_n))) \\
 &= \psi(\phi(d(fx_{n-1}, fx_n)), \phi(d(fx_{n-1}, fx_{n+1})), \\
 & \quad \phi(d(fx_n, fx_{n+1})), \phi(d(fx_{n-1}, fx_{n+1})), \\
 & \quad \phi(d(fx_n, fx_n))) \\
 &\leq \psi(\phi(d(fx_n, fx_{n+1})), \phi(d(fx_n, fx_{n+1})), \\
 & \quad \phi(d(fx_n, fx_{n+1})), 2\phi(d(fx_n, fx_{n+1})), \phi(0)) \\
 &\leq \phi(d(fx_n, fx_{n+1})), \\
 &\xi(\max\{d(gx_n, gx_{n+1}), d(gx_n, fx_n), \\
 & \quad d(gx_{n+1}, fx_{n+1})\}) \\
 &= \xi(\max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), \\
 & \quad d(fx_n, fx_{n+1})\}) \\
 &= \xi(d(fx_n, fx_{n+1})),
 \end{aligned} \tag{16}$$

and hence

$$\phi(d(fx_n, fx_{n+1})) \leq \phi(d(fx_n, fx_{n+1})) - \xi(d(fx_n, fx_{n+1})). \tag{17}$$

Since $\varphi(t) - \phi(t) + \xi(t) > 0$ for all $t > 0$, we have that $d(fx_n, fx_{n+1}) = 0$, which contradicts to (15). Therefore, we conclude that

$$d(fx_n, fx_{n+1}) \leq d(fx_{n-1}, x_n) \quad \forall n \in \mathbb{N}. \tag{18}$$

From above argument, we also have that for each $n \in \mathbb{N}$

$$\phi(d(fx_n, fx_{n+1})) \leq \phi(d(fx_{n-1}, fx_n)) - \xi(d(fx_{n-1}, fx_n)). \tag{19}$$

It follows (18) that the sequence $\{d(fx_n, fx_{n+1})\}$ is monotone decreasing, it must converge to some $\eta \geq 0$. Taking limit as $n \rightarrow \infty$ in (19) and using the continuities of φ and ϕ and the lower semicontinuity of ξ , we get

$$\varphi(\eta) \leq \phi(\eta) - \xi(\eta), \tag{20}$$

which implies that $\eta = 0$. So we conclude that

$$\lim_{n \rightarrow \infty} d(fx_n, fx_{n+1}) = 0. \tag{21}$$

We next claim that $\{fx_n\}$ is a Cauchy sequence, that is, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $p, q \geq n$, then $d(fx_p, fx_q) < \varepsilon$.

Suppose the above statement is false. Then there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$, there are $p_n, q_n \in \mathbb{N}$ with $p_n > q_n \geq n$ satisfying

$$d(fx_{q_n}, fx_{p_n}) \geq \varepsilon. \tag{22}$$

Further, corresponding to $q_n \geq n$, we can choose p_n in such a way that it is the smallest integer with $p_n > q_n \geq n$ and $d(fx_{q_n}, fx_{p_n}) \geq \varepsilon$. Therefore $d(fx_{q_n}, fx_{p_n-1}) < \varepsilon$. Now we have that for all $n \in \mathbb{N}$

$$\begin{aligned}
 \varepsilon &\leq d(fx_{p_n}, fx_{q_n}) \\
 &\leq d(fx_{p_n}, fx_{p_n-1}) + d(fx_{p_n-1}, fx_{q_n}) \\
 &< d(fx_{p_n}, fx_{p_n-1}) + \varepsilon.
 \end{aligned} \tag{23}$$

Letting $n \rightarrow \infty$, then we get

$$\lim_{n \rightarrow \infty} d(fx_{p_n}, fx_{q_n}) = \varepsilon. \tag{24}$$

On the other hand, we have

$$\begin{aligned}
 d(fx_{p_n}, fx_{q_n}) &\leq d(fx_{p_n}, fx_{p_n-1}) + d(fx_{p_n-1}, fx_{q_n-1}) \\
 &\quad + d(fx_{q_n-1}, fx_{q_n}), \\
 d(fx_{p_n-1}, fx_{q_n-1}) &\leq d(fx_{p_n-1}, fx_{p_n}) + d(fx_{p_n}, fx_{q_n}) \\
 &\quad + d(fx_{q_n}, fx_{q_n-1}).
 \end{aligned} \tag{25}$$

