Research Article
On Fuzzy Modular Spaces

Yonghong Shen¹² and Wei Chen³

¹ School of Mathematics, Beijing Institute of Technology, Beijing 100081, China
² School of Mathematics and Statistics, Tianshui Normal University, Tianshui 741001, China
³ School of Information, Capital University of Economics and Business, Beijing 100070, China

Correspondence should be addressed to Yonghong Shen; shenyonghong2008@hotmail.com

Received 12 November 2012; Revised 30 January 2013; Accepted 18 February 2013

Academic Editor: Luis Javier Herrera

Copyright © 2013 Y. Shen and W. Chen. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The concept of fuzzy modular space is first proposed in this paper. Afterwards, a Hausdorff topology induced by a 𝛽-homogeneous fuzzy modular is defined and some related topological properties are also examined. And then, several theorems on 𝜇-completeness of the fuzzy modular space are given. Finally, the well-known Baire’s theorem and uniform limit theorem are extended to fuzzy modular spaces.

1. Introduction and Preliminaries

In the 1960s, the concept of modular space was introduced by Nakano [1]. Soon after, Musielak and Orlicz [2] redefined and generalized the notion of modular space. A real function ρ on an arbitrary vector space X is said to be a modular if it satisfies the following conditions:

(M-1) ρ(x) = 0 if and only if x = θ (i.e., x is the null vector θ),
(M-2) ρ(x) = ρ(−x),
(M-3) ρ(𝛼x + 𝛽y) ≤ 𝛽ρ(x) + 𝛼ρ(y) for all x, y ∈ X and 𝛼, 𝛽 ≥ 0 with 𝛼 + 𝛽 = 1.

A modular space Xρ is defined by a corresponding modular ρ, that is, Xρ = {x ∈ X : ρ(λx) → 0 as λ → 0}.


In 2007, Nourouzi [14] proposed probabilistic modular spaces based on the theory of modular spaces and some researches on the Menger’s probabilistic metric spaces. A pair (X, ρ) is called a probabilistic modular space if X is a real vector space, ρ is a mapping from X into the set of all distribution functions (for x ∈ X, the distribution function ρ(x) is denoted by ρx, and ρx(t) is the value ρx at t ∈ ℝ) satisfying the following conditions:

(PM-1) ρx(0) = 0,
(PM-2) ρx(t) = 1 for all t > 0 if and only if x = θ,
(PM-3) ρ−x(t) = ρx(t),
(PM-4) ρx+βy(s + t) ≥ 𝛽ρx(s) ∧ ρy(t) for all x, y ∈ X and 𝛼, 𝛽, s, t ∈ ℝ⁺, 𝛼 + 𝛽 = 1.

Especially, for every x ∈ X, t > 0 and α ∈ ℝ\{0}, if

ραx(t) = ρx t |α|β , where β ∈ (0, 1],

then we say that (X, ρ) is 𝛽-homogeneous.

Recently, further studies have been made on the probabilistic modular spaces. Nourouzi [15] extended the well-known Baire’s theorem to probabilistic modular spaces by using a special condition. Fallahi and Nourouzi [16] investigated the continuity and boundedness of linear operators...
defined between probabilistic modular spaces in the probabilistic sense.

In this paper, following the idea of probabilistic modular space and the definition of fuzzy metric space in the sense of George and Veeramani [17], we apply fuzzy concept to the classical notions of modular and modular spaces and propose a novel concept named fuzzy modular spaces.

2. Fuzzy Modular Spaces

In this section, following the idea of probabilistic modular space, we will introduce the concept of fuzzy modular space by using continuous t-norm and present some related notions.

Definition 1 (Schweizer and Sklar [18]). A binary operation \( \ast : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is called a continuous t-norm if it satisfies the following conditions:

(TN-1) \( \ast \) is commutative and associative;
(TN-2) \( \ast \) is continuous;
(TN-3) \( a \ast 1 = a \) for every \( a \in [0, 1] \);
(TN-4) \( a \ast b \leq c \ast d \) whenever \( a \leq c, b \leq d \) and \( a, b, c, d \in [0, 1] \).

Three common examples of the continuous t-norm are:
1. \( a \ast_b = \min\{a, b\} \);
2. \( a \ast_b = a \cdot b \);
3. \( a \ast_b = \max\{a + b - 1, 0\} \).

For more examples, the reader can be referred to [19].

Definition 2 (George and Veeramani [17]). A fuzzy metric space is an ordered triple \((X, M, \ast)\) such that \(X\) is a nonempty set, \(\ast\) is a continuous t-norm, and \(M\) is a fuzzy set on \(X \times X \times (0, \infty)\) satisfying the following conditions, for all \(x, y, z \in X, s, t > 0\):

(F-1) \( M(x, y, t) > 0 \);
(F-2) \( M(x, y, t) = 1 \) if and only if \( x = y \);
(F-3) \( M(x, y, t) = M(y, x, t) \);
(F-4) \( M(x, y, t) * M(y, z, s) \leq M(x, z, t + s) \);
(F-5) \( M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1] \) is continuous.

