Research Article

Exact Number of Positive Solutions for a Class of Two-Point Boundary Value Problems

Yanmin Niu and Baoqiang Yan

School of Mathematical Sciences, Shandong Normal University, Jinan 250014, China

Correspondence should be addressed to Baoqiang Yan; yanbaoqiang666@gmail.com

Received 25 September 2013; Accepted 4 November 2013

Academic Editor: Yansheng Liu

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This paper considers the existence of positive solutions of the boundary value problems

\[ V'' + \lambda (V' + V^p - V^q) = 0 \]

and

\[ V(-1) = V(1) = 0, \]

where \( p > r > q > -1 \) and \( \lambda \) is a positive parameter. Using a time-map approach, we obtain the exact number of positive solutions in different cases.

1. Introduction and Main Results

The study of multiplicity results to boundary value problem

\[ V'' + \lambda f(V) = 0, \quad t \in (a, b), \]

\[ V(a) = V(b) = 0, \]

where \( \lambda > 0 \) is a positive parameter, is very interesting because of its applications. As we know, when \( f(V) = V^p + V^q \), the boundary value problem

\[ -V''(x) = \mu V^p(x) + V^q(x), \quad a \leq x \leq b, \]

\[ V(x) > 0, \quad x \in (a, b), \]

\[ V(a) = V(b) = 0, \]

with \( 0 < q < 1 < p \) and \( k \geq 0 \) are fixed given numbers and \( \mu > 0 \) is a parameter, comes from the elliptic equation

\[ -\Delta u = \lambda u^p + u^q, \quad x \in \Omega, \]

\[ u > 0, \quad x \in \Omega, \]

\[ u = 0, \quad x \in \partial \Omega, \]

which was raised by Ambrosetti et al. in [1].

Under different assumptions on \( f \), there are many results for the above problems and elliptic equations (see [2–8]).

In [9, 10], Liu considered the case of \( f(V) = V^p + (1/\lambda) V^q \) and \( f(V) = V^p + V^q + kV \) and gave the exact number of solutions and many interesting properties of the solutions.

Cheng [11] investigated the following two-point boundary value problem:

\[ -y'' = \lambda (y^p - y^q), \quad t \in (-1, 1), \]

\[ y(-1) = y(1) = 0, \]

where \( \lambda > 0 \) is a positive parameter and \( p > q > -1 \) and got the exact number of positive solutions.

Now, in this paper we consider the more general case

\[ -V'' = \lambda (V' + V^p - V^q), \quad t \in (-1, 1), \]

\[ V(-1) = V(1) = 0, \]

where \( \lambda > 0 \) is a positive parameter, \( p > r > q > -1 \), and \( (1/(r+1)) + (1/(p+1)) - (1/(q+1)) < 0 \).

Define \( \beta \), where \( \beta \) satisfies

\[ \frac{\beta^r}{r+1} + \frac{\beta^p}{p+1} - \frac{\beta^q}{q+1} = 0. \]

For \( p > r > q \) and \( -1 < q < 1 \), let \( \lambda_1 \) be given by

\[ \lambda_1 = \frac{r+1}{2} \int_0^1 \frac{dt}{\sqrt{t^{r+1} ([1-t^{r-q}]+(r+1)/(p+1) \beta^{p-q} (1-t^{p-q})}]}. \]

The main results of this paper are as follows.
Theorem 1. If $p > r > q \geq 1$, (5) has exactly one positive solution for any $\lambda > 0$.

Theorem 2. If $p > r \geq 1 > q > -1$, (5) has exactly one positive solution for $\lambda \in (0, \lambda_1)$ and none for $\lambda > \lambda_1$.

Theorem 3. If $p > 1 > r > q \geq 0$, (5) has exactly one positive solution for $\lambda \in (0, \lambda_1)$ and none for $\lambda > \lambda_1$.

Theorem 4. If $(1/3) \geq p > r > q \geq 0$, (5) has exactly one positive solution for $\lambda \in (\lambda_1, +\infty)$ and none for $\lambda < \lambda_1$.