Letting $n \rightarrow \infty$, then we get

$$\lim_{n \rightarrow \infty} d(fx_{p_n-1}, fx_{q_n-1}) = \varepsilon. \tag{26}$$

By (14), we have that the elements gx_{p_n} and gx_{q_n} are comparable. Substituting $x = x_{p_n}$ and $y = x_{q_n}$ in (10), we have that for all $n \in \mathbb{N}$,

$$\begin{aligned}
 & \psi(\phi(d(gx_{p_n}, gx_{q_n})), \phi(d(gx_{p_n}, fx_{p_n})), \\
 & \quad \phi(d(gx_{q_n}, fx_{q_n})), \phi(d(gx_{p_n}, fx_{q_n})), \\
 & \quad \phi(d(gx_{q_n}, fx_{p_n}))) \\
 &\leq \psi(\phi(d(fx_{p_n-1}, fx_{q_n-1})), \phi(d(fx_{p_n-1}, fx_{p_n})), \\
 & \quad \phi(d(fx_{q_n-1}, fx_{q_n})), \phi(d(fx_{p_n-1}, fx_{q_n})), \\
 & \quad \phi(d(fx_{q_n-1}, fx_{p_n}))) \\
 &\leq \psi(\phi(d(fx_{p_n-1}, fx_{q_n-1})), \phi(d(fx_{p_n-1}, fx_{p_n})), \\
 & \quad \phi(d(fx_{q_n-1}, fx_{q_n})), \phi(d(fx_{p_n-1}, fx_{p_n})) \\
 & \quad + \phi(d(fx_{p_n}, fx_{q_n})), \phi(d(fx_{q_n-1}, fx_{q_n})) \\
 & \quad + \phi(d(fx_{q_n}, fx_{p_n}))),
 \end{aligned}$$

$$M(x_{p_n}, x_{q_n}) = \max\{d(gx_{p_n}, gx_{q_n}),$$

$$\begin{aligned}
 & \quad d(gx_{p_n}, fx_{p_n}), d(gx_{q_n}, fx_{q_n})\} \\
 &= \max\{d(fx_{p_n-1}, fx_{q_n-1}), d(fx_{p_n-1}, fx_{p_n}), \\
 & \quad d(fx_{q_n-1}, fx_{q_n})\}.
 \end{aligned} \tag{27}$$

By above argument and using inequality (10), we can conclude that

$$\begin{aligned}\varphi(\epsilon) &\leq \psi(\phi(\epsilon), 0, 0, \phi(\epsilon), \phi(\epsilon)) - \xi(\epsilon) \\ &\leq \phi(\epsilon) - \xi(\epsilon),\end{aligned}\quad (28)$$

which implies that $\epsilon = 0$, a contradiction. Therefore, the sequence $\{fx_n\}$ is a Cauchy sequence.

Since X is complete and gX is closed, there exists $\nu \in X$ such that

$$\lim_{n \rightarrow \infty} gx_n = \lim_{n \rightarrow \infty} fx_n = g\nu. \quad (29)$$

Later, we prove that ν is a coincidence point of f and g . From (14) and (29), we deduce that

$$gx_n \sqsubseteq g\nu, \quad \forall n \in \mathbb{N}. \quad (30)$$

Substituting $x = x_n$ and $y = \nu$ in (10), we have that

$$\begin{aligned}\varphi(d(fx_n, f\nu)) &\leq \psi(\phi(d(gx_n, g\nu)), \phi(d(gx_n, fx_n)), \\ &\quad \phi(d(g\nu, f\nu)), \phi(d(gx_n, f\nu)), \\ &\quad \phi(d(g\nu, fx_n))) \\ &\quad - \xi(\max\{d(gx_n, g\nu), d(gx_n, fx_n), d(g\nu, f\nu)\}).\end{aligned}\quad (31)$$

Taking $n \rightarrow \infty$ in the above inequality, we have

$$\begin{aligned}\varphi(d(g\nu, f\nu)) &\leq \psi(0, 0, \phi(d(g\nu, f\nu)), \phi(d(g\nu, f\nu)), 0) \\ &\quad - \xi(d(g\nu, f\nu)) \leq \phi(d(g\nu, f\nu)) \\ &\quad - \xi(d(g\nu, f\nu)),\end{aligned}\quad (32)$$

which implies that $d(g\nu, f\nu) = 0$, that is, $g\nu = f\nu$. So we complete the proof. $\square$

We give the following example to illustrate Theorem 5.

