Viscosity Method for Hierarchical Fixed Point Problems with an Infinite Family of Nonexpansive Nonself-Mappings

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Received 4 January 2013; Accepted 22 March 2013

A viscosity method for hierarchical fixed point problems is presented to solve variational inequalities, where the involved mappings are nonexpansive nonself-mappings. Solutions are sought in the set of the common fixed points of an infinite family of nonexpansive nonself-mappings. The results generalize and improve the recent results announced by many other authors.

1. Introduction and Preliminaries

Let $X$ a real Banach space and $J$ be the normalized duality mapping from $X$ into $2^{X^*}$ given by

$$J(x) = \{ x^* \in X^* : \langle x, x^* \rangle = \| x \| \| x^* \|, \| x \| = \| x^* \| \}$$

for all $x \in X$, where $X^*$ denotes the dual space of $X$ and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between $X$ and $X^*$. If $X = H$ is a Hilbert space, then $J$ becomes the identity mapping on $H$. A point $x \in C$ is a fixed point of $T : C \subset X \to X$ provided $Tx = x$. Denote by $F(T)$ the set of fixed points of $T$; that is, $F(T) = \{ x \in C : Tx = x \}$.

Let $X$ be a normed linear space with $\dim X \geq 2$. The modulus of smoothness of $X$ is the function $\rho_X : [0, +\infty) \to [0, +\infty)$ defined by

$$\rho_X(\tau) := \sup \left\{ \frac{\| x + y \| + \| x - y \|}{2} - 1 : \| x \| = 1, \| y \| = \| x \| = \| x^* \| \} \right\}.$$  

The space $X$ is said to be smooth if $\rho_X(\tau) > 0$, for all $\tau > 0$. It is well known that if $X$ is smooth then $J$ is single valued. A Banach space $X$ is said to be strictly convex if $\| x \| = \| y \| = 1, x \neq y$, implies $\| x + y \|/2 < 1$.

Let $C$ be a nonempty closed convex subset of a real Banach space $X$. Recall the following concepts.

**Definition 1.** (i) A mapping $f : C \to C$ is a $\rho$-contraction if $\rho \in [0, 1)$ and if the following property is satisfied

$$\| f(x) - f(y) \| \leq \rho \| x - y \|, \quad \forall x, y \in C.$$  

(ii) A mapping $T : C \to E$ is nonexpansive provided

$$\| Tx - Ty \| \leq \| x - y \|, \quad \forall x, y \in C.$$  

(iii) A mapping $S : C \to X$ is

(a) accretive if for any $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Sx - Sy, j(x - y) \rangle \geq 0;$$  

(b) $\beta$-strongly accretive if for any $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Sx - Sy, j(x - y) \rangle \geq \beta \| x - y \|^2,$$

for some real constant $\beta > 0$.

Noting that if $S : C \to X$ is nonexpansive, then $I - S$ is accretive; if $f : C \to C$ is a $\rho$-contraction, then $I - f$ is $(1 - \rho)$-strongly accretive. Particularly, if $X = H$ is a Hilbert space, then (strongly) accretive mappings become (strongly) monotone mappings.
Definition 2. Let C and D be nonempty subsets of a Banach space X such that C is nonempty closed convex and D ⊂ C.

(i) A mapping \( Q : C \to D \) is called sunny, if \( Q(\text{Qx} + t(x - \text{Qx})) = \text{Qx} \) for each \( x \in C \) and \( t \geq 0 \) with \( Q(\text{Qx} + t(x - \text{Qx})) \in C \).

(ii) A mapping \( Q : C \to D \) is called a retraction from C to D if \( Q \) is continuous and \( F(Q) = D \).

(iii) A subset \( D \subset E \) is said to be a sunny nonexpansive retraction of \( C \) if there exists a sunny nonexpansive retraction \( Q \) of \( C \) onto \( D \). For details, see [1–3].

Note that if \( X = H \) is a Hilbert space, \( Q \) becomes the projection on \( C \), denoted by \( P_C \).

