Research Article

Solving Optimization Problems on Hermitian Matrix Functions with Applications

Xiang Zhang and Shu-Wen Xiang

Department of Computer Science and Information, Guizhou University, Guiyang 550025, China

Correspondence should be addressed to Xiang Zhang; zxjnsc@163.com

Received 17 October 2012; Accepted 20 March 2013

Academic Editor: K. Sivakumar

Copyright © 2013 X. Zhang and S.-W. Xiang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We consider the extremal inertias and ranks of the matrix expressions \( f(X,Y) = A_1 - B_Y X - (B_X Y)'^* - C_3 Y D_3 - (C_3 Y D_3)^* \), where \( A_1 = A_1^*, B_Y, C_3, \) and \( D_3 \) are known matrices and \( X \) and \( Y \) are the solutions to the matrix equations \( A_1 Y = C_1, Y B_1 = D_1, \) and \( A_2 X = C_2, \) respectively. As applications, we present necessary and sufficient condition for the previous matrix function \( f(X,Y) \) to be positive (negative), non-negative (positive) definite or nonsingular. We also characterize the relations between the Hermitian part of the solutions of the above-mentioned matrix equations. Furthermore, we establish necessary and sufficient conditions for the solvability of the system of matrix equations \( A_1 Y = C_1, Y B_1 = D_1, A_2 X = C_2, \) and \( B_X + (B_X Y)' + C_3 Y D_3 + (C_3 Y D_3)^* = A_3, \) and give an expression of the general solution to the above-mentioned system when it is solvable.

1. Introduction

Throughout, we denote the field of complex numbers by \( \mathbb{C} \), the set of all \( m \times n \) matrices over \( \mathbb{C} \) by \( \mathbb{C}^{m \times n} \), and the set of all \( m \times m \) Hermitian matrices by \( \mathbb{C}_h^{m \times m} \). The symbols \( A^* \) and \( \mathbb{H}(A) \) stand for the conjugate transpose, the column space of a complex matrix \( A \) respectively. \( I_n \) denotes the \( n \times n \) identity matrix. The Moore-Penrose inverse [1] \( A^+ \) of \( A \), is the unique solution \( X \) to the four matrix equations:

\[
\begin{align*}
(1) & \quad AXA = A, \\
(2) & \quad XAX = X, \\
(3) & \quad (AX)^* = AX, \\
(4) & \quad (XA)^* =XA.
\end{align*}
\]

Moreover, \( L_A \) and \( R_A \) stand for the projectors \( L_A = I - A^* A \), \( R_A = I - AA^* \) induced by \( A \). It is well known that the eigenvalues of a Hermitian matrix \( A \in \mathbb{C}_h^{m \times m} \) are real, and the inertia of \( A \) is defined to be the triplet

\[
i_+(A) = \{i_-(A), i_-(A), i_0(A)\},
\]

where \( i_+(A) \), \( i_-(A) \), and \( i_0(A) \) stand for the numbers of positive, negative, and zero eigenvalues of \( A \), respectively. The symbols \( i_+(A) \) and \( i_-(A) \) are called the positive index and the negative index of inertia, respectively. For two Hermitian matrices \( A \) and \( B \) of the same sizes, we say \( A \succeq B \) \((A \preceq B)\) in the Löwner partial ordering if \( A - B \) is positive (negative) semidefinite. The Hermitian part of \( X \) is defined as \( H(X) = X + X^* \). We will say that \( X \) is Re-nd (Re-nonnegative semidefinite) if \( H(X) \succeq 0 \), \( X \) is Re-pd (Re-positive definite) if \( H(X) > 0 \), and \( X \) is Re-ns if \( H(X) \) is nonsingular.