Example 6. Let $X = [0, 1]$. We define a partial order " $\sqsubseteq$ " on X as $x \sqsubseteq y$ if and only if $x \geq y$ for all $x, y \in X$. We take the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Let $f, g : X \rightarrow X$ be defined as

$$f(x) = \frac{1}{16}x^2, \quad g(x) = \frac{1}{4}x^2. \quad (33)$$

Let $\varphi, \phi, \xi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be defined as

$$\varphi(t) = \phi(t) = t, \quad \xi(t) = \frac{t}{8} \quad \forall t \in [0, 1], \quad (34)$$

and let $\psi : \mathbb{R}^{+5} \rightarrow \mathbb{R}^+$ denote

$$\psi(t_1, t_2, t_3, t_4, t_5) = \frac{1}{2} \cdot \max\left\{t_1, t_2, t_3, \frac{t_4}{2}, \frac{t_5}{2}\right\}. \quad (35)$$

Without loss of generality, we assume that $x > y$ and verify inequality (10).

For all $x, y \in [0, 1]$ with $x > y$, we have

$$\begin{aligned}\varphi(d(fx, fy)) &= \frac{1}{16}(x^2 - y^2), \\ \phi(d(gx, gy)) &= \frac{1}{4}(x^2 - y^2), \\ \phi(d(gx, fx)) &= \frac{1}{4}x^2 - \frac{1}{16}x^2 = \frac{3}{16}x^2, \\ \phi(d(gy, fy)) &= \frac{1}{4}y^2 - \frac{1}{16}y^2 = \frac{3}{16}y^2, \\ \phi(d(gx, fy)) &= \frac{1}{4}x^2 - \frac{1}{16}y^2 > \frac{3}{16}x^2, \\ \phi(d(fx, gy)) &= \left|\frac{1}{16}x^2 - \frac{1}{4}y^2\right|, \\ \xi(\max\{d(gx, gy), d(gx, fx), d(gy, fy)\})\end{aligned}\quad (36)$$

$$= \begin{cases} \frac{1}{4}(x^2 - y^2), & \text{if } x > 2y, \\ \frac{3}{16}x^2, & \text{if } x \leq 2y, \end{cases}$$

$$\begin{aligned}&\psi(\phi(d(gx, gy)), \phi(d(gx, fx)), \phi(d(gy, fy)), \\ &\quad \phi(d(gx, fy)), \phi(d(fx, gy))) \\ &= \frac{1}{8}x^2 - \frac{1}{32}y^2.\end{aligned}$$

Therefore, inequality (10) is satisfied and all the conditions of Theorem 5 are satisfied, and we obtained that 0 is a coincidence point of f and g .

Applying Definition 4, Theorem 5, and Example 6, if we let

$$\begin{aligned}&\psi(\phi(d(gx, gy)), \phi(d(gx, fx)), \phi(d(gy, fy)), \\ &\quad \phi(d(gx, fy)), \phi(d(gy, fx))) \\ &= \max\left\{\phi(d(gx, gy)), \phi(d(gx, fx)), \phi(d(gy, fy)), \right. \\ &\quad \left. \frac{1}{2}\phi(d(gx, fy)), \frac{1}{2}\phi(d(fx, gy))\right\},\end{aligned}\quad (37)$$

we are easy to get the following theorem.