Theorem 5. Assume that $1 > p > r > 0 > q > -1$. Define
\[
\theta(r, p, q) = \int_{1}^{\frac{1}{1 - r}} \left[ t^{p+1} (1 - m_1 \beta - q t^{r-q} - m_2 \beta^p q r^{p} - q) \right]^{1/2} dt
\]
where
\[
m_1 = (q + 1)/(r + 1) \quad \text{and} \quad m_2 = (q + 1)/(p + 1). \quad \text{One has the following.}
\]
(1) If $\theta(r, p, q) \geq 0$, (5) has exactly one positive solution for $\lambda \in (\lambda_1, +\infty)$ and none for $\lambda \in (0, \lambda_1)$.
(2) If $\theta(r, p, q) < 0$, there exists $\lambda_0 \in (0, \lambda_1)$ such that (5) has exactly two positive solutions for $\lambda \in (\lambda_0, \lambda_1)$, exactly one for $\lambda \in (\lambda_1, +\infty)$ or $\lambda = \lambda_0$, and none for $\lambda \in (0, \lambda_0)$.

2. The Proofs of Theorems 1–4

We assume throughout this section that $p > r > q > -1$ and
\[
\frac{1}{r + 1} + \frac{1}{p + 1} - \frac{1}{q + 1} < 0. \quad (9)
\]
Denote that $E = (\beta, +\infty)$, where $\beta$ is given by (6). For $a \geq \beta$ and $t \in (0, 1)$, let
\[
P(a, t) = \int_{a}^{t} (z^r + z^p) \, dz, \quad Q(a, t) = \int_{a}^{t} z^q \, dz. \quad (10)
\]
Define a function $F : E \rightarrow (0, +\infty)$ as
\[
F(a) = \int_{0}^{a} \left[ 2 \int_{v}^{a} (z^r + z^p - z^q) \, dz \right]^{-1/2} \, dv \quad \text{for } a \in E. \quad (11)
\]
It is clear that $\int_{0}^{b} (z^r + z^p - z^q) \, dz = 0$. Now we claim that $a^r + a^p - a^q \geq 0$ for $a \geq \beta$ and $\int_{0}^{a} [2 \int_{v}^{a} (z^r + z^p - z^q) \, dz]^{-1/2} \, dv < \infty$ if and only if $a \in E$. Let $g_1(z) = z^p$ and $g_2(z) = z^r - z^q$; then $f(z) = g_1(z) + g_2(z)$. Immediately, we get that $g_1'(z) = rz^{r-1} - qz^{q-1} = z^{r-1}(r z^{r-1} - q)$. In order to judge the sign of $g_1'(z)$, we just judge the sign of $T(z) = rz^{r-1} - q$. Since $T(0) = -q, T(1) = r - q, and T'(z) = (r - q)z^{r-2}$, we can get the following two results.

(1) For $r > q > 0$, the function $g_2(z)$ has a stable point $z_0 = (q/r)^{1/(r-q)}$. When $z \in (0, z_0)$, we have $g_2'(z) < 0$, and when $z \in (z_0, +\infty)$, we have $g_2'(z) > 0$.

(2) For $r > 0 > q$, we have $g_2'(z) > 0$ on $z \in (0, +\infty)$. Combining $g_1'(x) = p x^{p-1}$ and $g_2''(x) = p(p-1) x^{p-2}$ with the above two results, we obtain the monotony of $f(z)$.

(3) For $p \geq 1 > r > q > 0$, the function $f(z)$ has a stable point $z_0 \in (0, 1)$. When $z \in (0, z_0)$, we have $f'(z) < 0$ and when $z \in (0, z_0)$, we have $f'(z) > 0$.

(4) For $1 > p > r > q > 0$, we have that $f'(z) > 0$ on $(0, +\infty)$.

(5) For $p > r > 0 > q$ or $p > 0 > r > q$, we have that $f'(z) > 0$ on $(0, +\infty)$.

