Approximate Solutions of Nonlinear Partial Differential Equations by Modified $q$-Homotopy Analysis Method

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A modified $q$-homotopy analysis method (mq-HAM) was proposed for solving $n$th-order nonlinear differential equations. This method improves the convergence of the series solution in the $n$HAM which was proposed in (see Hassan and El-Tawil 2011, 2012). The proposed method provides an approximate solution by rewriting the $n$th-order nonlinear differential equation in the form of $n$ first-order differential equations. The solution of these $n$ differential equations is obtained as a power series solution. This scheme is tested on two nonlinear exactly solvable differential equations. The results demonstrate the reliability and efficiency of the algorithm developed.

1. Introduction

Homotopy analysis method (HAM) initially proposed by Liao in his Ph.D. thesis [1] is a powerful method to solve nonlinear problems. In recent years, this method has been successfully employed to solve many types of nonlinear problems in science and engineering [2–17]. HAM contains a certain auxiliary parameter $h$, which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. Moreover, by means of the so-called $h$-curve, a valid region of $h$ can be studied to gain a convergent series solution. More recently, a powerful modification of HAM was proposed [18–20]. Hassan and El-Tawil [21, 22] presented a new technique of using homotopy analysis method for solving nonlinear initial value problems ($n$HAM). El-Tawil and Huseen [23, 24] established a method, namely, $q$-homotopy analysis method ($q$-HAM) which is a more general method of HAM. The $q$-HAM contains an auxiliary parameter $n$ as well as $h$ such that the case of $n = 1$ ($q$-HAM; $n = 1$) and the standard homotopy analysis method (HAM) can be reached. In this paper, we present the modification of $q$-homotopy analysis method (mq-HAM) for solving nonlinear problems by transforming the $n$th-order nonlinear differential equation to a system of $n$ first-order equations. we note that the $n$HAM is a special case of $mq$-HAM ($mq$-HAM; $n = 1$).

2. Analysis of the $q$-Homotopy Analysis Method ($q$-HAM)

Consider the following nonlinear partial differential equation:

$$N \left[u(x,t)\right] = 0,$$

(1)

where $N$ is a nonlinear operator, $(x,t)$ denotes independent variables, and $u(x,t)$ is an unknown function. Let us construct the so-called zero-order deformation equation as follows:

$$(1 - nq) L \left[\varnothing(x,t;q) - u_0(x,t)\right] = qhH(x,t) N \left[\varnothing(x,t;q)\right],$$

(2)

where $n \geq 1, q \in [0, 1/n]$ denotes the so-called embedded parameter, $L$ is an auxiliary linear operator with the property $L[f] = 0$ when $f = 0, h \neq 0$ is an auxiliary parameter, and
\( H(x, t) \) denotes a non-zero auxiliary function. It is obvious that when \( q = 0 \) and \( q = 1/n \), (2) becomes
\[
\theta (x, t; 0) = u_0(x, t), \quad \theta \left( x, t; \frac{1}{n} \right) = u(x, t), \quad (3)
\]
respectively. Thus, as \( q \) increases from 0 to \( 1/n \), the solution \( \theta(x, t; q) \) varies from the initial guess \( u_0(x, t) \) to the solution \( u(x, t) \). We may choose \( u_0(x, t), L, h, \) and \( H(x, t) \) and assume that all of them can be properly chosen so that the solution \( \theta(x, t; q) \) of (2) exists for \( q \in [0, 1/n] \).