Theorem 7. Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, and let $f, g : X \rightarrow X$ be such that $fX \subset gX$, f is g -nondecreasing, gX is closed, and

$$\begin{aligned}&\varphi(d(fx, fy)) \\ &\leq \max\{\phi(d(gx, gy)), \phi(d(gx, fx)), \\ &\quad \phi(d(gy, fy)), \phi(d(gx, fy)), \psi(d(fx, gy))\} \\ &\quad - \xi(\max\{d(gx, gy), d(gx, fx), d(gy, fy)\}),\end{aligned}\quad (38)$$

for all $x, y \in X$ such that $gx \sqsubseteq gy$, where $\varphi \in \Theta$, $\psi \in \Psi$, $\phi \in \Phi$ and $\xi \in \Xi$, and

$$\begin{aligned} \varphi(t) - \phi(t) + \xi(t) &> 0 \quad \forall t > 0, \\ \varphi(t) = 0 \quad \text{iff } t = 0, \quad \phi(0) = \xi(0) = 0. \end{aligned} \quad (39)$$

Also, if any nondecreasing sequence $\{x_n\}$ in X converges to v , then one assumes that

$$x_n \sqsubseteq v \quad \forall n \in \mathbb{N}. \quad (40)$$

If there exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$, then f and g have a coincidence point in X .

In the other research of this paper, we recall the Meir-Keeler-type contraction [10] and α -admissible mapping [11]. In 1969, Meir and Keeler [10] introduced the following notion of Meir-Keeler-type contraction in a metric space (X, d) .

Definition 8. Let (X, d) be a metric space, $f : X \rightarrow X$. Then f is called a Meir-Keeler-type contraction whenever for each $\eta > 0$ there exists $\gamma > 0$ such that

$$\eta \leq d(x, y) < \eta + \gamma \implies d(fx, fy) < \eta. \quad (41)$$

And, the following definition was introduced in [11].

Definition 9. Let $f : X \rightarrow X$ be a self-mapping of a set X and $\alpha : X \times X \rightarrow \mathbb{R}^+$. Then f is called a α -admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \implies \alpha(fx, fy) \geq 1. \quad (42)$$

We introduce the notion of α - g -admissible mapping, as follows.

Definition 10. Let $f, g : X \rightarrow X$ be a self-mapping of a set X and $\alpha : X \times X \rightarrow \mathbb{R}^+$. Then f is called a α - g -admissible mapping if

$$x, y \in X, \quad \alpha(gx, gy) \geq 1 \implies \alpha(fx, fy) \geq 1. \quad (43)$$

We give the following example to illustrate Definition 10.

Example 11. Let $X = \mathbb{R}^+$ and we define

$$\begin{aligned} g(x) &= x + \frac{1}{2}, \quad f(x) = g(x) + \frac{1}{x+1}, \\ \alpha(x, y) &= x + y. \end{aligned} \quad (44)$$

Then f is a α - g -admissible mapping.

We now state the new notions of generalized Meir-Keeler-type ψ -contractions and generalized Meir-Keeler-type α - ψ -contractions in partially ordered complete metric spaces, as follows.

Definition 12. Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, and let $f, g : X \rightarrow X$. Then the pair (f, g) is called a generalized Meir-Keeler-type ψ -contraction whenever for each $\eta > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} \eta \leq \psi(d(gx, gy), d(gx, fx), d(gy, fy)), \\ d(gx, fy), d(gy, fx)) \end{aligned} \quad (45)$$

$$< \eta + \delta \implies d(fx, fy) < \eta,$$

for all $x, y \in X$ with $gx \sqsubseteq gy$, where $\psi \in \Psi$.

Definition 13. Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, let $f, g : X \rightarrow X$, and $\alpha : X \times X \rightarrow \mathbb{R}^+$. Then (f, g) is called a generalized Meir-Keeler-type α - ψ -contraction if the following conditions hold:

- (1) f is α - g -admissible;
- (2) for each $\eta > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} \eta \leq \psi(d(gx, gy), d(gx, fx), d(gy, fy), \\ d(gx, fy), d(gy, fx)) \\ < \eta + \delta \implies \alpha(fx, fx) \alpha(gy, gy) d(fx, fy) < \eta, \end{aligned} \quad (46)$$

for all $x, y \in X$ with $gx \sqsubseteq gy$, where $\psi \in \Psi$.

Remark 14. Note that if f is a generalized Meir-Keeler-type α - ψ -contraction, then we have that for all $x, y \in X$

$$\begin{aligned} \alpha(fx, fx) \alpha(gy, gy) d(fx, fy) \\ \leq \psi(d(gx, gy), d(gx, fx), d(gy, fy), \\ d(gx, fy), d(gy, fx)). \end{aligned} \quad (47)$$

Further, if

$$\begin{aligned} \psi(d(gx, gy), d(gx, fx), d(gy, fy), \\ d(gx, fy), d(gy, fx)) = 0, \end{aligned} \quad (48)$$

then $d(fx, fy) = 0$.