Let \( P : C \to C \) a nonexpansive self-mapping on \( C \) and \( \{T_n\} \) be a countable family of nonexpansive nonself-mappings of \( C \) into \( X \) such that \( \mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset \). Then we consider the following problem: find hierarchically a common fixed point of the infinite family \( \{T_n\} \) with respect to a nonexpansive mapping \( P \); namely, find \( x^* \in \mathcal{F} \), such that

\[ \langle x^* - Px^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{7} \]

Particularly, if \( \{T_n\} \) is a finite family of nonexpansive nonself-mappings, problem (7) has been studied by Ceng and Petruşel [4]. If \( X = H \) and \( \{T_n\} \) is an infinite family of nonexpansive self-mappings, Problem (7) reduces to the following problem: find hierarchically a common fixed point of \( \{T_n\} \) with respect to a nonexpansive mapping \( P \); namely, find \( x^* \in \mathcal{F} \), such that

\[ \langle x^* - Px^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{F}, \tag{8} \]

which was studied by Zhang et al. [5]. If \( X = H \) is a Hilbert space and \( T_n = T \), for all \( n \geq 1 \), where \( T \) is a nonexpansive mapping on \( C \), then problem (7) reduces to the following problem: finding hierarchically a fixed point of \( T \) with respect to another nonexpansive mapping \( P \); namely, find \( x^* \in F(T) \) such that

\[ \langle x^* - Px^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \tag{9} \]

Problem (7) includes many problems as special cases, so it is very important in the area of optimization and related fields, such as signal processing and image reconstruction (see [6–9]).

In 2007, Moudafi [10] introduced the following Krasnoselski-Mann’s algorithm in Hilbert spaces:

\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(\sigma_nPx_n + (1 - \sigma_n)Tx_n), \quad \forall n \geq 0, \tag{10} \]

where \( \{\alpha_n\} \) and \( \{\sigma_n\} \) are two real sequences in \((0, 1)\) and \( T \) and \( P \) are two nonexpansive mappings of \( C \) into itself. Furthermore, he established a weak convergence result for Algorithm (10) for solving problem (9).

Subsequently, Yao and Liou [11] derived a weak convergence result of algorithm (10) under the restrictions on parameters weaker than those in [10, Theorem 2.1].

Recently, Marino and Xu [12] introduced the following explicit hierarchical fixed point algorithm in Hilbert spaces:

\[ x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n)(\alpha_n Vx_n + (1 - \alpha_n)Tx_n), \quad \forall n \geq 0, \tag{11} \]

where \( f \) is a contraction on \( C \) and \( V, T \) are two nonexpansive mappings of \( C \) into itself and proved that the sequence \( \{x_n\} \) generated by (11) converges strongly to a solution of problem (9).

Very recently, Zhang et al. [5] introduced the following iterative algorithm in order to find hierarchically a fixed point of Problem (8):

\[ x_0 \in C, \]

\[ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)y_n, \tag{12} \]

\[ y_n = \beta_n P(x_n) + (1 - \beta_n)Tx_n, \]

where \( f : C \to C \) is a contraction, \( P : C \to C \) is a nonexpansive mapping, \( \{T_n\} : C \to C \) is a countable family of nonexpansive mappings, and \( T : C \to C \) is a mapping defined by

\[ T = \sum_{n=1}^{\infty} \lambda_n T_n, \quad \lambda_n \geq 0 \quad (n = 1, 2, \ldots) \text{ with } \sum_{n=1}^{\infty} \lambda_n = 1. \tag{13} \]

Under suitable conditions on parameters \( \{\alpha_n\} \) and \( \{\beta_n\} \), they established some strong and weak convergence theorems. Note that, in [5], \( \{T_n\} \) is an infinite family of self-mappings and \( P \) is also a self-mapping. And they obtained the results in the setting of Hilbert spaces.

Motivated and inspired by the above researches, in a reflexive Banach space which admits a weakly sequentially continuous duality mapping \( J \), we propose and analyze an iteration process for a countable family of nonexpansive nonself-mappings \( \{T_n\} : C \to X \) and \( S : C \to X \) is a nonexpansive nonself-mapping as follows:

\[ x_0 \in C, \]

\[ x_{n+1} = Q(\alpha_n f(x_n) + (1 - \alpha_n)y_n), \tag{14} \]

\[ y_n = \beta_n Sx_n + (1 - \beta_n)Tx_n, \quad n \geq 0, \]

where \( Q \) is a sunny nonexpansive retraction of \( X \) onto \( C \) and establishes a convergence theorem. Particularly, if \( X = H \) is a Hilbert space, we obtain some convergence results.

To prove the main results, we need the following lemmas.