Now, by expanding \( \theta(x, t; q) \) in Taylor series, we have
\[
\theta(x, t; q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) q^m, \quad (4)
\]
where
\[
u_m(x, t) = \frac{1}{m!} \frac{\partial^m \theta(x, t; q)}{\partial q^m} \bigg|_{q=0} \quad (5)
\]
Next, we assume that \( h, H(x, t), u_0(x, t), \) and \( L \) are properly chosen such that the series (4) converges at \( q = 1/n \) and that
\[
u(x, t) = \theta \left( x, t; \frac{1}{n} \right) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) \left( \frac{1}{n} \right)^m \quad (6)
\]
Let
\[
u_r(x, t) = \{u_0(x, t), u_1(x, t), u_2(x, t), \ldots, u_r(x, t)\} \quad (7)
\]
Differentiating equation (2) for \( m \) times with respect to \( q \) and then setting \( q = 0 \) and dividing the resulting equation by \( m! \), we have the so-called \( m \)th order deformation equation as follows:
\[
L\left[u_m(x, t) - k_m u_{m-1}(x, t)\right] = hH(x, t) R_m \left(\overline{u_{m-1}}(x, t)\right), \quad (8)
\]
where
\[
R_m \left(\overline{u_{m-1}}(x, t)\right) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \left[N \left[\theta(x, t; q)\right] - f(x, t)\right]}{\partial q^{m-1}} \bigg|_{q=0}, \quad (9)
\]
\[
k_m = \begin{cases} 0 & m \leq 1, \\ n & \text{otherwise.} \end{cases}
\]
It should be emphasized that \( u_m(x, t) \) for \( m \geq 1 \) is governed by the linear equation (8) with linear boundary conditions that come from the original problem. Due to the existence of the factor \((1/n)^m\), more chances for convergence may occur or even much faster convergence can be obtained better than the standard HAM. It should be noted that the case of \( n = 1 \) in (2), standard HAM, can be reached.

The \( q \)-homotopy analysis method (\( q \)-HAM) can be reformatted as follows.

We rewrite the nonlinear partial differential equation (1) in the following form:
\[
Lu(x, t) + Au(x, t) + Bu(x, t) = 0, \quad (10)
\]
\[
u(x, 0) = f_0(x), \quad \frac{\partial u(x, t)}{\partial t} \bigg|_{t=0} = f_1(x), \quad (11)
\]
where \( L = \partial^2/\partial t^2, z = 1, 2, \ldots \) is the highest partial derivative with respect to \( t \), \( A \) is a linear term, and \( B \) is a nonlinear term. The so-called zero-order deformation equation (2) becomes
\[
(1 - nq)L \left[\theta(x, t; q) - u_0(x, t)\right] = q h H(x, t) \left(\sum_{z=1}^{n} c_z t^{z-1} + \sum_{z=1}^{n} c_z t^{z-2} + \cdots + c_z\right) \quad (12)
\]
Hence,
\[
u_m(x, t) = k_m u_{m-1}(x, t)
\]
\[
+ h L^{-1} \left[H(x, t) \left(L u_{m-1}(x, t) + A u_{m-1}(x, t) + B \left(\overline{u_{m-1}}(x, t)\right)\right)\right] \quad (13)
\]
Now, the inverse operator \( L^{-1} \) is an integral operator which is given by
\[
L^{-1} (\cdot) = \int \cdots \int (\cdot) \, dt \, dt \cdots dt + c_1 t^{z-1} + c_2 t^{z-2} + \cdots + c_z \quad (14)
\]
where \( c_1, c_2, \ldots, c_z \) are integral constants.

To solve (10) by means of \( q \)-HAM, we choose the following initial approximation:
\[
u_0(x, t) = f_0(x) + f_1(x) t
\]
\[
+ f_2(x) t^2 + \cdots + f_{z-1}(x) \frac{t^{z-1}}{(z-1)!} \quad (15)
\]
Let \( H(x, t) = 1, \) by means of (14) and (15); then (13) becomes
\[
u_m(x, t) = k_m u_{m-1}(x, t)
\]
\[
+ h \int_0^t \int_0^t \cdots \int_0^t \left(\frac{\partial^z u_{m-1}(x, \tau) \partial \tau^z}{\partial \tau^z} + A u_{m-1}(x, \tau) + B \left(\overline{u_{m-1}}(x, \tau)\right)\right) \, d\tau \, d\tau \cdots d\tau \quad (16)
\]
where the \( z \)-times integral operator is given by
\[
\int_0^t \int_0^t \cdots \int_0^t (\cdot) \, dt \, dt \cdots dt + c_1 t^{z-1} + c_2 t^{z-2} + \cdots + c_z \quad (17)
\]
where \( c_1, c_2, \ldots, c_z \) are integral constants.
Now from \( \int_0^t \int_0^t \int_0^t \frac{\partial^z u_{m-1}(x, \tau)}{\partial \tau^z} \, d\tau \, d\tau \, d\tau \), we observe that there are repeated computations in each step which caused more consuming time. To cancel this, we use the following modification to (16):

\[
  u_m(x, t) = k_m u_{m-1}(x, t) + h \int_0^t \int_0^t \int_0^t (A u_{m-1}(x, \tau) + B(u_{m-1}(x, \tau))) \, d\tau \, d\tau \, d\tau. 
\]  (17)