On the other hand, if

$$\begin{aligned} \psi(d(gx, gy), d(gx, fx), d(gy, fy), \\ d(gx, fy), d(gy, fx)) > 0, \end{aligned} \quad (49)$$

then

$$\begin{aligned} \alpha(fx, fx) \alpha(gy, gy) d(fx, fy) \\ < \psi(d(gx, gy), d(gx, fx), d(gy, fy), \\ d(gx, fy), d(gy, fx)). \end{aligned} \quad (50)$$

We now state our main result for the generalized Meir-Keeler-type α - ψ -contraction, as follows.

Theorem 15. Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, let $\alpha : X \times X \rightarrow \mathbb{R}^+$ be continuous in each coordinate, and let $f, g : X \rightarrow X$ be such that $fX \subset gX$, f is g -nondecreasing, and gX is closed. Suppose the pair (f, g) is a generalized Meir-Keeler-type α - ψ -contraction and the following conditions hold.

- (i) If any nondecreasing sequence $\{x_n\}$ in X converges to v , then we assume that

$$x_n \sqsubseteq v \quad \forall n \in \mathbb{N}. \quad (51)$$

- (ii) There exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$ and $\alpha(fx_0, fx_0) \geq 1$.

(iii) If $\alpha(fx_n, fx_n) \geq 1$ for all $n \in \mathbb{N}$, then $\lim_{n \rightarrow \infty} \alpha(fx_n, fx_n) \geq 1$.

Then f and g have a coincidence point in X .

Proof. Since $fX \subset gX$ and by (ii), there exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$ and $\alpha(fx_0, fx_0) \geq 1$, we can choose $x_1 \in X$ such that $gx_1 = fx_0$. Since f is g -nondecreasing, we have $fx_0 \sqsubseteq fx_1$. In this process, we construct the sequence $\{x_n\}$ recursively as

$$fx_n = gx_{n+1} \quad \forall n \in \mathbb{N}. \quad (52)$$

Thus, we also conclude that

$$\begin{aligned} gx_0 \sqsubseteq fx_0 = gx_1 \sqsubseteq fx_1 = gx_2 \sqsubseteq \cdots \sqsubseteq fx_{n-1} \\ = gx_n \sqsubseteq fx_n = gx_{n+1} \sqsubseteq \cdots \end{aligned} \quad (53)$$

If any two consecutive terms in (53) are equal, then the conclusion of the theorem follows. So we may assume that

$$d(fx_{n-1}, fx_n) \neq 0, \quad \forall n \in \mathbb{N}. \quad (54)$$

On the other hand, since f is α - g -admissible and $\alpha(fx_0, fx_0) = \alpha(gx_1, gx_1) \geq 1$, we have

$$\alpha(fx_1, fx_1) = \alpha(gx_2, gx_2) \geq 1. \quad (55)$$

By continuing this process, we get

$$\alpha(fx_n, fx_n) = \alpha(gx_{n+1}, gx_{n+1}) \geq 1 \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (56)$$

By (53), (54), and (56), substituting $x = x_n$ and $y = x_{n+1}$ in (50), we have

$$\begin{aligned} d(fx_n, fx_{n+1}) \\ \leq \alpha(fx_n, fx_n) \alpha(gx_{n+1}, gx_{n+1}) d(fx_n, fx_{n+1}) \\ < \psi(d(gx_n, gx_{n+1}), d(gx_n, fx_n), d(gx_{n+1}, fx_{n+1}), \\ &\quad (gx_n, fx_{n+1}), d(gx_{n+1}, fx_n)) \\ = \psi(d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \\ &\quad d(fx_{n-1}, fx_{n+1}), d(fx_n, fx_n)). \end{aligned} \quad (57)$$

If $d(fx_{n-1}, fx_n) \leq d(fx_n, fx_{n+1})$, then the inequality (57) becomes

$$\begin{aligned} d(fx_n, fx_{n+1}) \\ < \psi(d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \\ &\quad d(fx_{n-1}, fx_{n+1}), d(fx_n, fx_n)) \\ \leq \psi(d(fx_n, fx_{n+1}), d(fx_n, fx_{n+1}), d(fx_n, fx_{n+1}), \\ &\quad 2d(fx_n, fx_{n+1}), 0) \\ \leq d(fx_n, fx_{n+1}), \end{aligned} \quad (58)$$

which implies a contradiction, and we get that $d(fx_n, fx_{n+1}) < d(fx_{n-1}, fx_n)$.

