Research Article

Bounds for the Arithmetic Mean in Terms of the Neuman-Sándor and Other Bivariate Means

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We present the largest values α_1 , α_2 , and α_3 and the smallest values β_1 , β_2 , and β_3 such that the double inequalities $\alpha_1 M(a, b) + (1 - \alpha_1)H(a, b) < A(a, b) < \beta_1 M(a, b) + (1 - \beta_1)H(a, b)$, $\alpha_2 M(a, b) + (1 - \alpha_2)\bar{H}(a, b) < A(a, b) < \beta_2 M(a, b) + (1 - \beta_2)\bar{H}(a, b)$, and $\alpha_3 M(a, b) + (1 - \alpha_3)He(a, b) < A(a, b) < \beta_3 M(a, b) + (1 - \beta_3)He(a, b)$ hold for all $a, b > 0$ with $a \neq b$, where $M(a, b)$, $A(a, b)$, $He(a, b)$, $H(a, b)$ and $\bar{H}(a, b)$ denote the Neuman-Sándor, arithmetic, Heronian, harmonic, and harmonic root-square means of a and b , respectively.

1. Introduction

For $a, b > 0$ with $a \neq b$ the Neuman-Sándor mean $M(a, b)$ [1] is defined by

$$M(a, b) = \frac{a - b}{2 \sinh^{-1}((a - b)/(a + b))}, \quad (1)$$

where $\sinh^{-1}(x) = \log(x + \sqrt{x^2 + 1})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the object intensive research. In particular, many remarkable inequalities for the Neuman-Sándor mean $M(a, b)$ can be found in the literature [1–10].

Let $\bar{H}(a, b) = \sqrt{2ab}/\sqrt{a^2 + b^2}$, $H(a, b) = 2ab/(a + b)$, $G(a, b) = \sqrt{ab}$, $He(a, b) = (a + \sqrt{ab} + b)/3$, $L(a, b) = (b - a)/(\log b - \log a)$, $P(a, b) = (a - b)/[4 \arctan \sqrt{a/b} - \pi]$, $A(a, b) = (a + b)/2$, $T(a, b) = (a - b)/[2 \arctan((a - b)/(a + b))]$, $Q(a, b) = \sqrt{(a^2 + b^2)}/2$, and $C(a, b) = (a^2 + b^2)/(a + b)$ be the harmonic root-square, harmonic, geometric, Heronian, logarithmic, first Seiffert, arithmetic,

second Seiffert, quadratic, and contraharmonic means of a and b , respectively. Then it is known that the inequalities

$$\begin{aligned} \bar{H}(a, b) &< H(a, b) < G(a, b) < L(a, b) \\ &< He(a, b) < P(a, b) < A(a, b) \\ &< M(a, b) < T(a, b) < Q(a, b) < C(a, b) \end{aligned} \quad (2)$$

hold for all $a, b > 0$ with $a \neq b$.

Neuman and Sándor [1, 2] proved that the inequalities

$$\begin{aligned} \frac{\pi}{4 \log(1 + \sqrt{2})} T(a, b) &< M(a, b) < \frac{A(a, b)}{\log(1 + \sqrt{2})}, \\ \sqrt{2T^2(a, b) - Q^2(a, b)} &< M(a, b) < \frac{T^2(a, b)}{Q(a, b)}, \end{aligned}$$