Now, for \( m = 1, k_m = 0 \), and

\[
  u_0(x, 0) + t \frac{\partial u_0(x, 0)}{\partial t} + \frac{t^2}{2!} \frac{\partial^2 u_0(x, 0)}{\partial t^2} + \cdots + \frac{t^{z-1}}{(z-1)!} \frac{\partial^{z-1} u_0(x, 0)}{\partial t^{z-1}} = f_0(x) + f_1(x) t + f_2(x) \frac{t^2}{2!} + \cdots + f_{z-1}(x) \frac{t^{z-1}}{(z-1)!} = u_0(x, t). 
\]  (18)

Substituting this equality into (17), we obtain

\[
  u_1(x, t) = h \int_0^t \int_0^t \int_0^t (A u_0(x, \tau) + B(u_0(x, \tau))) \, d\tau \, d\tau \, d\tau. 
\]  (19)

For \( m > 1, k_m = n \), and

\[
  u_m(x, 0) = 0, \quad \frac{\partial u_m(x, 0)}{\partial t} = 0, \\
  \frac{\partial^2 u_m(x, 0)}{\partial t^2} = 0, \cdots, \frac{\partial^{z-1} u_m(x, 0)}{\partial t^{z-1}} = 0. 
\]  (20)

The standard \( q \)-HAM is powerful when \( z = 1 \), and the series solution expression by \( q \)-HAM can be written in the following form:

\[
  u(x, t; n; h) \equiv U_M(x, t; n; h) = \sum_{i=0}^{M} u_i(x, t; n; h) \left( \frac{1}{h} \right)^i. 
\]  (22)

But when \( z \geq 2 \), there are too many additional terms where harder and more time consuming computations are performed. So, the closed form solution needs more numbers of iteration.

### 3. The Proposed Modified \( q \)-Homotopy Analysis Method (\( mq \)-HAM)

When \( z \geq 2 \), we rewrite (1) as in the following system of first-order differential equations:

\[
  u_i = u_1, \\
  u_1 = u_2, \\
  \vdots \\
  u_{\{z-1\}} = -Au(x, t) - Bu(x, t). 
\]  (23)

Set the initial approximation

\[
  u_0(x, t) = f_0(x), \\
  u_1(x, t) = f_1(x), \\
  \vdots \\
  u_{\{z-1\}}(x, t) = f_{z-1}(x). 
\]  (24)

Using the iteration formulas (19) and (21) as follows:

\[
  u_1(x, t) = h \int_0^t (-u_0(x, \tau)) \, d\tau, \\
  u_1(x, t) = h \int_0^t (-u_2(x, \tau)) \, d\tau, \\
  \vdots \\
  u_{\{z-1\}}(x, t) = h \int_0^t (A u_0(x, \tau) + B(u_0(x, \tau))) \, d\tau. 
\]  (25)
For $m > 1$, $k_m = n$, and
\[
\begin{align*}
 u_m(x,0) &= 0, \quad u_1(x,0) = 0, \\
 u_2(x,0) &= 0, \ldots, u_{\lfloor z - 1 \rfloor}(x,0) = 0.
\end{align*}
\]
(26)

Substituting in (17), we obtain
\[
\begin{align*}
 u_m(x,t) &= (n + h) u_{m-1}(x,t) + h \int_0^t (-u_{m-1}(x,\tau)) d\tau, \\
 u_1(x,t) &= (n + h) u_{1}(x,t) + h \int_0^t (-u_{2}(x,\tau)) d\tau, \\
 & \quad \vdots \\
 u_{\lfloor z - 1 \rfloor}(x,t) &= (n + h) u_{\lfloor z - 1 \rfloor}(x,t) + h \int_0^t (-u_{2\lfloor z - 1 \rfloor+1}(x,\tau)) d\tau.
\end{align*}
\]
(27)

It should be noted that the case of $n = 1$ in (27), the $n$HAM, can be reached.