Lemma 3 (see [1]). Let \( C \) be a nonempty and convex subset of a smooth Banach space \( X, D \subset C, f : X \to X^* \) the normalized duality mapping of \( X \), and \( Q : C \to D \) a retraction. Then the following conditions are equivalent:

(i) \( \langle x - Qx, f(y - Qx) \rangle \leq 0, \) for all \( x \in C \) and \( y \in D; \)

(ii) \( Q \) is both sunny and nonexpansive.

Lemma 4 (see [13, Lemma 3.1, 3.3]). Let \( X \) be a real smooth and strictly convex Banach space and \( C \) a nonempty closed and
convex subset of $X$ which is also a sunny nonexpansive retract of $X$. Assuming that $T: C \to X$ is a nonexpansive mapping and $Q$ is a sunny nonexpansive retraction of $X$ onto $C$, then $F(T) = F(QT)$.

**Lemma 5** (see [1]). Let $X$ be a real Banach space and $J: X \to 2^{X^*}$ the normalized duality mapping. Then for any $x, y \in X$, the following hold:

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle j(x + y), y \rangle$, for all $j(x + y) \in J(x + y)$;

(ii) $\|x\|^2 + 2\langle j(x), j(x) \rangle \leq \|x + y\|^2$, for all $j(x) \in J(x)$.

**Lemma 6** (see [14]). Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying

$$\sum_{n=0}^{\infty} b_n < \infty, \quad a_{n+1} \leq a_n + b_n, \quad n = 0, 1, 2, \ldots$$

Then $\lim_{n \to \infty} a_n$ exists.

**Lemma 7** (see [15]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying

$$a_{n+1} \leq (1 - \lambda) a_n + \lambda a_n + c_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}$, $\{b_n\}$ and $\{c_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0, 1], \sum_{n=0}^{\infty} \lambda_n = \infty$ or, equivalently, $\Pi_{n=0}^{\infty} (1 - \lambda_n) = 0$;

(ii) $\lim \sup_{n \to \infty} b_n \leq 0$;

(iii) $c_n \geq 0$ (or $n \geq 0$), $\sum_{n=0}^{\infty} c_n < \infty$.

Then $\lim_{n \to \infty} a_n = 0$.

If Banach space $X$ admits sequentially continuous duality mapping $J$ from weak topology to weak * topology, then by [16, Lemma 1] we get that duality mapping $J$ is single-valued. In this case, duality mapping $J$ is also said to be weakly sequentially continuous, that is, for each $\{x_n\} \subset X$ with $x_n \to x$, then $J(x_n) \to Jx$ [16, 17].

Recall that a Banach space $X$ is said to be satisfying Opial’s condition if for any sequence $\{x_n\}$ in $E$, $x_n \to x$ (or $n \to \infty$) implies that

$$\lim\sup_{n \to \infty} \|x_n - x\| < \lim\sup_{n \to \infty} \|x_n - y\|, \quad \forall y \in E, \text{ with } y \neq x.$$  \hspace{1cm} (17)

By [16, Lemma 1], we know that if $X$ admits a weakly sequentially continuous duality mapping, then $X$ satisfies Opial’s condition.

In the sequel, we also need the following lemmas.

**Lemma 8** (see [17]). Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $X$ which satisfies Opial’s condition and $T: C \to X$ a nonexpansive mapping. Then the mapping $I - T$ is demiclosed at zero, that is,

$$x_n \to x \quad \text{and} \quad x_n - Tx_n \rightharpoonup 0 \quad \text{implies} \quad x = Tx.$$

Let $C$ be a nonempty and convex subset of a Banach space $X$. Then for $x \in C$, one defines the inward set $I_C(x)$ as follows [2, 3]:

$$I_C(x) = \{ y \in X : y = x + \lambda (z - x), \; z \in C, \; \lambda \geq 0 \}.$$  \hspace{1cm} (19)

A mapping $T: C \to X$ is said to satisfy the inward condition if $Tx \in I_C(x)$ for all $x \in C$. $T$ is also said to satisfy the weakly inward condition if for each $x \in C$, $Tx \in \overline{I_C(x)} (\overline{I_C(x)}$ is the closure of $I_C(x)$). Clearly $C \subset I_C(x)$ and it is not hard to show that $I_C(x)$ is a convex set if $C$ does.