Based on the notion of probabilistic modular space and Definition 2, we will propose a novel concept named fuzzy modular spaces.

Definition 3. The triple \((X, \mu, \ast)\) is said to be a fuzzy modular space (shortly, \(F\)-modular space) if \(X\) is a real or complex vector space, \(\ast\) is a continuous t-norm, and \(\mu\) is a fuzzy set on \(X \times (0, \infty)\) satisfying the following conditions, for all \(x, y \in X, s, t > 0\) and \(\alpha, \beta \geq 0\) with \(\alpha + \beta = 1\):

(FM-1) \( \mu(x, t) > 0 \);
(FM-2) \( \mu(x, t) = 1 \) for all \( t > 0 \) if and only if \( x = \theta \);
(FM-3) \( \mu(x, t) = \mu(-x, t) \);
(FM-4) \( \mu(\alpha x + \beta y, s + t) \geq \mu(x, s) * \mu(y, t) \);
(FM-5) \( \mu(x, \cdot) : (0, \infty) \rightarrow (0, 1] \) is continuous.

Generally, if \((X, \mu, \ast)\) is a fuzzy modular space, we say that \((\mu, \ast)\) is a fuzzy modular on \(X\). Moreover, the triple \((X, \mu, \ast)\) is called \(\beta\)-homogeneous if for every \(x \in X, t > 0\) and \(\lambda \in \mathbb{R} \setminus \{0\}\),

\[
\mu(\lambda x, t) = \mu\left(x, \frac{t}{|\lambda|^\beta}\right), \quad \text{where } \beta \in (0, 1].
\]  

Example 4. Let \(X\) be a real or complex vector space and let \(\rho\) be a modular on \(X\). Take \(t\)-norm \(a \ast b = a \ast_L b\). For every \(t \in (0, \infty)\), define \(\mu(x, t) = t/(t + \rho(x))\) for all \( x \in X\). Then \((X, \mu, \ast)\) is a \(F\)-modular space.

Remark 5. Note that the above conclusion still holds even if the \(t\)-norm is replaced by \(a \ast b = a \ast_P b\) and \(a \ast b = a \ast_L b\), respectively.

Example 6. Let \(X = \mathbb{R}, \rho\) is a modular on \(X\), which is defined by \(\rho(x) = |x|^\beta\), where \(\beta \in (0, 1]\). Take \(t\)-norm \(a \ast b = a \ast_P b\). For every \(t \in (0, \infty)\), we define

\[
\mu(x, t) = \frac{1}{e^{\rho(x)/t}}
\]

for all \(x \in X\). Then \((X, \mu, \ast)\) is a \(\beta\)-homogeneous \(F\)-modular space.

Proof. We just need to prove the condition (FM-4) of Definition 3 and formula (2), because other conditions hold obviously. In the following, we first verify \(\mu(ax + \beta y, s + t) \geq \mu(x, s) * \mu(y, t)\), as \(\alpha, \beta \geq 0\) with \(\alpha + \beta = 1\).

Since \(\rho\) is a modular on \(X\), for all \(x, y \in X\), we have

\[
\rho(ax + \beta y) \leq \rho(x) + \rho(y).
\]  

Then, we can obtain

\[
\rho(ax + \beta y) \leq \frac{t + s}{t} \rho(x) + \frac{t + s}{s} \rho(y),
\]

that is,

\[
\frac{1}{t + s} \rho(ax + \beta y) \leq \frac{1}{t} \rho(x) + \frac{1}{s} \rho(y).
\]

Therefore

\[
e^{\rho(ax+\beta y)/(t+s)} \leq e^{\rho(x)/t} \cdot e^{\rho(y)/s} = e^{\rho(x)/t} \ast_P e^{\rho(y)/s}.
\]

Thus, we have \(\mu(ax + \beta y, s + t) \geq \mu(x, s) * \mu(y, t)\).