To illustrate the effectiveness of the proposed $m_q$-HAM, comparison between $m_q$-HAM and the $n$HAM are illustrated by the following examples.

4. Illustrative Examples

Example 1. Consider the following nonlinear sine-Gordon equation:
\[
u_{tt} - u_{xx} + \sin u = 0,
\]
(28)
subject to the following initial conditions:
\[
u(x,0) = 0, \quad \nu_t(x,0) = 4 \sech x.
\]
(29)
The exact solution is
\[
u(x,t) = 4 \tan^{-1}(t \sech x).
\]
(30)
In order to prevent suffering from the strongly nonlinear term $\sin u$ in the frame of $q$-HAM, we can use Taylor series expansion of $\sin u$ as follows:
\[
\sin u = u - \frac{u^3}{6} + \frac{u^5}{120},
\]
(31)
Then, (28) becomes
\[
u_{tt} - \nu_{xx} - \frac{u^3}{6} + \frac{u^5}{120} = 0.
\]
(32)
In order to solve (28) by $m_q$-HAM, we construct system of differential equations as follows:
\[
\begin{align*}
u_1(x,t) &= v(x,t), \\
\nu_i(x,t) &= \frac{\partial^2 u(x,t)}{\partial x^2} - u + \frac{u^3}{6} - \frac{u^5}{120},
\end{align*}
\]
(33)
with the following initial approximations:
\[
u_0(x,t) = 0, \quad v_0(x,t) = 4 \sech x,
\]
(34)
and the following auxiliary linear operators:
\[
\begin{align*}
 L u(x,t) &= \frac{\partial u(x,t)}{\partial t}, \\
 L v(x,t) &= \frac{\partial v(x,t)}{\partial t}, \\
 A u_{m-1}(x,t) &= -\frac{\partial^3 u_{m-1}(x,t)}{\partial x^2} + u_{m-1}(x,t), \\
 B v_{m-1}(x,t) &= -\frac{1}{6} \sum_{j=0}^{m-1} \sum_{i=0}^{j} \sum_{k=0}^{i} \sum_{l=0}^{k} u_{i-k} u_{k-l}.
\end{align*}
\]
(35)
From (25) and (27), we obtain
\[
\begin{align*}
u_1(x,t) &= h \int_0^t (-v_0(x,\tau)) d\tau, \\
v_1(x,t) &= h \int_0^t \left( -\frac{\partial^2 u_0(x,\tau)}{\partial x^2} + u_0 - \frac{u_0^3}{6} + \frac{u_0^5}{120} \right) d\tau.
\end{align*}
\]
(36)
Now, for $m \geq 2$, we get
\[
\begin{align*}
u_m(x,t) &= (n + h) \nu_{m-1}(x,t) + h \int_0^t (-\nu_{m-1}(x,\tau)) d\tau, \\
v_m(x,t) &= (n + h) \nu_{m-1}(x,t) + h \int_0^t \left( -\frac{\partial^2 \nu_{m-1}(x,\tau)}{\partial x^2} + \nu_{m-1}(x,\tau) \right) d\tau.
\end{align*}
\]
(37)
And the following results are obtained:
\[
\begin{align*}
u_1(x,t) &= -4ht \sech x, \\
v_1(x,t) &= 0, \\
u_2(x,t) &= -4h(h + n)t \sech x, \\
v_2(x,t) &= -4h^2 t^2 \sech^3 x,
\end{align*}
\]
(38)
\[
u_3(x,t) = -4h(h + n)^2 t \sech x + \frac{4}{3} h^3 t^3 \sech^3 x,
\]
$u_m(x,t)$, $(m = 4, 5, \ldots)$ can be calculated similarly. Then, the series solution expression by $m_q$-HAM can be written in the following form:
\[
u(x,t;n,h) \equiv U_M(x,t;n,h) = \sum_{j=0}^{M} u_j(x,t;n,h) \left( \frac{1}{n} \right)^j.
\]
(39)
Equation (39) is a family of approximation solutions to the problem (28) in terms of the convergence parameters $h$ and $n$. To find the valid region of $h$, the $h$-curves given by the 6th-order $n$HAM ($m_q$-HAM; $n = 1$) approximation and
the 6th-order \( m_q \)-HAM (\( n = 13 \)) approximation at different values of \( x, t \) are drawn in Figures 1 and 2, respectively, and these figures show the interval of \( h \) in which the value of \( U_6 \) is constant at certain \( x, t \), and \( n \); we chose the horizontal line parallel to \( x \)-axis (\( h \)) as a valid region which provides us with a simple way to adjust and control the convergence region. Figure 3 shows the comparison between \( U_6 \) of nHAM and \( U_6 \) of \( m_q \)-HAM using different values of \( n \) with the solution (30). The absolute errors of the 6th-order solutions nHAM approximate and the 6th-order solutions \( m_q \)-HAM approximate using different values of \( n \) are shown in Figure 4. The results obtained by \( m_q \)-HAM indicate that the speed of convergence for \( m_q \)-HAM with \( n > 1 \) is faster in comparison to \( n = 1 \) (nHAM). The results show that the convergence region of series solutions obtained by \( m_q \)-HAM is increasing as \( q \) is decreased as shown in Figures 3 and 4.