$$H(T(a, b), A(a, b)) < M(a, b) < L(A(a, b), Q(a, b)),$$

$$H(M(a, b), Q(a, b)) < T(a, b), \quad M(a, b) < \frac{A^2(a, b)}{P(a, b)},$$

$$\begin{aligned}
A^{2/3}(a, b) Q^{1/3}(a, b) &< M(a, b) < \frac{2A(a, b) + Q(a, b)}{3}, \\
\sqrt{A(a, b) T(a, b)} &< M(a, b) < \sqrt{A^2(a, b) + T^2(a, b)}, \\
\frac{G(x, y)}{G(1-x, 1-y)} &< \frac{L(x, y)}{L(1-x, 1-y)} < \frac{P(x, y)}{P(1-x, 1-y)} \\
&< \frac{A(x, y)}{A(1-x, 1-y)} < \frac{M(x, y)}{M(1-x, 1-y)} \\
&< \frac{T(x, y)}{T(1-x, 1-y)}, \\
\frac{1}{A(1-x, 1-y)} - \frac{1}{A(x, y)} &< \frac{1}{M(1-x, 1-y)} - \frac{1}{M(x, y)} \\
&< \frac{1}{T(1-x, 1-y)} - \frac{1}{T(x, y)}, \\
A(x, y) A(1-x, 1-y) &< M(x, y) M(1-x, 1-y) \\
&< T(x, y) T(1-x, 1-y)
\end{aligned} \tag{3}$$

hold for all $a, b > 0$ and $x, y \in (0, 1/2]$ with $a \neq b$ and $x \neq y$. All the results stated above are in fact particular cases of more general and stronger results for the Schwab-Borchardt means [1, 2]. Some of them are based on the sequential method of Sándor [11]. In particular, Neuman and Sándor [1] also found that the inequality

$$M(a, b) < L(a_n, b_n) \tag{4}$$

holds for all $n \geq 0$ and $a, b > 0$ with $a \neq b$, where $a_0 = Q(a, b)$, $b_0 = A(a, b)$, $a_{n+1} = (a_n + b_n)/2$, and $b_{n+1} = \sqrt{a_n b_n}$.

Li et al. [3] proved that the double inequality $L_{p_0}(a, b) < M(a, b) < L_2(a, b)$ holds for all $a, b > 0$ with $a \neq b$, where $L_p(a, b) = [(b^{p+1} - a^{p+1})/((p+1)(b-a))]^{1/p}$ ($p \neq -1, 0$), $L_0(a, b) = 1/e(b^b/a^a)^{1/(b-a)}$, and $L_{-1}(a, b) = (b-a)/(\log b - \log a)$ is the p th generalized logarithmic mean of a and b ; $p_0 = 1.843 \dots$ is the unique solution of the equation $(p+1)^{1/p} = 2 \log(1 + \sqrt{2})$.

In [4], Neuman proved that the double inequalities

$$\begin{aligned}
\alpha Q(a, b) + (1-\alpha) A(a, b) \\
&< M(a, b) < \beta Q(a, b) + (1-\beta) A(a, b), \\
\lambda C(a, b) + (1-\lambda) A(a, b) \\
&< M(a, b) < \mu C(a, b) + (1-\mu) A(a, b)
\end{aligned} \tag{5}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq [1 - \log(1 + \sqrt{2})]/[(\sqrt{2} - 1) \log(1 + \sqrt{2})] = 0.3249 \dots$, $\beta \geq 1/3$, $\lambda \leq [1 - \log(1 + \sqrt{2})]/\log(1 + \sqrt{2}) = 0.1345 \dots$, and $\mu \geq 1/6$.

The main purpose of this paper is to find the largest values α_1, α_2 , and α_3 and the smallest values β_1, β_2 , and β_3 such that the double inequalities

$$\begin{aligned}
&\alpha_1 M(a, b) + (1-\alpha_1) H(a, b) \\
&< A(a, b) < \beta_1 M(a, b) + (1-\beta_1) H(a, b), \\
&\alpha_2 M(a, b) + (1-\alpha_2) \bar{H}(a, b) \\
&< A(a, b) < \beta_2 M(a, b) + (1-\beta_2) \bar{H}(a, b), \\
&\alpha_3 M(a, b) + (1-\alpha_3) He(a, b) \\
&< A(a, b) < \beta_3 M(a, b) + (1-\beta_3) He(a, b)
\end{aligned} \tag{6}$$

hold for all $a, b > 0$ with $a \neq b$. All numerical computations are carried out using Mathematical software.

2. Lemmas

In order to establish our main results we need several lemmas, which we present in this section.

Lemma 1 (see [12, Lemma 1.1]). Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$ and $a_n, b_n > 0$ for all $n \in \{0, 1, 2, \dots\}$. Let $h(x) = f(x)/g(x)$; then the following hold.