Then from $\int_{0}^{b} (z^r + z^p - z^q) \, dz = 0$, we infer that $a^r + a^p - a^q > 0$ for $a \geq \beta$.

We consider the integration $\int_{0}^{a} \left[ 2 \int_{v}^{a} (z^r + z^p - z^q) \, dz \right]^{-1/2} \, dv$. It is clear that $v = a$ is a flaw point. Since $\int_{0}^{a} \left[ 2 \int_{v}^{a} (z^r + z^p - z^q) \, dz \right]^{-1/2} \, dv < \int_{0}^{a} \left[ 2 \int_{v}^{a} (z^r + z^p - z^q) \, dz \right]^{-1/2} \, dv$ on $E$, we consider the integration $\int_{0}^{a} \left[ 2 \int_{v}^{a} (z^r + z^p - z^q) \, dz \right]^{-1/2} \, dv$. Using Lagrange theorem, we obtain that
\[
\int_{0}^{a} \left[ 2 \int_{v}^{a} (z^r + z^p - z^q) \, dz \right]^{-1/2} \, dv
= \int_{0}^{a} \left[ 2 \left( \frac{a^{p+1}}{p+1} - \frac{v^{p+1}}{p+1} \right) - 2 \left( \frac{a^{q+1}}{q+1} - \frac{v^{q+1}}{q+1} \right) \right]^{-1/2} \, dv
= \int_{0}^{a} \left[ 2(v + \theta_1 (a - v)) \right]^{-1/2} \, dv
- \int_{0}^{a} \left[ 2(v + \theta_2 (a - v)) \right]^{-1/2} \, dv
\]
where $\theta_1, \theta_2 \in (0, 1)$ are constants.

The following Lemma 6 is listed to show that to study the number of positive solutions of (5) is equivalent to study the shape of the map $F(a)$ on $E$. Lemmas 7–9 show some properties of $F(a)$ on $E$. 

Journal of Applied Mathematics
Lemma 6. Let \( u(a, t) \) be the unique solution of the problem
\[
0 \leq u(t) \leq a,
\]
\[
\int_{\tau(t)}^{a} \left[ \int_{v}^{t} (z' + z^p - z^q) \, dz \right]^{-1/2} \, dv = F(a, t),
\]
where \( a \in E \). One has the following.

1. If \( \lambda > 0 \) and \( v \) is a positive solution of (5), \( v(0) \in E \), \( F(v(0)) = \sqrt{\lambda} \), and \( v(t) = u(v(0), |t|) \) for \( t \in [-1, 1] \).
2. If \( a \in E \) and \( F(a) = \sqrt{\lambda} \), \( v(t) = u(v(0), |t|) \) is a positive solution of (5) with \( v(0) = a \).

Proof. (1) Assume that \( \lambda > 0 \) and \( v \) is a positive solution of (5). Let \( r \in [-1, 1] \) satisfy \( v(r) = \max_{|t| \leq 1} v(t) = a \). It follows from \( \lambda > 0 \) that
\[
\left( v'(s) \right)^2 = \int_{\tau}^{s} 2v''(\eta) v'(\eta) \, d\eta = 2\lambda \int_{\tau(s)}^{a} (z' + z^p - z^q) \, dz, \quad s \in (-1, 1).
\]
This implies that \( a \geq \beta \) and \( v'(s) \neq 0 \) if \( v(s) < a \). And combine \( v''(r) = -\lambda a^p + a^q < 0 \) to obtain
\[
\begin{align*}
  v'(s) &> 0 \quad \text{for } s \in (-1, r), \\
  v'(s) &< 0 \quad \text{for } s \in (r, 1).
\end{align*}
\]
Then, we have that
\[
\begin{align*}
  v'(s) \left[ \int_{\tau(s)}^{a} (z' + z^p - z^q) \, dz \right]^{-1/2} = \sqrt{\lambda}, \quad s \in (-1, r), \\
  -v'(s) \left[ \int_{\tau(s)}^{a} (z' + z^p - z^q) \, dz \right]^{-1/2} = \sqrt{\lambda}, \quad s \in (r, 1).
\end{align*}
\]
It follows that
\[
\int_{\tau(t)}^{a} \left[ \int_{v}^{t} (z' + z^p - z^q) \, dz \right]^{-1/2} \, dv = (\tau - t) \sqrt{\lambda}, \quad t \in [-1, r],
\]
\[
\int_{\tau(t)}^{a} \left[ \int_{v}^{t} (z' + z^p - z^q) \, dz \right]^{-1/2} \, dv = (t - \tau) \sqrt{\lambda}, \quad t \in [r, 1],
\]
\[
(1 - \tau) \sqrt{\lambda} = \int_{\tau(t)}^{a} \left[ \int_{v}^{t} (z' + z^p - z^q) \, dz \right]^{-1/2} \, dv = (\tau + 1) \sqrt{\lambda}.
\]
From (19) and \( a \geq \beta \) we have that \( \tau = 0 \). With (17) and (18) we obtain the result (1) of this theorem.