On the other hand, for all \(\lambda \in \mathbb{R} \setminus \{0\}\), since \(\rho(\lambda x) = |\lambda x|^\beta = |\lambda|^\beta \cdot |x|^\beta = |\lambda|^\beta \rho(x)\), it follows that

\[
\mu(\lambda x, t) = \mu\left(x, \frac{t}{|\lambda|^\beta}\right).
\]

Hence, we know that \((X, \mu, \ast)\) is a \(\beta\)-homogeneous \(F\)-modular space.

\[\square\]

Theorem 7. If \((X, \mu, \ast)\) is a \(F\)-modular space, then \(\mu(x, \cdot)\) is nondecreasing for all \(x \in X\).
Proof. Suppose that \( \mu(x, t) < \mu(x, s) \) for some \( t > s > 0 \). Without loss of generality, we can take \( \alpha = 1, \beta = 0 \), and \( y = \theta \) is the null vector in \( X \). By Definition 3, we can obtain

\[
\mu(x, s) \ast \mu(\theta, t - s) = \mu(x, s) \ast \mu(y, t - s) \leq \mu(ax + by, t) = \mu(x, t) \ast \mu(x, s).
\]

(9)

Since \( \mu(\theta, t - s) = 1 \), we have \( \mu(x, s) < \mu(x, s) \). Obviously, this leads to a contradiction. \( \square \)

It should be noted that, in general, a fuzzy modular and a fuzzy metric (in the sense of George and Veeramani [17]) do not necessarily induce mutually when the triangular norm is the same one. In essence, the fuzzy modular and fuzzy metric can be viewed as two different characterizations for the same set. The former is regarded as a kind of fuzzy quantization on the classical vector modular, while the latter is regarded as a fuzzy measure on the distance between two points. Next, we construct two examples to show that there does not exist direct relationship between a fuzzy modular and a fuzzy metric.

Example 8. Let \( X = \mathbb{R} \). Take \( t \)-norm \( a \ast b = a \ast_M b \). For every \( t \in (0, \infty) \), we define

\[
\mu(x, t) = \frac{k}{k + |x|},
\]

(10)

where \( k > 0 \) is a constant.

Here, we only show that \( \mu(x, t) \) satisfies the condition (FM-4) of Definition 3, since other conditions can be easily verified.

For every \( x, y \in \mathbb{R} \), and \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \). Without loss of generality, we assume that \( |x| \leq |y| \). Since \( |ax + by| \leq |y| \), we then obtain

\[
\mu(ax + by, t + s) = \frac{k}{k + |ax + by|} \geq \frac{k}{k + |y|} = \min \left\{ \frac{k}{k + |x|}, \frac{k}{k + |y|} \right\} = \frac{k}{k + |x|} \mu(x, t) \ast_M \mu(y, s).
\]

(11)

Hence \((\mu, \ast_M)\) is a fuzzy modular on \( X \). However, if we set

\[
M(x, y, t) = \mu(x - y, t) = \frac{k}{k + |x - y|},
\]

(12)

it is easy to verify that \((M, \ast_M)\) is not a fuzzy metric on \( X \).

Example 9. Let \( X = \mathbb{R} \). Take \( t \)-norm \( a \ast b = a \ast_M b \). For every \( x, y \in X \) and \( t \in (0, \infty) \), we define

\[
M(x, y, t) = \begin{cases}
1, & x = y, \\
\frac{1}{2}, & x \neq y, x, y \in \mathbb{Z}, \\
\frac{1}{4}, & x \in \mathbb{Z}, y \in \mathbb{R} \setminus \mathbb{Z} \text{ or } x \in \mathbb{R} \setminus \mathbb{Z}, y \in \mathbb{Z}, \\
\frac{1}{4}, & x \neq y, x, y \in \mathbb{R} \setminus \mathbb{Z}.
\end{cases}
\]

(13)

It can easily be shown that \((M, \ast_M)\) is a fuzzy metric on \( X \).

3. Topology Induced by a \( \beta \)-Homogeneous Fuzzy Modular

In this section, we will define a topology induced by a \( \beta \)-homogeneous fuzzy modular and examine some topological properties. Let \( \mathbb{N} \) denote the set of all positive integers.

Definition 10. Let \((X, \mu, \ast)\) be a \( \mathcal{F} \)-modular space. The \( \mu \)-ball \( B(x, r, t) \) with center \( x \in X \) and radius \( r, 0 < r < 1, t > 0 \) is defined as

\[
B(x, r, t) = \{ y \in X : \mu(x - y, t) > 1 - r \}.
\]

(15)

An element \( x \in E \) is called a \( \mu \)-interior point of \( E \) if there exist \( r \in (0, 1) \) and \( t > 0 \) such that \( B(x, r, t) \subseteq E \). Meantime, we say that \( E \) is a \( \mu \)-open set in \( X \) if and only if every element of \( E \) is a \( \mu \)-interior point.

Lemma 11 (George and Veeramani [17]). If the \( t \)-norm \( \ast \) is continuous, then

(1) for every \( r_1, r_2 \in (0, 1) \) with \( r_1 > r_2 \), there exists \( r_3 \in (0, 1) \) such that \( r_1 \ast r_3 \geq r_2 \),

(2) for every \( r_4 \in (0, 1) \), there exists \( r_5 \in (0, 1) \) such that \( r_5 \ast r_5 \geq r_4 \).