**Lemma 9** (see [18, Theorem 2.4]). Let $X$ be a reflexive Banach space which admits a weakly sequentially continuous duality mapping $J$ from $X$ to $X^*$. Suppose $C$ is a nonempty closed convex subset of $X$ which is also a sunny nonexpansive retract of $X$, and $T: C \to X$ is a nonexpansive mapping satisfying the weakly inward condition and $F(T) \neq \emptyset$. Let $\{u_n\}$ be defined by

$$u_n = Q (\alpha_n f (u_n) + (1 - \alpha_n) Tu_n),$$

where $Q$ is a sunny nonexpansive retraction of $X$ onto $C$ and $\alpha_n \in (0, 1)$ satisfy the following conditions:

(i) $\alpha_n \to 0$, as $n \to \infty$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n \to \infty} (\alpha_n / \alpha_{n+1}) = 1$.

Then $\{x_n\}$ converges strongly to a fixed point $p$ of $T$ such that $p$ is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I - f) p, j(p - u) \rangle \leq 0, \quad \forall u \in F(T).$$  \hspace{1cm} (21)

**Remark 10.** If a Banach space $X$ admits a sequentially continuous duality mapping $J$ from weak topology to weak star topology, from Lemma 1 of [16] it follows that $X$ is smooth. So for Lemma 9, if $X$ is a reflexive and strictly convex Banach space which admits a weakly sequentially continuous duality mapping $J$, by Lemma 4, the weakly inward condition of $T$ can be removed.

2. Main Results

**Theorem 11.** Let $X$ be a reflexive and strictly convex Banach space which admits a weakly sequentially continuous duality mapping $J: X \to X^*$ and $C$ a nonempty, closed and convex subset of $X$ which is also a sunny nonexpansive retraction of $X$. Let $S: C \to X$ be a nonexpansive nonself-mapping, $f: C \to C$
a contractive mapping with a contractive constant \( \rho \in (0, 1) \) and \( T_i : C \to X \) \((i = \{1, 2, \ldots\})\) an infinite family of nonexpansive nonself-mappings such that \( \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset \). Let \( T : C \to X \) be defined by (13) and \( Q \) a sunny nonexpansive retraction of \( X \) onto \( C \). Let \( \{x_n\} \) be the sequence generated by (14), and \( \{\alpha_n\} \) and \( \{\beta_n\} \) the sequences in (0,1) satisfying the following conditions:

(i) \( \alpha_n \to 0 \) \((n \to \infty)\); 
(ii) \( \lim_{n \to \infty} \beta_n \alpha_n = 0 \); 
(iii) \( \sum_{n=0}^{\infty} |\alpha_n + \beta_n| < \infty \).

Then \( \{x_n\} \) converges strongly to some point \( x^* \in F(T) = \bigcap_{i=1}^{\infty} F(T_i) \), which is the unique solution to the following variational inequality:

\[
\langle (I - f) x^*, j(x - x^*) \rangle \geq 0, \quad \forall x \in F(T). \tag{22}
\]

**Proof.** From condition (ii), without loss of generality, we can assume that \( \beta_n \leq \beta_n \) for all \( n \geq 0 \).

First we prove that the sequence \( \{x_n\} \) is bounded. In fact, for any \( u \in F(T) \), we have

\[
\|x_{n+1} - u\| \leq \|Q(\alpha_n f(x_n) + (1 - \alpha_n) y_n) - Q u\| \leq \alpha_n \|f(x_n) - u\| + (1 - \alpha_n) \beta_n \|Sx_n + (1 - \beta_n) T x_n - u\| \leq \alpha_n \beta_n \|Sx_n - u\| + (1 - \alpha_n) \beta_n \|T x_n - u\|.
\]

By induction,

\[
\|x_{n+1} - u\| \leq \max \left\{ \|x_0 - u\|, \frac{\|f(u) - u\| + \|S u - u\|}{1 - \rho} \right\}.
\]

(23)

Thus \( \{x_n\} \) is bounded, so \( \{S x_n\} \) and \( \{T x_n\} \) are also bounded.