- (1) If the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is (strictly) increasing (decreasing), then $h(x)$ is also (strictly) increasing (decreasing) on $(0, r)$.
- (2) If the sequence $\{a_n/b_n\}$ is (strictly) increasing (decreasing) for $0 < n \leq n_0$ and (strictly) decreasing (increasing) for $n > n_0$, then there exists $x_0 \in (0, r)$ such that $h(x)$ is (strictly) increasing (decreasing) on $(0, x_0)$ and (strictly) decreasing (increasing) on (x_0, r) .

Lemma 2. The function

$$g(t) = \frac{t [\cosh(2t) + 2 \cosh(t) - 3]}{\sinh(2t) + t \cosh(2t) - 3t} \tag{7}$$

is strictly decreasing on $(0, \log(1 + \sqrt{2}))$, where $\sinh(t) = (e^t - e^{-t})/2$ and $\cosh(t) = (e^t + e^{-t})/2$ denote the hyperbolic sine and hyperbolic cosine functions, respectively.

Proof. Making use of power series $\sinh(t) = \sum_{n=0}^{\infty} t^{2n+1}/(2n+1)!$ and $\cosh(t) = \sum_{n=0}^{\infty} t^{2n}/(2n)!$, the function $g(t)$ can be written as

$$g(t) = \frac{\sum_{n=0}^{\infty} [(2^{2n+2} + 2)/(2n+2)!] t^{2n}}{\sum_{n=0}^{\infty} [(2n+5) 2^{2n+2}/(2n+3)!] t^{2n}}. \tag{8}$$

Let

$$a_n = \frac{2^{2n+2} + 2}{(2n+2)!}, \quad b_n = \frac{(2n+5) 2^{2n+2}}{(2n+3)!}. \tag{9}$$

Then simple computation leads to

$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = \frac{c_n}{(2n+5)(2n+7)2^{2n+3}}, \tag{10}$$

where

$$c_n = 2^{2n+5} - (12n^2 + 60n + 59). \quad (11)$$

It follows from (11) that

$$c_0 = -27, \quad c_1 = -3, \quad (12)$$

$$c_n \geq 128n^2 - (12n^2 + 60n + 59) = 116n^2 - 60n - 59 > 0 \quad (13)$$

for all $n \geq 2$.

Equations (10) and (12) together with inequality (13) lead to the conclusion that the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is strictly decreasing for $0 \leq n \leq 1$ and strictly increasing for $n \geq 2$. Then from Lemma 1(2) and (8) together with (9) we clearly see that there exists $t_0 \in (0, \infty)$ such that $g(t)$ is strictly decreasing on $(0, t_0)$ and strictly increasing on (t_0, ∞) .

Let $t^* = \log(1 + \sqrt{2})$. Then simple computations lead to

$$\begin{aligned} \sinh(t^*) &= 1, & \cosh(t^*) &= \sqrt{2}, \\ \sinh(2t^*) &= 2\sqrt{2}, & \cosh(2t^*) &= 3. \end{aligned} \quad (14)$$

It is not difficult to verify that

$$g'(t^*) = -2t^{*2} + (2 - \sqrt{2})t^* + 1 = -0.03734 \dots < 0. \quad (15)$$

From the piecewise monotonicity of $g(t)$ and inequality (15) we clearly see that $t^* = \log(1 + \sqrt{2}) < t_0$, which implies that $g(t)$ is strictly decreasing on $(0, \log(1 + \sqrt{2}))$. $\square$

Lemma 3. *The inequality*

$$\sqrt{1-t^2} > 1 - \frac{t^2}{2} - \frac{t^4}{8} - \frac{t^6}{16} - \frac{5t^8}{128} - \frac{35t^{10}}{128} \quad (16)$$

holds for all $t \in (0, 1)$.