Theorem 12. If \((X, \mu, \ast)\) is a \( \beta \)-homogeneous \( \mathcal{F} \)-modular space, then \( B(x, r, t/2^{\beta+1}) \subseteq B(x, r, t/2) \).
Proof. By Theorem 7, for every \( r \in (0,1) \) and \( t > 0 \), since \( \mu(x-y,t/2) \geq \mu(x-y,t/2^{\beta+1}) \), it is obvious that \( \{ y \in X : \mu(x-y,t/2) > 1-r \} \subset \{ y \in X : \mu(x-y,t/2) > 1-r \} \). \( \square \)

**Theorem 13.** Let \((X,\mu,*)\) be a \( \beta \)-homogeneous \( \mathcal{F} \)-modular space. Every \( \mu \)-ball \( B(x,r,t) \) in \((X,\mu,*)\) is a \( \mu \)-open set.

Proof. By Definition 10, for every \( y \in B(x,r,t) \), we have \( \mu(x-y,t) > 1-r \). Without loss of generality, we may assume that \( t = 2t_1 \). Since \( \mu(x-y,:) \) is continuous, there exists an \( \varepsilon > 0 \) such that \( \mu(x-y,(t_1-\varepsilon)/2^{\beta+1}) > 1-r \) for some \( \varepsilon > 0 \) with \( (t_1-\varepsilon)/2^{\beta+1} > 0 \) and \( (t_1-\varepsilon)/2^{\beta+1} \in (0,\varepsilon) \). Set \( r_0 = \mu(x-y,(t_1-\varepsilon)/2^{\beta+1}) \). Since \( r_0 > 1-r \), there exists an \( s \in (0,1) \) such that \( r_0 > 1-s > 1-r \). According to Lemma 11, we can find an \( r_1 \in (0,1) \) such that \( r_0 * r_1 \geq 1-s \).

Next, we show that \( B(y,1-r_1,\varepsilon/2^{\beta+1}) \subset B(x,r,2t_1) \). For every \( z \in B(y,1-r_1,\varepsilon/2^{\beta+1}) \), we have \( \mu(y-z,\varepsilon/2^{\beta+1}) > r_1 \). Therefore,

\[
\mu(x-z,t) = \mu(x-z,2t_1) \geq \mu(2(x-y),2(t_1-\varepsilon)) \\
* \mu(2(y-z),2\varepsilon) \\
= \mu(x-y,t_1-\varepsilon) * \mu(y-z,\varepsilon) \geq r_0 * r_1 \geq 1-s > 1-r.
\]

Thus \( z \in B(x,r,t) \) and hence \( B(y,1-r_1,\varepsilon/2^{\beta+1}) \subset B(x,r,t) \). \( \square \)

**Theorem 14.** Let \((X,\mu,*)\) be a \( \beta \)-homogeneous \( \mathcal{F} \)-modular space. Define

\[
\mathcal{F}_\mu = \{ A \subset X : x \in A \text{ if and only if there exist } t > 0 \text{ and } r \in (0,1) \text{ such that } B(x,r,t) \subset A \}.
\]

Then \( \mathcal{F}_\mu \) is a topology on \( X \).

Proof. The proof will be divided into three parts.

(i) Obviously, \( \emptyset, X \in \mathcal{F}_\mu \).

(ii) Suppose that \( A, B \in \mathcal{F}_\mu \). If \( x \in A \cap B \), then \( x \in A \) and \( x \in B \).

Therefore, there exist \( 0 < r_1, r_2 < 1 \) and \( t_1, t_2 > 0 \) such that \( B(x,r_1,t_1) \subset A \) and \( B(x,r_2,t_2) \subset B \). Set \( t = \min\{r_1,r_2\}, t = \min\{t_1,t_2\} \). Now, we claim that \( B(x,r,t) \subset B(x,r_1,t_1) \cap B(x,r_2,t_2) \).

If \( y \in B(x,r,t) \), then we know that \( \mu(x-y,t) > 1-r \). According to Theorem 7, we can obtain

\[
\mu(x-y,t_1) \geq \mu(x-y,t) > 1-r \geq 1-r_1.
\]

(iii) Suppose that \( \mathcal{F}_\mu \subset \mathcal{F}_\mu \). If \( x \in \bigcup_{A \in \mathcal{F}_\mu} A \), then there exists \( U \in \mathcal{F}_\mu \) such that \( x \in U \). Since \( U \in \mathcal{F}_\mu \), there exist \( 0 < r < 1 \) and \( t > 0 \) such that \( B(x,r,t) \subset U \subset \bigcup_{A \in \mathcal{F}_\mu} A \). Hence, \( \bigcup_{A \in \mathcal{F}_\mu} A \in \mathcal{F}_\mu \). \( \square \)

Obviously, if we take \( r = t = (1/n) (n = 1,2,3,\ldots) \), then the family of \( \mu \)-ball \( B(x,1/n,1/n), (n = 1,2,3,\ldots) \) constitutes a countable local base at \( x \). Therefore, we can obtain Theorem 15.