(2) Since \( u(a, 0) = a > 0 \) and \( u(a, t) \) is a positive solution of the boundary value problem
\[
\begin{align*}
u'' + [F(a)]^2 (\nu' + u^p - u^q) &= 0, \quad t \in (0, 1), \\
u'(0) &= 0, \quad u(1) = 0,
\end{align*}
\]
we have that \( v(t) = u(a, |t|) \) is a positive solution of (5). □

Lemma 7. \( F \) is differentiable on \( (\beta, \infty) \), and
\[
\begin{align*}
F(a) &= \frac{1}{4\sqrt{2}} \int_{0}^{1} H_h(a, t) \, dt, \quad a > \beta, \quad (21) \\
F'(a) &= \frac{1}{4\sqrt{2}} \int_{0}^{1} H_1(a, t) \, dt, \quad a > \beta, \quad (22)
\end{align*}
\]
where
\[
H_h(a, t) = 4a[P(a, t) - Q(a, t)]^{-1/2},
\]
\[
H_1(a, t) = 2[P(a, t) - Q(a, t)]^{-1/2} \times \left[ \frac{1 - r}{r + 1} d^r + \frac{1 - p}{p + 1} d^p \right] \left[ 1 - (r + 1)^{r+1} \right] \left[ 1 - (p + 1)^{p+1} \right].
\]

Proof. Equation (21) can be obtained by (11), immediately. From
\[
\frac{\partial P(a, t)}{\partial a} = \frac{\partial a}{\partial a},
\]
\[
\int_{0}^{1} \left[ \frac{1 - r}{r + 1} d^r + \frac{1 - p}{p + 1} d^p \right] \left[ 1 - (r + 1)^{r+1} \right] \left[ 1 - (p + 1)^{p+1} \right],
\]
\[
\frac{\partial Q(a, t)}{\partial a} = \frac{\partial a}{\partial a}, \quad (24)
\]
we have that
\[
\frac{\partial H_h(a, t)}{\partial a} = 2[P(a, t) - Q(a, t)]^{-1/2} \times \left[ \frac{1 - r}{r + 1} d^r + \frac{1 - p}{p + 1} d^p \right] \left[ 1 - (r + 1)^{r+1} \right] \left[ 1 - (p + 1)^{p+1} \right] = H_1(a, t),
\]
\[
(25)
\]
It follows from (21) that \( F \) is differentiable on \( (\beta, \infty) \) and (22) is true. □
Lemma 8. Consider the following:

\[
\lim_{a \to \beta} F(a) = \begin{cases} 
+\infty, & q \geq 1, \\
\sqrt{\lambda_1}, & q < 1, 
\end{cases} 
\]  

where \( \lambda_1 \) is given by (7).