Proof. Simple computations lead to

$$\begin{aligned} & \left(1 - \frac{t^2}{2} - \frac{t^4}{8} - \frac{t^6}{16} - \frac{5t^8}{128} - \frac{35t^{10}}{128}\right)^2 \\ &= 1 - t^2 - \frac{t^{10}}{16384} \\ & \quad \times (8064 - 4704t^2 - 1200t^4 - 585t^6 - 350t^8 - 1225t^{10}) \\ &< 1 - t^2 - \frac{t^{10}}{16384} (8064 - 4704 - 1200 - 585 - 350 - 1225) \\ &< 1 - t^2 \end{aligned} \quad (17)$$

for all $t \in (0, 1)$. $\square$

Lemma 4. *The function $f(t) = [t - \sinh^{-1}(t)] / [(1 - \sqrt{1-t^2})\sinh^{-1}(t)]$ is strictly decreasing in $(0, 1)$.*

Proof. Differentiating $f(t)$ gives

$$\begin{aligned} f'(t) &= \left(t \left(1 - t^2 - \sqrt{1-t^2}\right) \right. \\ & \quad \left. + \sqrt{1+t^2} \left[t \sinh^{-1}(t) + \sqrt{1-t^2} - 1\right] \sinh^{-1}(t)\right) \\ & \quad \times \left(\left(1 - \sqrt{1-t^2}\right)^2 \sqrt{1-t^4} [\sinh^{-1}(t)]^2\right)^{-1}. \end{aligned} \quad (18)$$

Making use of the power series

$$\sqrt{1+t^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} (2n)!}{(2n-1) 4^n (n!)^2} t^{2n} \quad (19)$$

$$= 1 + \frac{t^2}{2} - \frac{t^4}{8} + \frac{t^6}{16} - \frac{5t^8}{128} + \frac{7t^{10}}{256} - \dots,$$

$$\sqrt{1-t^2} = \sum_{n=0}^{\infty} \frac{(2n)!}{(1-2n) 4^n (n!)^2} t^{2n} \quad (20)$$

$$= 1 - \frac{t^2}{2} - \frac{t^4}{8} - \frac{t^6}{16} - \frac{5t^8}{128} - \frac{7t^{10}}{256} - \dots,$$

$$\begin{aligned} \sinh^{-1}(t) &= \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(2n+1) 4^n (n!)^2} t^{2n+1} \\ &= t - \frac{t^3}{6} + \frac{3t^5}{40} - \frac{5t^7}{112} + \dots, \end{aligned} \quad (21)$$

we get

$$\begin{aligned} \sqrt{1+t^2} \sinh^{-1}(t) &= t + \frac{t^3}{3} - \frac{2t^5}{15} + \frac{8t^7}{105} - \frac{1091t^9}{13440} + \dots \\ &< t + \frac{t^3}{3} - \frac{2t^5}{15} + \frac{8t^7}{105}, \end{aligned} \quad (22)$$

$$\begin{aligned} t \sinh^{-1}(t) + \sqrt{1-t^2} - 1 &= \frac{t^2}{2} - \frac{7t^4}{24} + \frac{t^6}{80} - \frac{75t^8}{896} + \dots < \frac{t^2}{2} - \frac{7t^4}{34} + \frac{t^6}{80}, \end{aligned}$$

for $t \in (0, 1)$.

Let

$$\begin{aligned} g(t) &= t \left(1 - t^2 - \sqrt{1-t^2}\right) \\ & \quad + \sqrt{1+t^2} \left[t \sinh^{-1}(t) + \sqrt{1-t^2} - 1\right] \sinh^{-1}(t) \\ &= -\left(1 - \sqrt{1-t^2}\right) \left[t \sqrt{1-t^2} + \sqrt{1+t^2} \sinh^{-1}(t)\right] \\ & \quad + t \sqrt{1+t^2} [\sinh^{-1}(t)]^2. \end{aligned} \quad (23)$$

We divide the proof into two cases.