Next we prove that \( \|S x_n - u_n\| \to 0 \) as \( n \to \infty \), where the sequence \( \{u_n\} \) is defined by

\[
u_0 = x_0 \in C, \quad u_{n+1} = Q(\alpha_n f(u_n) + (1 - \alpha_n) T u_n).
\]

(25)

By Lemma 9 and Remark 10, \( \{u_n\} \) converges strongly to some point \( x^* \in F(T) \), which is the unique solution to the following variational inequality:

\[
\langle (I - f) x^*, j(x^* - x) \rangle \leq 0, \quad \forall x \in F(T). \tag{26}
\]

Furthermore, we obtain

\[
\begin{align*}
\|x_{n+1} - u_{n+1}\| & \leq \|Q(\alpha_n f(x_n) + (1 - \alpha_n) y_n) - Q(\alpha_n f(u_n) + (1 - \alpha_n) T u_n)\| \leq \alpha_n \|f(x_n) - f(u_n)\| + (1 - \alpha_n) \|y_n - T u_n\| \\
& \leq \alpha_n \rho \|x_n - u_n\| + (1 - \alpha_n) \|S x_n - u\| + (1 - \beta_n) \|T x_n - T u\|.
\end{align*}
\]

(27)

where \( M = \sup_{n \geq 0}\|S x_n - T u\|. \) It follows from conditions (i)-(ii) and Lemma 7 we have \( \lim_{n \to \infty} \|x_n - u_n\| = 0 \). Since as \( n \to \infty, u_n \to x^* \in F(T) \), we get \( x_n \to x^* (n \to \infty) \), which is the unique solution to the variational inequality (22).

**Remark 12.** Theorem 11 extends Theorem 2.1 in [5] from the following aspects: (i) from Hilbert spaces to reflexive and strictly convex Banach spaces which admits a weakly sequentially continuous duality mapping; (ii) for the infinite family of mappings \( \{T_i\} \) from self-mappings to nonself-mappings. In addition, the existence of the sunny nonexpansive retraction has been proved in [19, Theorem 3.10].

**Remark 13.** If we take

\[
\begin{align*}
\alpha_n &= \frac{1}{(1 + n)^\alpha}, \\
\beta_n &= \frac{1}{(1 + n)^\beta},
\end{align*}
\]

(28)

\[0 < \alpha < \beta < 1,\]

then since \( \|x_{n+1} - u_n\| = 1/n^{\alpha+1} \) and \( \|\beta_{n+1} - \beta_n\| = 1/n^{\beta+1} \) as \( n \to \infty \), it is not hard to find that the conditions (i)-(iii) are satisfied. For details, see [12, Remark 3.2].

In the sequel, we consider the result in the setting of Hilbert spaces.

**Theorem 14.** Let \( H \) be a Hilbert space and \( C \) a nonempty, closed and convex subset of \( H \). Let \( S : C \to H \) be a nonexpansive nonself-mapping, \( f : C \to C \) a contractive mapping with a contractive constant \( \rho \in (0, 1) \), and \( T_i : C \to H \) \((i = \{1, 2, \ldots\})\) an infinite family of nonexpansive nonself-mappings such that \( F(T) = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset \). Let \( \{x_n\} \) be the sequence generated by (14) and \( \{\alpha_n\} \) and \( \{\beta_n\} \) the sequences in (0,1) satisfying the following conditions:

(i) \( \alpha_n \to 0 \); 
(ii) \( \lim_{n \to \infty} \beta_n \alpha_n = \tau \in (0, +\infty) \); 
(iii) \( \lim_{n \to \infty} (\|\beta_n - \beta_{n+1}\| + |\alpha_n - \alpha_{n+1}|)/\alpha_n \beta_n = 0 \);
(iv) there exists a constant \( K > 0 \) such that \( 1/\alpha_n (\|1/\beta_n\| - 1/\beta_{n+1}) \| \leq K \) for all \( n > 0 \).
Then \( \{x_n\} \) converges strongly to some point \( x^* \in F(T) \), which is the unique solution to the following variational inequality:

\[
\left\langle \frac{1}{\tau} (I - f) x^* + (I - S) x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in F(T).
\]

(29)

**Proof.** By condition (ii), without loss of generality, we can assume that \( \beta_n \leq (\tau + 1)x_n \) for all \( n \geq 0 \). Similar to the proof of (24), for any \( u \in F(T) \), we have

\[
\|x_{n+1} - u\| \leq \max \left\{ \|x_0 - u\|, \frac{(\tau + 1)(\|f(u) - u\| + \|Su - u\|)}{1 - \rho} \right\}.
\]

(30)

Thus \( \{x_n\} \) is bounded. Furthermore, \( \{f(x_n)\}, \{Tx_n\}, \{y_n\}, \{Sx_n\} \) are all bounded. Put \( u_n = \alpha_nf(x_n) + (1 - \alpha_n)y_n \) and \( M = \sup_{n \geq 0} \|f(x_n)\| + \|y_n\|, \|Tx_n\| + \|Sx_n\| \). So \( \{u_n\} \) and \( \{P_C(u_n)\} \) are also bounded.