**Theorem 15.** The topology \( \mathcal{F}_\mu \) induced by a \( \beta \)-homogeneous \( \mathcal{F} \)-modular space is first countable.

**Theorem 16.** Every \( \beta \)-homogeneous \( \mathcal{F} \)-modular space is Hausdorff.

Proof. For the \( \beta \)-homogeneous \( \mathcal{F} \)-modular space \((X,\mu,*)\), let \( x, y \) be two distinct points in \( X \). By Definition 3, we can easily obtain \( 0 < \mu(x-y,t) < 1 \) for all \( t > 0 \). Set \( r = \mu(x-y,t) \). According to Lemma 11, for every \( r_0 \in (r,1) \), there exists \( r_1 \in (0,1) \) such that \( r_1 * r_1 \geq r_0 \).

Next, we consider the \( \mu \)-balls \( B(x,1-r_1,t/2^{\beta+1}) \) and \( B(y,1-r_1,t/2^{\beta+1}) \) and then show that \( B(x,1-r_1,t/2^{\beta+1}) \cap B(y,1-r_1,t/2^{\beta+1}) = \emptyset \) using reduction to absurdity. If there exists \( z \in B(x,1-r_1,t/2^{\beta+1}) \cap B(y,1-r_1,t/2^{\beta+1}) \), then

\[
r = \mu(x-y,t) \geq \mu(2(z-x),t/2) * \mu(2(z-y),t/2) \\
= \mu(x-z,t/2^{\beta+1}) * \mu(z-y,t/2^{\beta+1}) \\
\geq r_1 * r_1 \geq r_0,
\]

which is a contradiction. Hence \((X,\mu,*)\) is Hausdorff. \( \square \)

In order to obtain some further properties, several basic notions derived from general topology are introduced in the \( \mathcal{F} \)-modular space.

**Definition 17.** Let \((X,\mu,*)\) be a \( \mathcal{F} \)-modular space.

(i) A sequence \( \{x_n\} \) in \( X \) is said to be \( \mu \)-convergent to a point \( x \in X \), denoted by \( x_n \xrightarrow{\mu} x \), if for every \( r \in (0,1) \) and \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( x_n \in B(x,r,t) \) for all \( n \geq n_0 \).

(ii) A subset \( A \subset X \) is called \( \mu \)-bounded if and only if there exist \( t > 0 \) and \( r \in (0,1) \) such that \( \mu(x,t) > 1-r \) for all \( x \in A \).

(iii) A subset \( B \subset X \) is called \( \mu \)-compact if and only if every \( \mu \)-open cover of \( B \) has a finite subcover (or equivalently, every sequence in \( B \) has a \( \mu \)-convergent subsequence in \( B \)).

(iv) A subset \( C \subset X \) is called \( \mu \)-closed if and only if for every sequence \( \{x_n\} \subset C \), \( x_n \xrightarrow{\mu} x \) implies \( x \in C \).

**Theorem 18.** Every \( \mu \)-compact subset \( A \) of a \( \beta \)-homogeneous \( \mathcal{F} \)-modular space \((X,\mu,*)\) is \( \mu \)-bounded.
Proof. Suppose that $A$ is a $\mu$-compact subset of the given $\beta$-homogeneous $\mathcal{F}$-modular space $(X, \mu, *)$. Fix $t > 0$ and $r \in (0, 1)$, it is easy to see that the family of $\mu$-ball $\{B(x, r, t/2^{\beta+1}) : x \in A\}$ is a $\mu$-open cover of $A$. Since $A$ is $\mu$-compact, there exist $x_1, x_2, \ldots, x_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} B(x_i, r, t/2^{\beta+1})$. For every $x \in A$, there exists $i$ such that $x \in B(x_i, r, t/2^{\beta+1})$. Therefore, we have $\mu(x - x_i, t/2^{\beta+1}) > 1 - r$. Set $\alpha = \min\{\mu(x_i, t/2^{\beta+1}) : 1 \leq i \leq n\}$. Clearly, we know that $\alpha > 0$. Thus, we have

$$
\mu(x, t) = \mu((x - x_i) + x_i, t) \geq \mu\left(2(x - x_i), \frac{t}{2}\right) * \mu\left(2x_i, \frac{t}{2}\right) 
= \mu\left(x - x_i, \frac{t}{2^{\beta+1}}\right) * \mu\left(x_i, \frac{t}{2^{\beta+1}}\right) 
\geq (1 - r) * \alpha > 1 - s
$$

for some $s \in (0, 1)$. This shows that $A$ is $\mu$-bounded. \hfill \square

**Theorem 19.** Let $(X, \mu, *)$ be a $\beta$-homogeneous $\mathcal{F}$-modular space, and let $\mathcal{F}_\mu$ be the topology induced by the $\beta$-homogeneous modular. Then for a sequence $\{x_n\}$ in $X$, $x_n \xrightarrow{\mu} x$ if and only if $\mu(x - x_n, t) \to 1$ as $n \to \infty$.