It is well known that investigation on the solvability conditions and the general solution to linear matrix equations is very active (e.g., [2–9]). In 1999, Braden [10] gave the general solution to

\[
BX + (BX)^* = A.
\]

In 2007, Djordjević [11] considered the explicit solution to (3) for linear bounded operators on Hilbert spaces. Moreover, Cao [12] investigated the general explicit solution to

\[
BXC + (BXC)^* = A.
\]

some necessary and sufficient conditions for the consistence of the matrix equation
\[ A_1X + (A_1X)^* + B_1YC_1 + (B_1YC_1)^* = E_1 \]
and presented an expression of the general solution to (5).

Note that (5) is a special case of the following system:
\[ A_1Y = C_1, \quad YB_1 = D_1, \quad A_2X = C_2, \]
\[ B_3X + (B_3X)^* + C_3YD_3 + (C_3YD_3)^* = A_3. \]
(6)

To our knowledge, there has been little information about (6). One goal of this paper is to give some necessary and sufficient conditions for the solvability of the system of matrix equations (6) and present an expression of the general solution to system (6) when it is solvable.

In order to find necessary and sufficient conditions for the solvability of the system of matrix equations (6), we need to consider the extremal ranks and inertias of (10) subject to (11) and (13).

There have been many papers to discuss the extremal ranks and inertias of the following Hermitian expressions:
\[ p(X) = A_3 - B_3X - (B_3X)^*, \]
\[ g(Y) = A - BYC - (BYC)^*, \]
\[ h(X,Y) = A_1 - B_1XB_1^* - C_1YC_1^*, \]
\[ f(X,Y) = A_3 - B_3X - (B_3X)^* - C_3YD_3 - (C_3YD_3)^*. \]
(10)

Tian has contributed much in this field. One of his works [15] considered the extremal ranks and inertias of (7). He and Wang [16] derived the extremal ranks and inertias of (7) subject to \( A_1X = C_1, A_2XB_2 = C_2 \). Liu and Tian [17] studied the extremal ranks and inertias of (8). Chu et al. [18] and Liu and Tian [19] derived the extremal ranks and inertias of (9). Zhang et al. [20] presented the extremal ranks and inertias of (9), where \( X \) and \( Y \) are Hermitian solutions of
\[ A_2X = C_2, \]
\[ YB_2 = D_2, \]
(11)
(12) respectively. He and Wang [16] derived the extremal ranks and inertias of (10). We consider the extremal ranks and inertias of (10) subject to (11) and (13)
\[ A_1Y = C_1, \quad YB_1 = D_1, \]
(13)
which is not only the generalization of the above matrix functions, but also can be used to investigate the solvability conditions for the existence of the general solution to the system (6). Moreover, it can be applied to characterize the relations between Hermitian part of the solutions of (11) and (13).

The remainder of this paper is organized as follows. In Section 2, we consider the extremal ranks and inertias of (10) subject to (11) and (13). In Section 3, we characterize the relations between the Hermitian part of the solution to (11) and (13). In Section 4, we establish the solvability conditions for the existence of a solution to (6) and obtain an expression of the general solution to (6).

2. Extremal Ranks and Inertias of Hermitian Matrix Function (10) with Some Restrictions

In this section, we consider formulas for the extremal ranks and inertias of (10) subject to (11) and (13). We begin with the following Lemmas.

**Lemma 1 (see [21])**. (a) Let \( A_1, C_1, B_1, \) and \( D_1 \) be given. Then the following statements are equivalent:

(1) system (13) is consistent,
(2) \( R_{A_1} = 0, \quad D_1L_{B_1} = 0, \quad A_1D_1 = C_1B_1. \)
(14)

(3) \( r(A_1) = r(A_2), \quad r(B_1), \quad A_1D_1 = C_1B_1. \)
(15)

In this case, the general solution can be written as
\[ Y = A_1C_1 + L_{A_1}D_1B_1^* + L_{A_1}VW, \]
where \( V \) is arbitrary.