**Step 1.** We prove that \( \|x_{n+1} - x_n\| \to 0 \ (n \to \infty) \).

From (14), we obtain

\[
\|x_{n+1} - x_n\| = \|P_C(u_n) - P_C(u_{n-1})\| \leq u_n - u_{n-1} \\
\leq \alpha_n \|f(x_n) - f(x_{n-1})\| + (1 - \alpha_n) \|y_n - y_{n-1}\|
\]

\[
+ |\alpha_n - \alpha_{n-1}| (\|f(x_{n-1})\| + \|y_{n-1}\|)
\]

\[
\leq \alpha_n \rho \|x_n - x_{n-1}\| + (1 - \alpha_n)
\]

\[
\times \|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| M,
\]

\[
\|y_n - y_{n-1}\| \leq \beta_n \|Sx_n - Sx_{n-1}\| + (1 - \beta_n) \|Tx_n - Tx_{n-1}\|
\]

\[
+ |\beta_n - \beta_{n-1}| (\|Sx_{n-1}\| + \|Tx_{n-1}\|)
\]

\[
\leq \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| M.
\]

(31)

Substituting (32) into (31), we have

\[
\|x_{n+1} - x_n\| \leq (1 - \alpha_n(1 - \rho)) \|x_n - x_{n-1}\|
\]

\[
+ \alpha_n \left( |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| \right) M.
\]

(33)

By conditions (i), (iii), and Lemma 7, we have \( \|x_{n+1} - x_n\| \to 0 \ (n \to \infty) \).

**Step 2.** We prove that \( \omega_n(x_n) \subset F(T) \), where \( \omega_n(x_n) \) is the \( \omega \)-limit point set of \( \{x_n\} \) in the weak topology:

\[
\|x_{n+1} - QTx_n\| \leq \alpha_n \|f(x_n)\| + \beta_n \|Sx_n\| + (\alpha_n + \beta_n + \alpha_n \beta_n) \|Tx_n\|.
\]

(34)

Noting that \( \alpha_n \to 0 \) and \( \beta_n \to 0 \), we have \( \|x_{n+1} - QTx_n\| \to 0 \ (n \to \infty) \). Then from **Step 1** we have \( \|x_n - QTx_n\| \to 0 \ (n \to \infty) \). Furthermore, it follows from Lemmas 4 and 8 that \( \omega_n(x_n) \subset F(QT) = F(T) \), where \( Q = P_C \).

**Step 3.** We show that \( \|x_{n+1} - x_n\|/\beta_n \to 0 \ (n \to \infty) \).

It follows from (31) and (33) that

\[
\frac{\|x_{n+1} - x_n\|}{\beta_n} \leq \frac{\|u_n - u_{n-1}\|}{\beta_n} \leq (1 - \alpha_n(1 - \rho)) \frac{\|x_n - x_{n-1}\|}{\beta_n} + (1 - \alpha_n(1 - \rho)) \frac{\|x_n - x_{n-1}\|}{\beta_n} \frac{1}{\beta_n} - \frac{1}{\beta_n} + \frac{(\alpha_n - \alpha_{n-1}) + |\beta_n - \beta_{n-1}| M}{\alpha_n \beta_n}.
\]

(35)

By conditions (i) and (iii), \( \|x_n - x_{n-1}\| \to 0 \ (n \to \infty) \), and Lemma 7, we have

\[
\frac{\|x_{n+1} - x_n\|}{\beta_n} \to 0 \ (n \to \infty).
\]

(36)

Thus from (35), we get

\[
\frac{\|u_n - u_{n-1}\|}{\beta_n} \to 0 \ (n \to \infty).
\]

(37)

**Step 4.** We show that \( \{x_n\} \) converges strongly to some point \( x' \in F(T) \), which is the unique solution of (29).