Proof. From

\[
\lim_{a \to \beta} \frac{d}{da} \left[ \int_{a}^{\beta} \frac{d\nu}{\sqrt{2 \int_{a}^{\beta} (x^r + z^p - z^q) dz}} \right] = 0
\]  

and the Lebesgue theorem, we have that

\[
\lim_{a \to \beta} F(a) = \lim_{a \to \beta} \int_{0}^{\beta} \frac{d\nu}{\sqrt{2 \int_{0}^{\beta} (x^r + z^p - z^q) dz}}
\]

On the other hand, from \( \int_{0}^{\beta} (z^r + z^p - z^q) dz = 0 \), we can obtain that

\[
\int_{0}^{\beta} (z^r + z^p - z^q) dz = 0
\]

Hence from Lemma 7, we have

\[
F' (a) = \frac{1}{4 \sqrt{2}} \int_{0}^{1} 2(P-Q)^{-3/2} I(a, t) dt, \quad \text{for } a > \beta.
\]  

Proof of Theorem 1. From \( p > r > q \geq 1 \), we obtain that

\[
I(a, t) < \frac{1-r}{r+1} \frac{a^{p+1} (1-t^{p+1}) + \frac{p}{p+1} a^{p+1} (1-t^{p+1})}{a^{r+1} (1-t^{r+1}) + \frac{r}{r+1} a^{r+1} (1-t^{r+1}) + (r+1)/(p+1) a^{p+1} (1-t^{p+1})}.
\]  

Hence from Lemma 7, we have that

\[
F' (a) = \frac{1}{4 \sqrt{2}} \int_{0}^{1} 2(P-Q)^{-3/2} I(a, t) dt, \quad \text{for } a > \beta.
\]  

Proof of Theorem 2. From \( p > r \geq 1 > q > -1 \), (33), and (34), we have that \( F'(a) < 0 \) for \( a > \beta \). It follows from Lemmas 6, 8, and 9 that the results of Theorem 2 hold.
Proof of Theorem 3. Conditions \( p > 1 > r > q \geq 0 \) and \( \frac{1}{(r+1)} + \frac{1}{(p+1)} - \frac{1}{(q+1)} < 0 \) imply that \( p-1 > q(r+p+2) > 0 \). With (33), we obtain that

\[
I(a, t) < \frac{1-r}{r+1} a^{p+1} (1-t^{p+1}) + \frac{1-p}{p+1} a^{q+1} (1-t^{q+1})
\]

\[
- \frac{1-q}{q+1} a^{q+1} (1-t^{q+1})
\]

\[
< \frac{2 (1-rp)}{(p+1)(r+1)} a^{q+1} (1-t^{q+1}) - \frac{1-q}{q+1} a^{q+1} (1-t^{q+1})
\]

\[
\times (1-t^{q+1}) < 0,
\]

for \( a > \beta, t \in (0, 1) \), which means that \( F'(a) < 0 \) for \( a > \beta \). Thus by Lemmas 6, 8, and 9, we have the results of this theorem.

Proof of Theorem 4. From \( \frac{1}{3} \geq p > r > q \geq 0 \) and (33), we have that

\[
I(a, t) > \frac{1-r}{r+1} a^{q+1} (1-t^{q+1}) + \frac{1-p}{p+1} a^{q+1} (1-t^{q+1})
\]

\[
- \frac{1-q}{q+1} a^{q+1} (1-t^{q+1})
\]

\[
> \left( \frac{2}{p+1} - \frac{1-q}{q+1} \right) a^{q+1} (1-t^{q+1})
\]

\[
= \frac{q(3-p) + (1-3p)}{(p+1)(q+1)} a^{q+1} (1-t^{q+1}) > 0,
\]

for \( a > \beta, t \in (0, 1) \). Hence \( F'(a) > 0 \) for \( a > \beta \). By Lemmas 6, 8, and 9, we have the results of this theorem.

3. The Proof of Theorem 5

In this section we always assume that \( 1 > p > r > 0 > q > -1 \).