(b) Let \( A_2 \) and \( C_2 \) be given. Then the following statements are equivalent:

(1) equation (11) is consistent,
(2) \( R_{A_2} = 0. \)
(17)

(3) \( r(A_2) = r(A_2). \)
(18)

In this case, the general solution can be written as
\[ X = A_1^* C_1 + L_{A_1} W, \]
where \( W \) is arbitrary.

**Lemma 2 ([22, Lemma 1.5, Theorem 2.3])**. Let \( A \in \mathbb{C}_h^{m	imes n}, \)
\( B \in \mathbb{C}_h^{m\times r}, \) and \( D \in \mathbb{C}_h^{n\times r}, \) and denote that
\[ M = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \]
\[ N = \begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix}, \]
\[ L = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}, \]
\[ G = \begin{bmatrix} P & ML_N \\ L_NM^* & 0 \end{bmatrix}. \]
Then one has the following
(a) the following equalities hold
\[ i_k(M) = r(B) + i_k(R_B A R_B), \quad (21) \]
\[ i_k(N) = r(Q), \quad (22) \]
(b) if \( \mathcal{R}(B) \subseteq \mathcal{R}(A) \), then \( i_k(L) = i_k(A) + i_k(D - B^* A^1 B) \). Thus \( i_k(L) = i_k(A) \) if and only if \( \mathcal{R}(B) \subseteq \mathcal{R}(A) \) and \( i_k(D - B^* A^1 B) = 0 \),
(c) \[ i_k(G) = \begin{bmatrix} P & M & 0 \\ M^* & 0 & N^* \\ 0 & N & 0 \end{bmatrix} - r(N). \quad (23) \]

Lemma 3 (see [23]). Let \( A \in \mathbb{C}^{m \times n} \), \( B \in \mathbb{C}^{m \times m} \), and \( C \in \mathbb{C}^{n \times n} \). Then they satisfy the following rank equalities:
(a) \( r[A \ B] = r(A) + r(E_B) \), \( r(E_B) = r(B) + r(E_B A) \),
(b) \( r[\begin{bmatrix} A & C \end{bmatrix}] = r(A) + r(CF_A) \),
(c) \( r[\begin{bmatrix} A & B \end{bmatrix}] = r(B) + r(C) + r(E_B A F_C) \),
(d) \( r[\begin{bmatrix} B & A F_C \end{bmatrix}] = r[\begin{bmatrix} B & A \end{bmatrix}] - r(C) \),
(e) \( r[\begin{bmatrix} C & B \end{bmatrix}] = r[\begin{bmatrix} C & 0 \end{bmatrix}] - r(B) \),
(f) \( r[\begin{bmatrix} A & B F_D \end{bmatrix}] = r[\begin{bmatrix} A & B \end{bmatrix}] - r(D - E) \).

Lemma 4 (see [15]). Let \( A \in \mathbb{C}_h^{m \times n} \), \( B \in \mathbb{C}^{m \times m} \), \( C \in \mathbb{C}_h^{n \times n} \), \( Q \in \mathbb{C}^{n \times m} \), and \( P \in \mathbb{C}^{p \times n} \) be given, and \( T \in \mathbb{C}^{m \times m} \) be nonsingular. Then one has the following
(1) \( i_k(T A T^*) = i_k(A) \),
(2) \( i_k(\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}) = i_k(A) + i_k(C) \),
(3) \( i_k(\begin{bmatrix} 0 & Q \end{bmatrix}) = r(Q) \),
(4) \( i_k(\begin{bmatrix} A & B L P \\ L P & 0 \end{bmatrix}) = r(P) = i_k(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}). \)