Setting \( W_n = \beta_n S + (1 - \beta_n)T \), we have

\[
x_{n+1} = P_C(u_n) - u_n + \alpha_n f(x_n) + (1 - \alpha_n) W_n x_n.
\]

(38)

Then

\[
x_n - x_{n+1} = u_n - P_C(u_n) + \alpha_n (I - f) x_n + (1 - \alpha_n) (I - W_n) x_n.
\]

(39)

Letting \( v_n = (x_n - x_{n+1})/(1 - \alpha_n) \beta_n \), from condition (i) and (36), we have \( v_n \to 0 \ (n \to \infty) \). Noting that \( I - W_n \) is
monotone and $I - f$ is $(1 - \rho)$-strongly monotone, for any $x^* \in F(T)$, from Lemma 3 we obtain
\[
\langle v_n, x_n - x^* \rangle \\
= \frac{1}{(1 - \alpha_n) \beta_n} \langle u_n - P_C(u_n), x_n - x^* \rangle \\
+ \frac{\alpha_n}{(1 - \alpha_n) \beta_n} \langle (I - f)x_n, x_n - x^* \rangle \\
+ \frac{1}{\beta_n} \langle (I - W_n)x_n, x_n - x^* \rangle \\
= \frac{1}{(1 - \alpha_n) \beta_n} \langle u_n - P_C(u_n), x_n - x^* \rangle \\
+ \frac{\alpha_n}{(1 - \alpha_n) \beta_n} \langle (I - f)x_n, x_n - x^* \rangle \\
+ \frac{1}{\beta_n} \langle (I - W_n)x_n - (I - W_n)x^*, x_n - x^* \rangle \\
+ \frac{1}{\beta_n} \langle (I - W_n)x^*, x_n - x^* \rangle \\
\geq \frac{1}{(1 - \alpha_n) \beta_n} \langle u_n - P_C(u_n), P_C(u_{n-1}) - P_C(u_n) \rangle \\
+ \frac{\alpha_n}{(1 - \alpha_n) \beta_n} \|x_n - x^*\|^2 \\
+ \frac{\alpha_n}{(1 - \alpha_n) \beta_n} \langle (I - f)x^*, x_n - x^* \rangle \\
+ \langle (I - S)x^*, x_n - x^* \rangle.
\] (40)

Thus we have
\[
\|x_n - x^*\|^2 \\
\leq \frac{(1 - \alpha_n) \beta_n}{\alpha_n (1 - \rho)} \langle v_n, x_n - x^* \rangle
\]
\[
- \frac{(1 - \alpha_n) \beta_n}{\alpha_n (1 - \rho)} \langle (I - S)x^*, x_n - x^* \rangle \\
- \frac{1}{\alpha_n (1 - \rho)} \langle u_n - P_C(u_n), P_C(u_{n-1}) - P_C(u_n) \rangle \\
- \frac{1}{(1 - \rho)} \langle (I - f)x^*, x_n - x^* \rangle
\]
\[
\leq \frac{(1 - \alpha_n) \beta_n}{\alpha_n (1 - \rho)} \|v_n\| \|x_n - x^*\| \\
- \frac{(1 - \alpha_n) \beta_n}{\alpha_n (1 - \rho)} \langle (I - S)x^*, x_n - x^* \rangle \\
+ \frac{1}{\alpha_n (1 - \rho)} \|u_n - P_C(u_n)\| \frac{\|u_{n-1} - u_n\|}{\alpha_n} \\
- \frac{1}{(1 - \rho)} \langle (I - f)x^*, x_n - x^* \rangle.
\] (41)

Since $\beta_n \leq (\tau + 1)\alpha_n$, by (37) we have
\[
\frac{\|u_n - u_{n-1}\|}{\alpha_n} \to 0 \quad (n \to \infty).
\] (42)