Case 1 ($t \in (0, \sqrt{10}/5)$). Then from Lemma 3, (22), and (23) we have

$$\begin{aligned} g(t) &< t \left[1 - t^2 - \left(1 - \frac{t^2}{2} - \frac{t^4}{8} - \frac{t^6}{16} - \frac{5t^8}{128} - \frac{35t^{10}}{128} \right) \right] \\ &\quad + \left(t + \frac{t^3}{3} - \frac{2t^5}{15} + \frac{8t^7}{105} \right) \times \left(\frac{t^2}{2} - \frac{7t^4}{24} + \frac{t^6}{80} \right) \\ &= \frac{t^2}{201600} (24235t^2 + 50309t^4 + 192t^6 - 17920) \\ &< \frac{t^2}{201600} \left[24235 \left(\frac{\sqrt{10}}{5} \right)^2 + 50309 \left(\frac{\sqrt{10}}{5} \right)^4 \right. \\ &\quad \left. + 192 \left(\frac{\sqrt{10}}{5} \right)^6 - 17920 \right] < 0. \end{aligned} \quad (24)$$

Case 2 ($t \in [\sqrt{10}/5, 1)$). Then from Lemma 3, (19), (20), and (23) together with $\sinh^{-1}(\sqrt{10}/5) = 0.5964 \dots$ we get

$$\begin{aligned} g(t) &< - \left[1 - \left(1 - \frac{t^2}{2} - \frac{t^4}{8} - \frac{t^6}{16} \right) \right] \\ &\quad \times \left[t \left(1 - \frac{t^2}{2} - \frac{t^4}{8} - \frac{t^6}{16} - \frac{5t^8}{128} - \frac{35t^{10}}{128} \right) \right. \\ &\quad \left. + 0.596 \left(1 + \frac{t^2}{2} - \frac{t^4}{8} \right) \right] \\ &\quad + (0.596)^2 \left(1 + \frac{t^2}{2} - \frac{t^4}{8} + \frac{t^6}{16} \right) := \\ g_1(t) &= 0.356409t - 0.298t^2 - 0.3217955t^3 - 0.2235t^4 \\ &\quad + 0.08448875t^5 - 0.03725t^6 + 0.0847755625t^7 \\ &\quad - 0.0093125t^8 + 0.078125t^9 + 0.00465625t^{10} \\ &\quad + 0.03515625t^{11} + 0.1455078125t^{13} \\ &\quad + 0.03662109375t^{15} + 0.01708984375t^{17}. \end{aligned} \quad (25)$$

Numerical computations show that

$$g_1 \left(\frac{\sqrt{10}}{5} \right) = -0.0000435, \quad g_1(1) = -0.0510683 \dots, \quad (26)$$

$$g_1' \left(\frac{\sqrt{10}}{5} \right) = -0.52459 \dots, \quad g_1'(1) = 2.4665 \dots, \quad (27)$$

$$g_1'' \left(\frac{\sqrt{10}}{5} \right) = -1.86198 \dots, \quad g_1''(1) = 43.2711 \dots, \quad (28)$$

$$g_1''' \left(\frac{\sqrt{10}}{5} \right) = 4.63578 \dots, \quad (29)$$

$$\begin{aligned} g_1^{(4)}(t) &> (-5.364 + 9.653865t) \\ &\quad + t^2 (-13.41 + 71.2114725t - 15.645t^2) \\ &> \left[-5.364 + 9.653865 \times \left(\frac{\sqrt{10}}{5} \right) \right] \\ &\quad + t^2 \left[\left(-13.41 + 71.2114725 \times \left(\frac{\sqrt{10}}{5} \right) - 15.645 \right) \right] \\ &= 0.7416 \dots + t^2 \times 15.9830 \dots > 0. \end{aligned} \quad (30)$$

It follows from (29) and (30) that $g_1''(t)$ is strictly increasing in $[\sqrt{10}/5, 1)$. Then (28) leads to the conclusion that there exists $t_0 \in (\sqrt{10}/5, 1)$ such that $g_1'(t)$ is strictly decreasing in $[\sqrt{10}/5, t_0]$ and strictly increasing in $[t_0, 1)$.