Lemma 5 (see [22, Lemma 1.4]). Let \( S \) be a set consisting of (square) matrices over \( \mathbb{C}^{m \times m} \), and let \( H \) be a set consisting of (square) matrices over \( \mathbb{C}^{m \times n} \). Then one has the following
(a) \( S \) has a nonsingular matrix if and only if \( \max_{X \in S} r(X) = m \);
(b) \( X \in S \) is nonsingular if and only if \( \min_{X \in S} r(X) = m \);
(c) \( \{0\} \in S \) if and only if \( \min_{X \in S} r(X) = 0 \);
(d) \( S = \{0\} \) if and only if \( \max_{X \in S} r(X) = 0 \);
(e) \( H \) has a matrix \( X > 0 \) \( (X < 0) \) if and only if \( \max_{X \in H} i_k(X) = m (\max_{X \in H} i_k(X) = m) \);
(f) \( X \in H \) satisfies \( X > 0 \) \( (X < 0) \) if and only if \( \min_{X \in H} i_k(X) = m (\min_{X \in H} i_k(X) = m) \);
(g) \( H \) has a matrix \( X \geq 0 \) \( (X \leq 0) \) if and only if \( \min_{X \in H} i_k(X) = 0 (\min_{X \in H} i_k(X) = 0) \);
(h) any \( X \in H \) satisfies \( X \geq 0 \) \( (X \leq 0) \) if and only if \( \max_{X \in H} i_k(X) = 0 (\max_{X \in H} i_k(X) = 0) \).

Lemma 6 (see [16]). Let \( p(X, Y) = A - BX - (BX)^* - CYD - (CYD)^* \), where \( A, B, C, \) and \( D \) are given with appropriate sizes, and denote that
\[ M_1 = \begin{bmatrix} A & B & C \\ B^* & 0 & 0 \\ C^* & 0 & 0 \end{bmatrix}, \]
\[ M_2 = \begin{bmatrix} A & B & D^* \\ B^* & 0 & 0 \\ D & 0 & 0 \end{bmatrix}, \]
\[ M_3 = \begin{bmatrix} A & B & C & D^* \\ B^* & 0 & 0 & 0 \\ C^* & 0 & 0 & 0 \\ D & 0 & 0 & 0 \end{bmatrix}, \]
\[ M_4 = \begin{bmatrix} A & B & C & D^* \\ B^* & 0 & 0 & 0 \\ C^* & 0 & 0 & 0 \\ D & 0 & 0 & 0 \end{bmatrix}. \]

Then one has the following:
(1) the maximal rank of \( p(X, Y) \) is
\[ \max_{X \in \mathbb{C}^{m \times n}, Y} r[p(X, Y)] = \min \{ m, r(M_1), r(M_2), r(M_3) \}, \]
(2) the minimal rank of \( p(X, Y) \) is
\[ \min_{X \in \mathbb{C}^{m \times n}, Y} r[p(X, Y)] = 2r(M_3) - 2r(B) + \max \{ u_+ + u_-, v_+ + v_-, u_+ + v_- \}, \]
(3) the maximal inertia of \( p(X, Y) \) is
\[ \max_{X \in \mathbb{C}^{m \times n}, Y} i_k[p(X, Y)] = \min \{ i_k(M_1), i_k(M_2) \}, \]
(4) the minimal inertia of \( p(X, Y) \) is
\[ \min_{X \in \mathbb{C}^{m \times n}, Y} i_k[p(X, Y)] = r(M_3) - r(B) + \max \{ i_k(M_1) - r(M_4), i_k(M_2) - r(M_3) \}. \]

Now we present the main theorem of this section.
Theorem 7. Let $A_1 \in \mathbb{C}^{m \times n}, C_1 \in \mathbb{C}^{m \times k}, B_1 \in \mathbb{C}^{k \times l}, D_1 \in \mathbb{C}^{n \times l}, A_2 \in \mathbb{C}^{t \times q}, C_2 \in \mathbb{C}^{t \times p}, A_3 \in \mathbb{C}^{p \times p}$.