Combining condition (ii), $v_n \to 0 \quad (n \to \infty)$, (41), and (42), every weak cluster point of $\{x_n\}$ is also a strong cluster point. From (40), we obtain
\[
\langle (I - f)x_n, x_n - x^* \rangle \\
= \frac{(1 - \alpha_n) \beta_n}{\alpha_n} \langle v_n, x_n - x^* \rangle \\
- \frac{1}{\alpha_n} \langle u_n - P_C(u_n), x_n - x^* \rangle \\
- \frac{(1 - \alpha_n)}{\alpha_n} \langle (I - W_n)x_n, x_n - x^* \rangle \\
= \frac{(1 - \alpha_n) \beta_n}{\alpha_n} \langle v_n, x_n - x^* \rangle \\
- \frac{1}{\alpha_n} \langle u_n - P_C(u_n), P_C(u_{n-1}) - P_C(u_n) \rangle \\
- \frac{1}{\alpha_n} \langle u_n - P_C(u_n), P_C(u_{n-1}) \rangle \\
- \frac{1}{\alpha_n} \langle u_{n-1} - u_n \|u_{n-1} - u_n\| \frac{\|u_{n-1} - u_n\|}{\alpha_n} \\
- \frac{1}{(1 - \rho)} \langle (I - S)x^*, x_n - x^* \rangle \\
- \frac{1}{(1 - \rho)} \langle (I - f)x^*, x_n - x^* \rangle
\]
\[
\times \langle (I - W_n)x_n - (I - W_n)x^*, x_n - x^* \rangle
\]
\[
- \frac{(1 - \alpha_n)}{\alpha_n} \langle (I - W_n)x^*, x_n - x^* \rangle.
\] (43)
\[
\frac{(1 - \alpha_n)\beta_n}{\alpha_n} \|v_n\| \|x_n - x^*\| \\
+ \frac{1}{\alpha_n} \|u_n - P_C(u_n)\| P_C(u_{n-1}) - P_C(u_n)\| \\
- \frac{(1 - \alpha_n) \beta_n}{\alpha_n} \langle (I - S)x^*, x_n - x^* \rangle \\
\leq \frac{(1 - \alpha_n) \beta_n}{\alpha_n} \|v_n\| \|x_n - x^*\| \\
+ \frac{\|u_{n-1} - u_n\|}{\alpha_n} \|u_n - P_C(u_n)\| \\
- \frac{(1 - \alpha_n) \beta_n}{\alpha_n} ((I - S)x^*, x_n - x^*). \\
\]  
(43)

Note that the sequence \(\{x_n\}\) is bounded; thus there exists a subsequence \(\{x_{n_j}\}\) converging to a point \(x' \in H\). From Step 2, we have \(x' \in F(T)\). Then it follows from the above inequality, (42), and \(v_n \to 0 \ (n \to \infty)\) that

\[
\langle (I - f)x', x' - x^* \rangle \\
\leq -\tau \langle (I - S)x^*, x' - x^* \rangle, \quad \forall x^* \in F(T). \\
\]  
(44)

Replacing \(x^*\) with \(x' + \mu(x^* - x')\), where \(\mu \in (0, 1)\) and \(x^* \in F(T)\), we have

\[
\langle (I - f)x', x' - x^* \rangle \\
\leq -\tau \langle (I - S)(x' + \mu(x^* - x')), x' - x^* \rangle, \\
\forall x^* \in F(T). \\
\]  
(45)

Letting \(\mu \to 0\), we have

\[
\langle (I - f)x', x' - x^* \rangle \\
\leq -\tau \langle (I - S)x', x' - x^* \rangle, \quad \forall x^* \in F(T). \\
\]  
(46)

If there exists another subsequence \(\{x_{n_j}\}\) of \(\{x_n\}\) converging to a point \(x'' \in H\). From Step 2, we also have \(x'' \in F(T)\). Then from (46) we obtain

\[
\langle (I - f)x', x' - x'' \rangle \leq -\tau \langle (I - S)x', x' - x'' \rangle \\
\]  
(47)

and, via interchanging \(x'\) and \(x''\),

\[
\langle (I - f)x'', x'' - x' \rangle \leq -\tau \langle (I - S)x'', x'' - x' \rangle. \\
\]  
(48)

Adding up these two inequalities yields

\[
(1 - \rho) \|x' - x''\|^2 \leq \langle (I - f)x' - (I - f)x'', x' - x'' \rangle \leq 0, \\
\]  
(49)

which implies \(x' = x''\). Then \(\{x_n\}\) converges strongly to \(x' \in F(T)\), which is the solution to the following variational inequality:

\[
\frac{1}{\tau} \langle (I - f)x' + (I - S)x', x - x' \rangle \geq 0, \quad \forall x \in F(T). \\
\]  
(50)

Since \(I - f\) is \((1 - \rho)\)-strongly monotone and \(I - S\) is monotone, it is easy to see that the above variational inequality has a unique solution. \(\square\)

**Remark 15.** Theorem 14 extends Theorem 3.2 in [12] from the following aspects: (i) from a nonexpansive mapping \(T\) to an infinite family of nonexpansive mappings \(\{T_i\}\); (ii) from self-mappings to nonself-mappings.

**Acknowledgments**

The author is extremely grateful to the referees for their useful suggestions that improved the content of the paper. Supported by the China Postdoctoral Science Foundation Funded Project (no. 2012M51928).

**References**


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