By increasing the number of iterations by \( m_q \)-HAM, the series solution becomes more accurate, more efficient, and

Example 2. Consider the following Klein-Gordon equation:

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{3}{4} \frac{\partial u}{\partial t} - \frac{3}{2} u^3 = 0, \quad (40)
\]

subject to the following initial conditions:

\[
u(x, 0) = -\text{sech} \, x, \quad (41)\]

\[
u_t(x, 0) = \frac{1}{2} \text{sech} \, x \, \tanh \, x.
\]
The exact solution is
\[ u(x, t) = \text{sech} \left( x + \frac{t}{2} \right). \] \hspace{1cm} (42)

In order to solve (40) by \( mq \)-HAM, we construct system of differential equations as follows:
\[ u_t(x, t) = v(x, t), \]
\[ v_t(x, t) = \frac{\partial^2 u(x, t)}{\partial x^2} - \frac{3}{4} u + \frac{3}{2} u^3, \] \hspace{1cm} (43)

with the following initial approximations:
\[ u_0(x, t) = -\text{sech}x, \quad v_0(x, t) = \frac{1}{2} \text{sech}x \tanh x, \] \hspace{1cm} (44)

and the following auxiliary linear operators:
\[ Lu(x, t) = \frac{\partial u(x, t)}{\partial t}, \quad L v(x, t) = \frac{\partial v(x, t)}{\partial t}, \]
\[ A u_{m-1}(x, t) = -\frac{\partial^3 u_{m-1}(x, t)}{\partial x^2} + \frac{3}{4} u_{m-1}(x, t), \] \hspace{1cm} (45)
\[ B u_{m-1}(x, t) = -\frac{3}{2} \sum_{j=0}^{m-1} u_{m-1-j} \sum_{i=0}^{j} u_i u_{j-i}. \]

From (25) and (27), we obtain
\[ u_1(x, t) = h \int_0^t (-v_0(x, \tau)) d\tau, \]
\[ v_1(x, t) = h \int_0^t \left( -\frac{\partial u_0}{\partial x^2} + \frac{3}{4} u_0 - \frac{3}{2} u_0^3 \right) d\tau. \] \hspace{1cm} (46)
For $m \geq 2$, we get

$$u_m (x, t) = (n + h) u_{m-1} (x, t) + h \int_0^t \left( -v_{m-1} (x, \tau) \right) d\tau,$$

$$v_m (x, t) = (n + h) v_{m-1} (x, t) + h \int_0^t \left( -\frac{\partial^2 u_{m-1} (x, \tau)}{\partial x^2} + \frac{3}{4} u_{m-1} (x, \tau) \right) d\tau,$$

$$- \frac{3}{2} \sum_{j=0}^{m-1} u_{m-1-j} \sum_{i=0}^{j} u_i u_{j-i} d\tau. \tag{47}$$