+max\{s_+ + s_-, t_+ + t_- s_+ + t_-, s_+ + t_+\},

(c) the maximal inertia of (10) subject to (13) and (11) is

$$\max_{X \in G, Y \in S} f(X, Y) = \min \{i_+(E_1) - r(A_1) - r(A_2), i_-(E_2) - r(B_1) - r(A_2)\}.$$ (33)

(d) the minimal inertia of (10) subject to (13) and (11) is

$$\min_{X \in G, Y \in S} f(X, Y) = r(E_3) - r \begin{bmatrix} B_3 \\ A_2 \end{bmatrix} + \max \{i_+(E_1) - r(E_4), i_-(E_2) - r(E_5)\}.$$ (34)

Proof. By Lemma 1, the general solutions to (13) and (11) can be written as

$$X = A_2^t C_2 + L A_1 W,$$

$$Y = A_1^t C_1 + L A_1 D_1 \tilde{B}_1^t + L A_1 Z R b_1,$$

where $W$ and $Z$ are arbitrary matrices with appropriate sizes. Put

$$Q = B_3 L A_1, \quad T = C_1 L A_1, \quad J = R b_1 D_3,$$

$$P = A_3 - B_3 A_2^t C_2 - (B_3 A_1^t C_2) \tilde{B}_1$$

$$- C_3 (A_1^t C_1 + L A_1 D_1 \tilde{B}_1) D_3$$

= $C_3 (A_1^t C_1 + L A_1 D_1 \tilde{B}_1) D_3.$ (36)

Substituting (36) into (10) yields

$$f(X, Y) = P - Q W - (Q W)^* - T Z J - (T Z J)^*.$$ (37)

Clearly $P$ is Hermitian. It follows from Lemma 6 that

$$\max_{X \in G, Y \in S} f(X, Y) = \max \{m, r(N_1), r(N_2), r(N_3)\},$$

$$\min_{X \in G, Y \in S} f(X, Y) = \max \{m, r(N_1), r(N_2), r(N_3)\}$$

$$= \max \{m, r(N_1), r(N_2), r(N_3)\},$$

$$= 2r(N_3) - 2r(Q) + \max \{s_+ + s_-, t_+ + t_- s_+ + t_-, s_+ + t_+\},$$

$$= 2r(N_3) - 2r(Q) + \max \{s_+ + s_-, t_+ + t_- s_+ + t_-, s_+ + t_+\},$$

Then one has the following:

(a) the maximal rank of (10) subject to (13) and (11) is

$$\max \{p, r(E_1) - 2r(A_1) - 2r(A_2)\},$$

$$r(E_2) - 2r(B_1) - 2r(A_2),$$

$$r(E_3) - 2r(A_2) - r(E_4) - r(B_1)\},$$

(b) the minimal rank of (10) subject to (13) and (11) is

$$2r(E_3) - 2r \begin{bmatrix} B_3 \\ A_2 \end{bmatrix} + \max \{r(E_1) - 2r(E_4), r(E_2) - 2r(E_3), i_+(E_1) + i_-(E_2) - r(E_4) - r(E_3), i_-(E_1) + i_+(E_2) - r(E_4) - r(E_3)\},$$

$$2r(N_3) - 2r(Q) + \max \{s_+ + s_-, t_+ + t_- s_+ + t_-, s_+ + t_+\},$$
\[
\max_{X \in G, Y \in S} i_{\pm} [f(X, Y)] = \max_{W, Z} r (P - Q W - (Q W)^* - T Z J - (T Z J)^*) \quad (40)
\]
\[
\min_{X \in G, Y \in S} i_{\pm} [f(X, Y)] = \max_{W, Z} r (P - Q W - (Q W)^* - T Z J - (T Z J)^*) \quad (41)
\]
\[
r(N_3) = r (P - Q W - (Q W)^* - T Z J - (T Z J)^*) - r(Q) + \max \{s_k, t_k\},
\]
where
\[
N_1 = \begin{bmatrix} P & Q & T \\ Q^* & 0 & 0 \\ T^* & 0 & 0 \end{bmatrix},
\]
\[
N_2 = \begin{bmatrix} P & Q & J \\ Q^* & 0 & 0 \\ J^* & 0 & 0 \end{bmatrix},
\]
\[
N_3 = \begin{bmatrix} P & Q & T & J \\ Q^* & 0 & 0 & 0 \\ T^* & 0 & 0 & 0 \end{bmatrix},
\]
\[
N_4 = \begin{bmatrix} P & Q & T & J \\ Q^* & 0 & 0 & 0 \\ J^* & 0 & 0 & 0 \end{bmatrix},
\]
\[
N_5 = \begin{bmatrix} P & Q & T & J \\ Q^* & 0 & 0 & 0 \\ J^* & 0 & 0 & 0 \end{bmatrix},
\]
\[
s_k = i_{\pm}(N_1) - r(N_4), \quad t_k = i_{\pm}(N_2) - r(N_3).
\]