Proof. Fix $t > 0$. Suppose that $x_n \xrightarrow{\mu} x$. Then for every $r \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x, t) > 1 - r$ for all $n \geq n_0$. Namely, $\mu(x_n - x, t) > 1 - r$ for all $n \geq n_0$. Thus, we have $1 - \mu(x_n - x, t) < r$ for all $n \geq n_0$. Because $r$ is arbitrary, we can verify that $\mu(x_n - x, t) \to 1$ as $n \to \infty$.

On the other hand, if for every $t > 0$, $\mu(x - x_n, t) \to 1$ as $n \to \infty$, then for every $r \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $1 - \mu(x - x_n, t) < r$ for all $n \geq n_0$. Therefore, we know that $\mu(x - x_n, t) > 1 - r$ for all $n \geq n_0$. Thus $x_n \in B(x, r, t/2^{\beta+1})$ for all $n \geq n_0$, and hence $x_n \xrightarrow{\mu} x$ as $n \to \infty$. \hfill \square

### 4. $\mu$-Completeness of a Fuzzy Modular Space

In this section, we will establish some related theorems of $\mu$-completeness of a fuzzy modular space.

**Definition 20.** Let $(X, \mu, *)$ be a $\mathcal{F}$-modular space.

(i) A sequence $\{x_n\}$ in $X$ is a $\mu$-Cauchy sequence if and only if for every $e \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x_m, t) > 1 - e$ for all $m, n \geq n_0$.

(ii) The $\mathcal{F}$-modular space $(X, \mu, *)$ is called $\mu$-complete if every $\mu$-Cauchy sequence is $\mu$-convergent.

In [16], Fallahi and Nourouzi proved that every $\mu$-convergent sequence is a $\mu$-Cauchy sequence in the $\beta$-homogeneous $\mathcal{F}$-modular space. Here we will propose a similar result in a $\mathcal{F}$-modular space. Noticing that the following theorem shows that a $\mu$-convergent sequence is not necessarily a $\mu$-Cauchy sequence in a general $\mathcal{F}$-modular space.

**Theorem 21.** Let $(X, \mu, *_{\mathcal{F}})$ be a $\beta$-homogeneous $\mathcal{F}$-modular space. Then every $\mu$-convergent sequence $(x_n)$ in $X$ is a $\mu$-Cauchy sequence.

Proof. Suppose that the sequence $(x_n)$ $\mu$-converges to $x \in X$. Therefore, for every $e \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x, t/2^{\beta+1}) > 1 - e$ for all $n \geq n_0$. For all $m, n \geq n_0$, we have

$$
\mu(x_m - x_n, t) \geq \mu\left(2(x_m - x_n), \frac{t}{2}\right) * \mu\left(2x, \frac{t}{2}\right) 
\geq \mu\left(x_m - x_n, \frac{t}{2^{\beta+1}}\right) * \mu\left(x_n, \frac{t}{2^{\beta+1}}\right) 
> (1 - e) * (1 - e) = 1 - e.
$$

Hence $(x_n)$ is a $\mu$-Cauchy sequence in $X$. \hfill \square

**Remark 22.** The proof of Theorem 21 shows that, in the $\mathcal{F}$-modular space, a $\mu$-convergent sequence is not necessarily a $\mu$-Cauchy sequence. However, the $\beta$-homogeneity and the choice of triangular norms are essential to guarantee the establishment of theorem.

**Theorem 23.** Every $\mu$-closed subspace of $\mu$-complete $\mathcal{F}$-modular space is $\mu$-complete.

Proof. From Definition 20, it is evident to see that the theorem holds. \hfill \square

**Theorem 24.** Let $(X, \mu, *_{\mathcal{F}})$ be a $\beta$-homogeneous $\mathcal{F}$-modular space, and let $Y$ be a subset of $X$. If every $\mu$-Cauchy sequence of $Y$ is $\mu$-convergent in $X$, then every $\mu$-Cauchy sequence of $Y$ is also $\mu$-convergent in $X$, where $\overline{Y}$ denotes the $\mu$-closure of $Y$.