The following results are obtained:

$$u_1 (x, t) = -\frac{1}{2} ht \operatorname{sech} x \tanh x,$$

$$v_1 (x, t) = ht \left( -\frac{3 \operatorname{sech} x}{4} + \frac{\operatorname{sech}^3 x}{2} + \operatorname{sech} x \tanh x \right),$$

$$u_2 (x, t) = h \left( \frac{3}{16} h^2 \operatorname{sech}^3 x - \frac{1}{16} h^2 \cosh (2x) \operatorname{sech}^3 x \right) - \frac{1}{2} h (n + h) t \operatorname{sech} x \tanh x,$$

$$u_m (x, t), (m = 3, 4, \ldots) \text{ can be calculated similarly. Then, the series solution expression by } m^q\text{-HAM can be written in the following form:}$$

$$u(x, t; n; h) \equiv U_M (x, t; n; h) = \sum_{i=0}^{M} u_i (x, t; n; h) \left( \frac{1}{n} \right)^i. \tag{49}$$

Equation (49) is a family of approximation solutions to the problem (40) in terms of the convergence parameters $h$ and $n$. To find the valid region of $h$, the $h$-curves given by the 6th-order $n$HAM ($m^q$-HAM; $n = 1$) approximation and the 6th-order $m^q$-HAM ($m^q$-HAM; $n > 1$) approximation at different values of $x, t$ are drawn in Figures 9 and 10; these figures show the interval of $h$ in which the value of $U_6$ is constant at certain $x, t$, and $n$; we chose the horizontal line parallel to $x$-axis ($h$) as a valid region which provides us with a simple way to adjust and control the convergence region. Figure 11 shows the comparison between $U_6$ of $n$HAM ($m^q$-HAM; $n = 1$) and $U_6$ of $m^q$-HAM ($n = 5, 20, 50, 100$) with exact solution of (40) at $x = 1$ with $(h = -1, h = -4.85, h = -18.55, h = -43.11, h = -79.5)$, respectively.
Figure 12: The absolute error of \( U_6 \) of \( n \)HAM (\( mq \)-HAM; \( n = 1 \)) and \( U_6 \) of \( mq \)-HAM (\( n = 5, 20, 50, 100 \)) for problem (40) at \( h = -1 \), \( x = 1 \), using \( (h = -1, h = -4.85, h = -18.55, h = -43.11, h = -79.5) \), respectively.

Figure 13: The comparison between the \( U_3 \), \( U_6 \) of \( n \)HAM (\( mq \)-HAM; \( n = 1 \)) and the exact solution of (40) at \( h = -1 \) and \( x = 1 \).

Figure 14: The comparison between the \( U_3 \) and \( U_6 \) of \( n \)HAM (\( mq \)-HAM; \( n = 100 \)) and the exact solution of (40) at \( h = -79.5 \) and \( x = 1 \).

obtained by \( mq \)-HAM is increasing as \( q \) is decreased as shown in Figures 11 and 12.

By increasing the number of iterations by \( mq \)-HAM, the series solution becomes more accurate, more efficient, and the interval of \( t \) (convergent region) increases as shown in Figures 13, 14, 15, and 16.

Figure 17 shows that the convergence of the series solutions obtained by the 3rd-order \( mq \)-HAM (\( n = 100 \)) is faster than that of the series solutions obtained by the 6th order \( n \)HAM. This fact shows the importance of the convergence parameters \( n \) in the \( mq \)-HAM.

5. Conclusion

In this paper, a modified \( q \)-homotopy analysis method was proposed (\( mq \)-HAM). This method provides an approximate solution by rewriting the \( n \)th-order nonlinear differential equations in the form of system of \( n \) first-order differential equations. The solution of these \( n \) differential equations is
obtained as a power series solution, which converges to a closed form solution. The mq-HAM contains two auxiliary parameters \( n \) and \( h \) such that the case of \( n = 1 \) (mq-HAM; \( n = 1 \)); the nHAM which is proposed in [21, 22] can be reached. In general, it was noticed from the illustrative examples that the convergence of mq-HAM is faster than that of nHAM.

References


[15] S. Liao, “An explicit solution of (40) at \( (h = -79.5, \ h = -1) \) and \( x = 1 \).