Now, we simplify the ranks and inertias of block matrices in (38)–(41).

By Lemma 4, block Gaussian elimination, and noting that
\[
L_S^* = (I - S^t S)^* = I - S^* (S^*)^t = R_S^t,
\]
we have the following:
\[
r(N_1) = r \begin{bmatrix} P & Q & T \\ Q^* & 0 & 0 \\ T^* & 0 & 0 \end{bmatrix} = r \begin{bmatrix} A_3 & D_3^* & C_2^* & B_3 & C_2^* \\ C_2^* & 0 & A_1^* & 0 & 0 \\ C_1 D_3 & A_1^* & 0 & 0 \\ B_1^* & 0 & 0 & 0 & A_2^* \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix} - 2r(A_1) - 2r(A_2),
\]
\[
r(N_3) = r \begin{bmatrix} P & Q & T & J^* \\ Q^* & 0 & 0 \\ T^* & 0 & 0 \\ J^* & 0 & 0 \end{bmatrix} = r \begin{bmatrix} A_3 & B_3 & C_2 & D_3^* & C_2^* \\ B_3^* & 0 & 0 & 0 & A_2^* \\ D_3^* C_2^* & 0 & 0 \\ C_1 D_3 & 0 & A_1 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 \end{bmatrix} - 2r(B_1) - 2r(A_2) - 2r(A_1),
\]
\[
r(N_4) = r \begin{bmatrix} P & Q & T & J^* \\ Q^* & 0 & 0 \\ T^* & 0 & 0 \\ J^* & 0 & 0 \end{bmatrix} = r \begin{bmatrix} A_3 & B_3 & C_2 & D_3^* & C_2^* \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_3 & 0 & 0 & 0 & A_1^* \\ C_2 & 0 & A_2 & 0 & 0 \end{bmatrix} - r(B_1) - 2r(A_2) - r(A_1),
\]
\[
r(N_5) = r \begin{bmatrix} P & Q & T & J^* \\ Q^* & 0 & 0 \\ T^* & 0 & 0 \\ J^* & 0 & 0 \end{bmatrix} = r \begin{bmatrix} A_3 & B_3 & C_3 & D_3^* & C_3^* \\ B_3^* & 0 & 0 & 0 & A_2^* \\ D_3^* C_3^* & 0 & 0 & 0 & A_1^* \\ C_3 & 0 & 0 & 0 & A_1^* \\ C_2 & A_2 & 0 & 0 & 0 \end{bmatrix} - 2r(B_1) - 2r(A_2) - r(A_1).
\]