Proof. Suppose that the sequence $(y_n)$ is a $\mu$-Cauchy sequence of $Y$. Therefore, for every $n \in \mathbb{N}$ and $t > 0$, there exists $y_n \in Y$ such that $\mu(y_n - y_m, t/4^{\beta+1}) > 1 - (n + 1)/(n + 2)$. According to Theorem 7, we have $\mu(x_n - y_m, t/4^{\beta+1}) > 1 - r$ for all $m, n \geq n_0$. That is to say, $\mu(x_n - y_n, t/4^{\beta+1}) \to 1$ as $m, n \to \infty$. Next, we will show that the sequence $(y_n)$ is a $\mu$-Cauchy sequence of $Y$. For every $m, n \geq n_0$, we have

$$
\mu(y_n - y_m, t) \geq \mu\left(2(y_n - y_m), \frac{t}{2}\right) * \mu\left(2y, \frac{t}{2}\right) 
\geq \mu\left(2y_n - y_m, \frac{t}{2^{\beta+1}}\right) * \mu\left(4y_n - y_m, \frac{t}{2^{\beta+1}}\right) 
* \mu\left(4(y_n - y_m), \frac{t}{4}\right).
$$
\[ z \in B(y, r', e/4^\beta), \text{ there exists a sequence } \{z_n\} \text{ in } B(y, r', e/4^\beta) \text{ such that } z_n \to z \text{ and hence we have} \]
\[ \mu(z - y, e/4^\beta) \geq \mu\left(2 \left(z - z_n\right), e/2^\beta \right) \ast_M \mu\left(2 \left(z - y\right), e/2^\beta \right) \]
\[ = \mu\left(z - z_n, e/4^\beta\right) \ast_M \mu\left(z_n - y, e/4^\beta\right) > 1 - r \]
\[ \text{for some } n \in \mathbb{N}. \text{ Therefore, we can obtain} \]
\[ \mu(x - z, 2t) = \mu\left(2 \left(z - y\right), 2e\right) \ast_M \mu\left(2 \left(x - y\right), 2 \left(t - e\right\right) \]
\[ = \mu\left(z - y, e/2^\beta \right) \ast_M \mu\left(x - y, t - e/2^\beta \right) \]
\[ > (1 - r) \ast_M (1 - r) = 1 - r. \]

This shows that \( B(y, r', e/4^\beta) \subseteq B(x, r, 2t) \). It means that if \( A \) is a nonempty \( \mu \)-open set of \( X \), then \( A \cap U_n \) is nonempty and \( \mu \)-open. Now, let \( x_k \in A \cap U_n \), there exist \( t_1 > 0 \) and \( t_2 > 0 \) such that \( B(x_k, t_1/2^\beta \cap A \cap U_n \). Choose \( r_1 > r_1 \) and \( r_2 = \min\{t_1, 1\} > 0 \) such that \( B(x_k, r, t_1/2^\beta \cap A \cap U_n \). Since \( U_2 \) is \( \mu \)-dense in \( X \), we can obtain \( B(x_k, r, t_1/2^\beta \cap A \cap U_n \). Choose \( r_2 < r_1 \) and \( t_2/2^\beta \)， \( B(x_k, r, t_2/2^\beta \cap A \cap U_n \). By induction, we can obtain a sequence \( \{x_k\} \) in \( X \) and two sequence \( \{r_k\}, \{t_k/2^\beta\} \) such that \( 0 < r_k < 1/n, 0 < t_k/2^\beta < 1/n \) and \( B(x_k, r_k, t_k/2^\beta \subseteq A \cap U_n \).

Next, we show that \( \{x_k\} \) is a \( \mu \)-Cauchy sequence. For given \( t_1 \) and \( r_k \), we have
\[ \mu(x - x_k, 2t_k \geq \mu \left(2 \left(x - x_k\right), 2t_k \right) \]
\[ = \mu\left(x - x_k, t_k/2^\beta \right) \ast_M \mu\left(x_k - x, t_k/2^\beta \right) \]
\[ > 1 - r_k > 1 - r. \]

According to the arbitrary of \( t \), it follows that \( \{x_k\} \) is a \( \mu \)-Cauchy sequence. Since \( X \) is \( \mu \)-complete, there exists \( x \in X \) such that \( x_n \mu \rightarrow x \). But \( x_k \in B(x_k, r_k, t_k/2^\beta) \) for all \( n \geq k \), and therefore \( x \in \bigcap_{n=1}^{\infty} U_n \cap U_k \) for all \( k \). Thus \( A \cap \bigcap_{n=1}^{\infty} U_n \) is \( \mu \)-dense in \( X \).