From (27) and the piecewise monotonicity of $g_1'(t)$ we clearly see that there exists $t_1 \in (t_0, 1)$ such that $g_1(t)$ is strictly decreasing in $[\sqrt{10}/5, t_1]$ and strictly increasing in $[t_1, 1)$. Therefore,

$$g(t) < g_1(t) < 0 \quad (31)$$

for $t \in [\sqrt{10}/5, 1)$ follows from (25) and (26) together with the piecewise monotonicity of $g_1(t)$. $\square$

Lemma 5. Let $p \in (0, 1)$, $\lambda_0 = \log(1 + \sqrt{2})/[3 - 2 \log(1 + \sqrt{2})] = 0.7123 \dots$, and

$$\varphi_p(t) = \frac{3p\sqrt{1-t^2}}{(p-1)t + 2p-1} - \sinh^{-1}(\sqrt{1-t^2}). \quad (32)$$

Then $\varphi_{1/2}(t) < 0$ and $\varphi_{\lambda_0}(t) > 0$ for all $t \in (0, 1)$.

Proof. We first prove that $\varphi_{1/2}(t) < 0$ for $t \in (0, 1)$. From (32) one has

$$\varphi_{1/2}(1) = 0, \quad (33)$$

$$\varphi_{1/2}'(t) = \frac{4t - 2t^2 + (1/4)t^3 - (3t - 3/4)\sqrt{2-t^2}}{\sqrt{1-t^2}\sqrt{2-t^2}(2-(1/2)t^2)}. \quad (34)$$

Let

$$f(t) = 4t - 2t^2 + \frac{1}{4}t^3 - \left(3t - \frac{3}{4} \right) \sqrt{2-t^2}. \quad (35)$$

Then

$$\begin{aligned} f(1) &= 0, \\ f'(t) &= \frac{24t^2 - 3t - 24 + (16 - 16t + 3t^2)\sqrt{2-t^2}}{4\sqrt{2-t^2}}. \end{aligned} \quad (36)$$

We divide the proof into two cases.

Case 1 ($t \in (0, 0.7)$). Then we clearly see that

$$\begin{aligned}
 & \frac{24t^2 - 3t - 24 + (16 - 16t + 3t^2)\sqrt{2-t^2}}{\sqrt{2-t^2}} \\
 & < 24t^2 - \frac{3t+24}{\sqrt{2}} + (16 - 16t + 3t^2) \\
 & = 27t^2 - \left(\frac{3}{2}\sqrt{2} + 16\right)t + 16 - 12\sqrt{2} \\
 & = 27 \left(t - \frac{3\sqrt{2} + 32 + \sqrt{5376\sqrt{2} - 5870}}{108} \right) \\
 & \quad \times \left(t - \frac{3\sqrt{2} + 32 - \sqrt{5376\sqrt{2} - 5870}}{108} \right) \\
 & = 27(t - 0.72105\cdots)(t + 0.04985\cdots) < 0.
 \end{aligned} \tag{37}$$

Case 2 ($t \in [0.7, 1)$). Then we get

$$\begin{aligned}
 & [24t^2 - 3t - 24 + (16 - 16t + 3t^2)\sqrt{2-t^2}]\sqrt{2-t^2} \\
 & = (24t^2 - 3t - 24)\sqrt{2-t^2} + 32 - 32t - 10t^2 + 16t^3 - 3t^4 \\
 & < 24t^2 - 3t - 24 + 32 - 32t - 10t^2 + 16t^3 - 3t^4 \\
 & = -(1-t)(27t + 13t^2 - 3t^3 - 8) < 0.
 \end{aligned} \tag{38}$$

It follows from (32) and (33) together with Cases 1 and 2 that $f(t) < 0$ for $t \in (0, 1)$. Then from (34) and (35) we know that $\varphi_{1/2}(t)$ is strictly decreasing in $(0, 1)$.

Therefore, $\varphi_{1/2}(t) < 0$ for $t \in (0, 1)$ follows from (33) and the monotonicity of $\varphi_{1/2}(t)$.