From the argument above, we have that the sequence $\{d(fx_n, fx_{n+1})\}$ is decreasing, and it must converge to some $\eta \geq 0$, that is,

$$\lim_{n \rightarrow \infty} d(fx_n, fx_{n+1}) = \eta. \quad (59)$$

It follow from that (57) and (59), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \psi(d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \\ d(fx_{n-1}, fx_{n+1}), d(fx_n, fx_n)) = \eta. \end{aligned} \quad (60)$$

Notice that $\eta = \inf \{d(fx_n, fx_{n+1}) : n \in \mathbb{N} \cup \{0\}\}$. We claim that $\eta = 0$. Suppose, to the contrary, that $\eta > 0$. Since (f, g) is a generalized Meir-Keeler-type α - ψ -contraction, corresponding to η use, and taking into account the above inequality (60), there exist $\delta > 0$ and a natural number k such that

$$\begin{aligned} \eta \leq \psi(d(fx_{k-1}, fx_k), d(fx_{k-1}, fx_k), \\ d(fx_k, fx_{k+1}), d(fx_{k-1}, fx_{k+1}), \\ d(fx_k, fx_k)) \\ < \eta + \delta \implies \alpha(fx_k, fx_k) \alpha(gx_{k+1}, gx_{k+1}) \\ &\quad \times d(fx_k, fx_{k+1}) < \eta, \end{aligned} \quad (61)$$

which implies

$$\begin{aligned} d(fx_k, fx_{k+1}) \leq \alpha(fx_k, fx_k) \\ \times \alpha(gx_{k+1}, gx_{k+1}) d(fx_k, fx_{k+1}) < \eta. \end{aligned} \quad (62)$$

So we get a contradiction, since $\eta = \inf \{d(fx_n, fx_{n+1}) : n \in \mathbb{N} \cup \{0\}\}$. Thus we have that

$$\lim_{n \rightarrow \infty} d(fx_n, fx_{n+1}) = 0. \quad (63)$$

We next claim that $\{fx_n\}$ is a Cauchy sequence, that is, for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that if $p, q \geq n$, then $d(fx_p, fx_q) < \epsilon$.

Suppose the above statement is false. Then there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, there are $p_n, q_n \in \mathbb{N}$ with $p_n > q_n \geq n$ satisfying

$$d(fx_{q_n}, fx_{p_n}) \geq \epsilon. \quad (64)$$

Further, corresponding to $q_n \geq n$, we can choose p_n in such a way that is it the smallest integer with $p_n > q_n \geq n$ and $d(fx_{q_n}, fx_{p_n}) \geq \epsilon$. Therefore $d(fx_{q_n}, fx_{p_n-1}) < \epsilon$. Now we have that for all $n \in \mathbb{N}$

$$\begin{aligned} \epsilon \leq d(fx_{p_n}, fx_{q_n}) \leq d(fx_{p_n}, fx_{p_n-1}) \\ + d(fx_{p_n-1}, fx_{q_n}) \\ < d(fx_{p_n}, fx_{p_n-1}) + \epsilon. \end{aligned} \quad (65)$$

Letting $n \rightarrow \infty$, then we get

$$\lim_{n \rightarrow \infty} d(fx_{p_n}, fx_{q_n}) = \epsilon. \quad (66)$$

On the other hand, we have

$$\begin{aligned} d(fx_{p_n}, fx_{q_n}) &\leq d(fx_{p_n}, fx_{p_{n-1}}) + d(fx_{p_{n-1}}, fx_{q_{n-1}}) \\ &\quad + d(fx_{q_{n-1}}, fx_{q_n}), \\ d(fx_{p_{n-1}}, fx_{q_{n-1}}) &\leq d(fx_{p_{n-1}}, fx_{p_n}) + d(fx_{p_n}, fx_{q_n}) \\ &\quad + d(fx_{q_n}, fx_{q_{n-1}}). \end{aligned} \quad (67)$$