Denote that

\[
S(a, t) = \frac{P(a, t)}{Q(a, t)}, \quad a > \beta, \ t \in (0, 1),
\]

\[
h_1(s) = (s-1)(s-n), \quad s \in (-\infty, \infty),
\]

\[
h_2(s) = -3(s-m)(s-n) + 2(s-1)(s-l), \quad s \in (-\infty, \infty),
\]

where \( P(a, t), Q(a, t), \) and \( \beta \) are given by (10), (6), and

\[
m = m(t) = \frac{q+1}{r+1} - \frac{(p-r)(q+1)}{(p+1)(r+1)} a^{p+1} \frac{1-t^{p+1}}{1-t^{r+1}},
\]

\[
a > \beta, \ t \in (0, 1),
\]

\[
n = n(t) = \frac{1-q}{1-r} + \frac{(p-r)(q+1)}{(p+1)(1-r)} a^{p+1} \frac{1-t^{p+1}}{1-t^{q+1}},
\]

\[
a > \beta, \ t \in (0, 1),
\]

\[
l = l(t) = \frac{1-q^2}{1-r^2} + \frac{(p^2-r^2)(q+1)}{(p+1)(1-r^2)} a^{p+1} \frac{1-t^{p+1}}{1-t^{q+1}},
\]

\[
a > \beta, \ t \in (0, 1),
\]

Remark 10. From \( 1 > p > r > 0 > q > -1 \), (6), and (41), it is obvious that \( m < 1, n > 1, \) and \( l < n \).

Lemma 11. For \( a > \beta \) and \( t \in (0, 1) \),

\[
\frac{q+1}{r+1} a^{r-q} + \frac{q+1}{p+1} a^{p-q} < \frac{P(a, t)}{Q(a, t)} < a^{-q} + a^{p-q}.
\]

Proof. Let

\[
K(t) = \frac{a^r}{r+1} \left( (r-q)(r+q+1)t^{r+q} + (q+1) t^r - (r+1) r^r \right)
\]

\[
+ \frac{a^p}{p+1} \left( (p-q)(p+q+1)t^{p+q} + (q+1) t^p - (p+1) p^p \right).
\]

Condition \( 1 > p > r > 0 > q > -1 \) implies that

\[
\begin{align*}
K'(t) &= \frac{a^r}{r+1} \left[ (r-q)(r+q+1)t^{r+q-1} + (q+1) t^r - (r+1) r^r \right]
\quad + \frac{a^p}{p+1} \left[ (p-q)(p+q+1)t^{p+q-1} + (q+1) t^p - (p+1) p^p \right]
\end{align*}
\]

\[
< \frac{a^r}{r+1} \left[ (r-q)(r+q+1) + (q+1) q \right] t^{r+q-1} - (p+1) p t^{p-1}
\]

\[
= \frac{a^r}{r+1} \left[ (r-q)(r+q+1) + (q+1) q \right] t^{r+q-1} - (p+1) p t^{p-1}
\]

\[
+ \frac{a^p}{p+1} \left[ (p-q)(p+q+1) + (q+1) q \right] t^{p+q-1} - (p+1) p t^{p-1}
\]

\[
= 0, \quad t \in (0, 1).
\]

With \( K(0) = 0 \) and (44), we have that \( K(t) > 0 \) for \( t \in (0, 1) \).

It follows that

\[
\frac{d}{dt} P(a, t) = \frac{q+1}{(1-t^{q+1})^2} \frac{a^{q+1}}{a^{p+1}} K(t) > 0, \quad t \in (0, 1).
\]

Combining

\[
\lim_{t \to 0} \frac{P(a, t)}{Q(a, t)} = \frac{q+1}{r+1} a^{r-q} + \frac{q+1}{p+1} a^{p-q},
\]

\[
\lim_{t \to 1} \frac{P(a, t)}{Q(a, t)} = a^{-q} + a^{p-q},
\]

we have the results of this lemma.