Definition 27. Let \( X \) be any nonempty set and let \((Y, \mu, \ast)\) be a \( \mathcal{F} \)-modular space. A sequence \( \{f_n\} \) of functions from \( X \) to
Y is said to \( \mu \)-converge uniformly to a function \( f \) from \( X \) to \( Y \) if given \( t > 0 \) and \( r \in (0,1) \); there exists \( n_0 \in \mathbb{N} \) such that

\[
\mu(f_n(x) - f(x), t) > 1 - r \quad \text{for all } n \geq n_0 \text{ and for every } x \in X.
\]

**Theorem 28** (Uniform limit theorem). Let \( f_n : X \to Y \) be a sequence of continuous functions from a topological space \( X \) to a \( \beta \)-homogeneous \( \mathcal{F} \)-modular space \( (Y, \mu, \ast) \). If \( \{f_n\} \) \( \mu \)-converges uniformly to \( f : X \to Y \), then \( f \) is continuous.

**Proof.** Let \( V \) be a \( \mu \)-open set of \( Y \) and \( x_0 \in f^{-1}(V) \). Since \( V \) is \( \mu \)-open, there exist \( r \in (0,1) \) and \( t > 0 \) such that \( B(f(x_0), r, t) \subset V \). Owing to \( r \in (0,1) \), we can choose \( s \in (0,1) \) such that \( (1-s) \ast (1-s) \ast (1-s) > 1 - r \). Since \( \{f_n\} \) \( \mu \)-converges uniformly to \( f \), given \( s \in (0,1) \) and \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( \mu(f_n(x) - f(x), t/4^{\beta+1}) > 1 - s \) for all \( n \geq n_0 \) and for every \( x \in X \). Moreover, \( f_n \) is continuous for every \( n \in \mathbb{N} \), there exists a neighborhood \( U \) of \( x_0 \) such that \( f_n(U) \subset B(f(x_0), s, t/4^{\beta+1}) \). Therefore, we know that \( \mu(f_n(x) - f_n(x_0), t/4^{\beta+1}) > 1 - s \) for every \( x \in U \). Thus, we have

\[
\mu(f(x) - f(x_0), t) \geq \mu(2(f(x) - f_n(x)), t/2) \ast \mu(2(f_n(x) - f(x_0)), t/2) \\
= \mu(f(x) - f_n(x), t/2^{\beta+1}) \ast \mu(f_n(x) - f(x_0), t/2^{\beta+1}) \\
\geq \mu(f(x) - f_n(x), t/2^{\beta+1}) \ast \mu(2(f_n(x) - f_n(x_0)), t/2^{\beta+1}) \\
= \mu(2(f_n(x_0) - f(x_0)), t/2^{\beta+1}) \\
= \mu(f(x) - f_n(x_0), t/2^{\beta+1}) \\
\geq \mu(f_n(x_0) - f(x_0), t/2^{\beta+1}) \\
\geq (1-s) \ast (1-s) \ast (1-s) > 1 - r.
\]

(27)

This shows that \( f(x) \in B(f(x_0), r, t) \subset V \). Hence \( f(U) \subset V \); that is, \( f \) is continuous.

**Remark 29.** All the results in this paper are still valid if the condition (FM-5) in Definition 3 is replaced by left continuity.

6. Conclusions

In this paper, we have proposed the concept of fuzzy modular space based on the (probabilistic) modular space and continuous \( t \)-norm, which can be regarded as a generalization of (probabilistic) modular space in the fuzzy sense. Meantime, two examples are given to show that a fuzzy modular and a fuzzy metric do not necessarily induce mutually when the triangular norm is the same one. In the sequel, we have defined \( \beta \)-homogeneity is essential to ensure the establishment of most important conclusions, and some properties also depend on the choice of triangular norms. Finally, we have extended the well-known Baire's theorem and uniform limit theorem to \( \beta \)-homogeneous fuzzy modular spaces.

Further research will focus on the following problems. (1) We first address the problem whether there is a relationship between a fuzzy modular and a fuzzy metric. If the aforementioned relationship exists, then the following issue should be simultaneously considered. (2) It has important theoretical values to explore what conditions a fuzzy modular and a fuzzy metric can induce mutually. (3) Similar to the fixed point theory in probabilistic or fuzzy metric spaces, it is an interesting and valuable research direction to construct fixed point theorems in fuzzy modular spaces. (4) Inspired by [3, 4, 20–22], a problem worthy to be considered is extending the modular sequence (function) space and the Orlicz sequence space to fuzzy setting by the method used in this paper.

Acknowledgments

This work was supported by “Qing Lan” Talent Engineering Funds by Tianshui Normal University. The second author acknowledge the support of the Beijing Municipal Education Commission Foundation of China (no. KM201210038001), the National Natural Science Foundation (no. 71240002) and the Funding Project for Academic Human Resources Development in Institutions of Higher Learning Under the Jurisdiction of Beijing Municipality (no. PHR2011088333).

References


Submit your manuscripts at http://www.hindawi.com