Next, we prove that $\varphi_{\lambda_0}(t) > 0$ for $t \in (0, 1)$. From (32) we clearly see that we only have to prove that

$$\begin{aligned}
 \varphi_{\lambda_0}(\sqrt{1-t^2}) &= \frac{3\lambda_0 t}{(\lambda_0 - 1)\sqrt{1-t^2} + 2\lambda_0 + 1} - \sinh^{-1}(t) \\
 &> 0
 \end{aligned} \tag{39}$$

for all $t \in (0, 1)$.

Inequality (39) can be rewritten as

$$\lambda_0 > \frac{(1 - \sqrt{1-t^2})\sinh^{-1}(t)}{3t - (2 + \sqrt{1-t^2})\sinh^{-1}(t)}. \tag{40}$$

Let

$$\begin{aligned}
 h(t) &= \frac{(1 - \sqrt{1-t^2})\sinh^{-1}(t)}{3t - (2 + \sqrt{1-t^2})\sinh^{-1}(t)} \\
 &= \frac{1}{1 + 3[t - \sinh^{-1}(t)] / [(1 - \sqrt{1-t^2})\sinh^{-1}(t)]}.
 \end{aligned} \tag{41}$$

Then inequality (40) follows from Lemma 4 and (41) together with $h(1) = \lambda_0$. $\square$

3. Main Results

Theorem 6. The double inequality

$$\begin{aligned}
 & \alpha_1 M(a, b) + (1 - \alpha_1) H(a, b) \\
 & < A(a, b) < \beta_1 M(a, b) + (1 - \beta_1) H(a, b)
 \end{aligned} \tag{42}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 6/7 = 0.8571\cdots$ and $\beta_1 \geq \log(1 + \sqrt{2}) = 0.8813\cdots$.

Proof. Since $H(a, b)$, $M(a, b)$, and $A(a, b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a > b$. Let $x = (a - b)/(a + b)$ and $t = \sinh^{-1}(x)$. Then $x \in (0, 1)$, $t \in (0, \log(1 + \sqrt{2}))$, and

$$\begin{aligned}
 \frac{A(a, b) - H(a, b)}{M(a, b) - H(a, b)} &= \frac{x^2}{x/\sinh^{-1}(x) - (1 - x^2)} \\
 &= \frac{t \cosh(2t) - t}{2 \sinh(t) + t \cosh(2t) - 3t}.
 \end{aligned} \tag{43}$$

Let

$$f(t) = \frac{t \cosh(2t) - t}{2 \sinh(t) + t \cosh(2t) - 3t}. \tag{44}$$

Then $f(t)$ can be rewritten as

$$\begin{aligned}
 f(t) &= \frac{t \sum_{n=0}^{\infty} (2^{2n}/(2n)!) t^{2n} - t}{2 \sum_{n=0}^{\infty} (t^{2n+1}/(2n+1)!) + \sum_{n=0}^{\infty} (2^{2n}/(2n)!) t^{2n+1} - 3t} \\
 &= \frac{\sum_{n=1}^{\infty} (2^{2n}/(2n)!) t^{2n}}{\sum_{n=1}^{\infty} [(2 + (2n+1)2^{2n})/(2n+1)!] t^{2n+1}} \\
 &= \frac{\sum_{n=0}^{\infty} (2^{2n+2}/(2n+2)!) t^{2n}}{\sum_{n=0}^{\infty} [(2 + (2n+3)2^{2n+2})/(2n+3)!] t^{2n}} \\
 &:= \frac{\sum_{n=0}^{\infty} a_n t^{2n}}{\sum_{n=0}^{\infty} b_n t^{2n}}, \\
 \frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} &= \frac{(12n+34)2^{2n+2}}{[(2n+3)2^{2n+2}+2][(2n+5)2^{2n+4}+2]} > 0
 \end{aligned} \tag{45}$$

for all $n \geq 0$.