Lemma 12. For \( 1 > p > r > 0 > q > -1 \),

\[
\lim_{a \to \beta} F'(a) = \frac{1}{2 \sqrt{2}} \left( \frac{q+1}{\beta^{q+1}} \right)^{1/2} \theta(r, p, q).
\]

(47)
Proof. From Lemma 7 we have that
\[ 2\sqrt{2} \dot{F}(a) = \int_0^1 \left( \frac{1-r}{r+1} a^{r+1} (1-t^{r+1}) + \frac{1-p}{p+1} a^{p+1} (1-t^{p+1}) \right) \times \left( [P(a,t) - Q(a,t)]^{3/2} \right)^{-1} dt \]
\[ \times \left( [P(\alpha,t) - Q(\alpha,t)]^{3/2} \right)^{-1} dt = \int_0^1 \frac{1-r}{[P(a,t) - Q(a,t)]^{1/2}} dt \]
\[ = \frac{r-q}{q+1} \int_0^1 \frac{1-t^{q+1}}{[P(a,t) - Q(a,t)]^{3/2}} dt \]
\[ - \frac{p-r}{p+1} \int_0^1 \frac{1-t^{p+1}}{[P(a,t) - Q(a,t)]^{3/2}} dt. \]
\[ (48) \]

It follows from Lebesgue theorem that
\[ \lim_{a \to \beta} \int_0^1 \frac{1-r}{[P(a,t) - Q(\alpha,t)]^{1/2}} dt = \int_0^1 \frac{1-r}{[P(\beta,t) - Q(\beta,t)]^{1/2}} dt, \]
\[ \lim_{a \to \beta} \frac{r-q}{q+1} \int_0^1 \frac{1-t^{q+1}}{[P(\beta,t) - Q(\beta,t)]^{3/2}} dt \]
\[ + \lim_{a \to \beta} \frac{p-r}{p+1} \int_0^1 \frac{1-t^{p+1}}{[P(\beta,t) - Q(\beta,t)]^{3/2}} dt \]
\[ = \frac{r-q}{q+1} \int_0^1 \frac{1-t^{q+1}}{[P(\beta,t) - Q(\beta,t)]^{3/2}} dt \]
\[ + \frac{p-r}{p+1} \int_0^1 \frac{1-t^{p+1}}{[P(\beta,t) - Q(a,t)]^{3/2}} dt. \]
\[ (49) \]

Finally, \( \int_0^\beta (z^r + z^p - z^q) dz = 0 \) implies that
\[ P(\beta,t) - Q(\beta,t) \]
\[ = \int_0^\beta \left( \frac{\beta^q}{q+1} - \frac{\beta^p}{p+1} \right) dt \]
\[ = \frac{\beta^q}{q+1} (1 - m_1 \beta^{q+1} t^{q+1} - m_2 \beta^{p+1} t^{p+1}). \]
\[ (50) \]

Combining (49) and (50), we complete the proof of this lemma. \( \square \)

**Lemma 13.** \( F(a) \) has continuous derivatives up to second order on \((\beta, \infty)\) as follows:
\[ F'(a) = \frac{1-r}{2\sqrt{2}} \int_0^1 G(a,t) h_1 (S(a,t)) dt, \quad a > \beta, \]
\[ (51) \]
\[ F''(a) = \frac{1-r^2}{4a\sqrt{2}} \int_0^1 G(a,t) h_2 (S(a,t)) dt, \quad a > \beta, \]
\[ (52) \]
where \( G(a,t) = [S(a,t) - 1]^{-5/2} [Q(a,t)]^{-1/2} \).