Letting $n \rightarrow \infty$, then we get

$$\lim_{n \rightarrow \infty} d(fx_{p_{n-1}}, fx_{q_{n-1}}) = \epsilon. \quad (68)$$

By (53), we have that the elements gx_{p_n} and gx_{q_n} are comparable. Substituting $x = x_{p_n}$ and $y = x_{q_n}$ in (50), we have that for all $n \in \mathbb{N}$,

$$\begin{aligned} d(fx_{p_n}, fx_{q_n}) &\leq \alpha(fx_{p_n}, fx_{p_n}) \alpha(gx_{q_n}, gx_{q_n}) d(fx_{p_n}, fx_{q_n}) \\ &< \psi(d(gx_{p_n}, gx_{q_n}), d(gx_{p_n}, fx_{p_n}), d(gx_{q_n}, fx_{q_n}), \\ &\quad d(gx_{p_n}, fx_{q_n}), d(gx_{q_n}, fx_{p_n})) \\ &\leq \psi(d(fx_{p_{n-1}}, fx_{q_{n-1}}), d(fx_{p_{n-1}}, fx_{p_n}), \\ &\quad d(fx_{q_{n-1}}, fx_{q_n}), d(fx_{p_{n-1}}, fx_{q_n}), \\ &\quad d(fx_{q_{n-1}}, fx_{p_n})) \\ &\leq \psi(d(fx_{p_{n-1}}, fx_{q_{n-1}}), d(fx_{p_{n-1}}, fx_{p_n}), \\ &\quad d(fx_{q_{n-1}}, fx_{q_n}), d(fx_{p_{n-1}}, fx_{p_n}) + d(fx_{p_n}, fx_{q_n}), \\ &\quad d(fx_{q_{n-1}}, fx_{q_n}) + d(fx_{q_n}, fx_{p_n})). \end{aligned} \quad (69)$$

Letting $n \rightarrow \infty$ in (69), then we get

$$\epsilon < \psi(\epsilon, 0, 0, \epsilon, \epsilon) \leq \epsilon, \quad (70)$$

which implies a contradiction. Thus, $\{fx_n\}$ is a Cauchy sequence.

Since X is complete and gX is closed, there exists $v \in X$ such that

$$\lim_{n \rightarrow \infty} gx_n = \lim_{n \rightarrow \infty} fx_n = gv. \quad (71)$$

Since α is continuous in each coordinate and by the condition (iii), we have

$$\alpha(gv, gv) = \lim_{n \rightarrow \infty} \alpha(fx_n, fx_n) \geq 1. \quad (72)$$

Later, we prove that v is a coincidence point of f and g . From (53) and (71), we deduce that

$$gx_n \sqsubseteq gv, \quad \forall n \in \mathbb{N}. \quad (73)$$

By (72) and substituting $x = x_n$ and $y = v$ in (50), we have that

$$\begin{aligned} d(fx_n, fv) &\leq \alpha(fx_n, fx_n) \alpha(gv, gv) d(fx_n, fv) \\ &< \psi(d(gx_n, gv), d(gx_n, fx_n), \\ &\quad d(gv, fv), d(gx_n, fv), d(gv, fx_n)). \end{aligned} \quad (74)$$

Taking $n \rightarrow \infty$ in the above inequality, we have

$$\begin{aligned} d(gv, fv) &< \psi(d(gv, gv), d(gv, gv), \\ &\quad d(gv, fv), d(gv, fv), d(gv, gv)) \\ &\leq d(gv, fv). \end{aligned} \quad (75)$$

This implies that $gv = fv$. So we complete the proof. $\square$

Apply Theorem 15, we are easy to get the following theorem.

Theorem 16. Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, and let $f, g : X \rightarrow X$ be such that $fX \subset gX$, f is g -nondecreasing, and gX is closed. Suppose the pair (f, g) is a generalized Meir-Keeler-type ψ -contraction and the following conditions hold.

(i) If any nondecreasing sequence $\{x_n\}$ in X converges to v , then we assume that

$$x_n \sqsubseteq v \quad \forall n \in \mathbb{N}. \quad (76)$$

(ii) There exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$.

Then f and g have a coincidence point in X .

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