It follows from (45) together with Lemma 1(1) that $f(t)$ is strictly increasing in $(0, \log(1 + \sqrt{2}))$. Note that

$$\lim_{t \rightarrow 0^+} f(t) = \frac{a_0}{b_0} = \frac{6}{7}, \quad (46)$$

$$f(\log(1 + \sqrt{2})) = \log(1 + \sqrt{2}).$$

Therefore, Theorem 6 follows from (43) and (44) together with (46) and the monotonicity of $f(t)$. $\square$

Theorem 7. *The double inequality*

$$\alpha_2 M(a, b) + (1 - \alpha_2) \bar{H}(a, b) < A(a, b) < \beta_2 M(a, b) + (1 - \beta_2) \bar{H}(a, b) \quad (47)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq \log(1 + \sqrt{2}) = 0.8813 \dots$ and $\beta_2 \geq 9/10$.

Proof. Since $\bar{H}(a, b)$, $M(a, b)$, and $A(a, b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a > b$. Let $x = (a - b)/(a + b)$ and $t = \sinh^{-1}(x)$. Then $x \in (0, 1)$, $t \in (0, \log(1 + \sqrt{2}))$, and

$$\begin{aligned} \frac{A(a, b) - \bar{H}(a, b)}{M(a, b) - \bar{H}(a, b)} &= \frac{(\sqrt{1+x^2} + x^2 - 1) \sinh^{-1}(x)}{x\sqrt{1+x^2} - (1-x^2) \sinh^{-1}(x)} \\ &= \frac{t [\cosh(2t) + 2 \cosh(t) - 3]}{\sinh(2t) + t \cosh(2t) - 3t}. \end{aligned} \quad (48)$$

Simple computations lead to

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{t [\cosh(2t) + 2 \cosh(t) - 3]}{\sinh(2t) + t \cosh(2t) - 3t} &= \frac{9}{10}, \\ \lim_{t \rightarrow \log(1+\sqrt{2})} \frac{t [\cosh(2t) + 2 \cosh(t) - 3]}{\sinh(2t) + t \cosh(2t) - 3t} &= \log(1 + \sqrt{2}). \end{aligned} \quad (49)$$

Therefore, Theorem 7 follows from Lemma 2, (48), and (49). $\square$

Theorem 8. *The double inequality*

$$\alpha_3 M(a, b) + (1 - \alpha_3) He(a, b) < A(a, b) < \beta_3 M(a, b) + (1 - \beta_3) He(a, b) \quad (50)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq 1/2$ and $\beta_3 \geq \log(1 + \sqrt{2})/[3 - 2 \log(1 + \sqrt{2})] = 0.7123 \dots$.

Proof. Since $H(a, b)$, $M(a, b)$, and $A(a, b)$ are symmetric and homogeneous of degree one, without loss of generality, we

assume that $a > b$. Let $x = (a - b)/(a + b) \in (0, 1)$, $0 < p < 1$, and $\lambda_0 = \log(1 + \sqrt{2})/[3 - 2 \log(1 + \sqrt{2})]$; then

$$\begin{aligned} \frac{A(a, b) - He(a, b)}{M(a, b) - He(a, b)} &= \frac{(1 - \sqrt{1-x^2}) \sinh^{-1}(x)}{3x - (2 + \sqrt{1-x^2}) \sinh^{-1}(x)}, \quad (51) \\ pM(a, b) + (1-p)He(a, b) - A(a, b) \\ &= \frac{A(a, b)}{3} \left[\frac{3px}{\sinh^{-1}(x)} - (p-1)\sqrt{1-x^2} - (1+2p) \right] \\ &= \frac{A(a, b) [(1+2p) - (1-p)\sqrt{1-x^2}]}{3 \sinh^{-1}(x)} \varphi_p(\sqrt{1-x^2}), \end{aligned} \quad (52)$$

where $\varphi_p(t)$ is defined as in Lemma 5.

Note that

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{(1 - \sqrt{1-x^2}) \sinh^{-1}(x)}{3x - (2 + \sqrt{1-x^2}) \sinh^{-1}(x)} &= \frac{1}{2}, \\ \lim_{x \rightarrow 1} \frac{(1 - \sqrt{1-x^2}) \sinh^{-1}(x)}{3x - (2 + \sqrt{1-x^2}) \sinh^{-1}(x)} &= \lambda_0. \end{aligned} \quad (53)$$

Therefore, Theorem 8 follows from (51)–(53) together with Lemma 5. $\square$

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