**Proof.** Equation (51) can be obtained by
\[ H_1(a,t) = 2[P(a,t) - Q(a,t)]^{-3/2} \]
\[ \times \left( 1-r \right) P(a,t) - (1-q) Q(a,t) \]
\[ - \frac{p-r}{p+1} a^{p+1} (1-t^{p+1}) \]
\[ = 2 \left[ \frac{P(a,t)}{Q(a,t)} - 1 \right]^{-5/2} [Q(a,t)]^{-1/2} \]
\[ \times (1-r) \left[ P(a,t) - \frac{1-q}{Q(a,t)} (Q(a,t) - \frac{p-r}{p+1} a^{p+1} \right] \]
\[ \times \frac{1-t^{p+1}}{1-t^{p+1}} \left[ \frac{P(a,t)}{Q(a,t)} - 1 \right] \]
\[ = 2 (1-r) G(a,t) h_1 (S(a,t)) \]
\[ (53) \]
and Lemma 7, immediately. From (24) we have that
\[ a \frac{\partial H_1(a,t)}{\partial a} = -3(P - Q)^{-5/2} \]
\[ \times \left( r+1 \right) P - (q+1) Q \]
\[ + \frac{p-r}{p+1} a^{p+1} (1-t^{p+1}) \]
\[ \times \left( 1-r \right) P - (1-q) Q - \frac{p-r}{p+1} a^{p+1} \]
\[ \times \left[ 1-t^{p+1} \right] + 2(P - Q)^{-3/2} \]
\[ \times \left[ (1-r^2) P - (1-q^2) Q \right. \]
\[ - \frac{p^2-r^2}{p+1} a^{p+1} (1-t^{p+1}) \]
\[ = \left( \frac{p}{Q-1} \right)^{-3/2} (1-r^2) \]
\[ \times \left\{ -3 \left[ P - \frac{q+1}{r+1} Q + \frac{(p-r)}{(p+1)(r+1)} \right. \right. \]
\[ \times a^{p+1} (1-t^{p+1}) \]
\[ \times \left( 1-t^{p+1} \right) + 2(P - Q) \]
This means that
\[
G(a, t) = [S(a, t) - 1]^{-5/2}[Q(a, t)]^{-1/2} > (a^{-q} + a^{p-q})^{-5/2} a^{-(1+q)/2} \sqrt{1+q},
\]
for \(a > \beta, t \in (0, 1)\). It follows from Lemma 13 and (58) that
\[
F''(a) + \delta(r, p, q) F'(a) \frac{1 + r}{2a}
\]
Now, from (59)–(62) we have (55), where
\[
\eta(r, p, q) = \frac{(1-r)(r-q) \sqrt{1+q}}{2 \sqrt{2}} \int_0^1 [n(t) - 1] \, dt.
\]

Lemma 15. If \(\theta(r, p, q) \geq 0\), then \(F'(a) > 0\) for \(a > \beta\). If \(\theta(r, p, q) < 0\), then there exists \(a^* > \beta\), such that \(F'(a) < 0\) for \(a \in (\beta, a^*)\) and \(F'(a) > 0\) for \(a > a^*\).

Proof. It follows from Lemma 14 that if
\[
F'(a) = 0, \quad \text{then } F''(a) > 0,
\]
and then
\[
F'(a) \text{ has at most one zero in } (\beta, \infty).
\]
By (40), (42), and (51) we can obtain that
\[
F'(a) > 0 \quad \text{for } \frac{P(a, t)}{Q(a, t)} > n(t).
\]
With (10) and (41), (66) implies that
\[
F'(a) > 0 \quad \text{for } \frac{q + 1}{r + 1} a^r + \left(\frac{q + 1}{p + 1} - \frac{p - q}{r - 1}\right) a^p > \frac{1 - q}{1 - r} a^q.
\]
If \(\theta(r, p, q) \neq 0\), then from Lemma 12 and (64)–(67) we have the results of this lemma, immediately. If \(\theta(r, p, q) = 0\), then by Lemmas 12 and 14 we have that \(F''(a) > 0\) for a near \(\beta\), and so \(F'(a) > 0\) for \(a\) near \(\beta\). Thus, it follows from (64)–(67) that \(F'(a) > 0\) for \(a > \beta\).

Proof of Theorem 5. From Lemmas 6, 8, 9, and 15, we can obtain the results of Theorem 5.

References