By Lemma 2, we can get the following:
\[
i_{\pm}(N_1) = i_{\pm} \begin{bmatrix} P & Q & T \\ Q^* & 0 & 0 \\ T^* & 0 & 0 \end{bmatrix} = i_{\pm} \begin{bmatrix} A_3 & B_3 & C_2 & D_3^* & C_2^* \\ B_3^* & 0 & 0 & 0 & A_2^* \\ D_3 & 0 & 0 & 0 & 0 \\ B_1^* & 0 & 0 & 0 & A_1^* \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix} - 2r(A_1) - 2r(A_2).
\[
A_3 \quad C_3 \quad D_3^* C_1^* \quad B_3 \quad C_2^* \\
C_3^* \quad 0 \quad A_1^* \quad 0 \quad 0 \\
C_1^* D_3 \quad A_1 \quad 0 \quad 0 \quad 0 \\
B_3^* \quad 0 \quad 0 \quad 0 \quad A_2^* \\
C_2 \quad 0 \quad 0 \quad 0 \quad A_2 \quad 0
\]
\]
\[-r(A_1) - r(A_2), \quad (46)\]

\[
\begin{align*}
i_+(N_2) &= i_+(P Q J^*) \\
&= i_+(A_3 \quad D_3^* \quad C_3 D_1 \quad B_3 \quad C_2^*) \\
&= i_+(D_3 \quad B_1\quad 0 \quad 0 \quad 0, \\
B_3^* \quad 0 \quad 0 \quad 0 \quad A_2^* \\
C_2 \quad 0 \quad 0 \quad 0 \quad A_2 \quad 0
\end{align*}
\]
\[-r(B_1) - r(A_2). \quad (47)\]

Substituting (44)-(47) into (38) and (41) yields (31)-(34), respectively.

\section*{Corollary 8}

Let \(A_1, \ C_1, \ B_1, \ D_1, \ A_2, \ C_2, \ A_3, \ B_3, \ C_3, \ D_3, \) and \(E_\i\) (\(i = 1, 2, \ldots, 5\)) be as in Theorem 7, and suppose that the system of matrix equations (13) and (11) is consistent, respectively. Denote the set of all solutions to (13) by S and (11) by G. Then, one has the following:

(a) there exist \(X \in G \) and \(Y \in S\) such that \(A_3 - B_3 X - (B_3 X)^* - C_3 Y D_3 - (C_3 Y D_3)^* > 0\) if and only if

\[
i_+(E_1) - r(A_1) - r(A_2) \geq p, \quad (48)
i_+(E_2) - r(B_1) - r(A_2) \geq p.
\]

(b) there exist \(X \in G \) and \(Y \in S\) such that \(A_3 - B_3 X - (B_3 X)^* - C_3 Y D_3 - (C_3 Y D_3)^* < 0\) if and only if

\[
i_-(E_1) - r(A_1) - r(A_2) \geq p, \quad (49)
i_-(E_2) - r(B_1) - r(A_2) \geq p.
\]

(c) there exist \(X \in G \) and \(Y \in S\) such that \(A_3 - B_3 X - (B_3 X)^* - C_3 Y D_3 - (C_3 Y D_3)^* \geq 0\) if and only if

\[
r(E_3) - r\left[\begin{array}{c}
B_3 \\
A_2
\end{array}\right] + i-(E_1) - r(E_4) \leq 0, \quad (50)
r(E_3) - r\left[\begin{array}{c}
B_3 \\
A_2
\end{array}\right] + i-(E_2) - r(E_4) \leq 0.
\]

(d) there exist \(X \in G \) and \(Y \in S\) such that \(A_3 - B_3 X - (B_3 X)^* - C_3 Y D_3 - (C_3 Y D_3)^* \leq 0\) if and only if

\[
r(E_3) - r\left[\begin{array}{c}
B_3 \\
A_2
\end{array}\right] + i+(E_1) - r(E_4) \leq 0, \quad (51)
r(E_3) - r\left[\begin{array}{c}
B_3 \\
A_2
\end{array}\right] + i+(E_2) - r(E_4) \leq 0.
\]

(e) \(A_3 - B_3 X - (B_3 X)^* - C_3 Y D_3 - (C_3 Y D_3)^* > 0\) for all \(X \in G \) and \(Y \in S\) if and only if

\[
r(E_3) - r\left[\begin{array}{c}
B_3 \\
A_2
\end{array}\right] + i_+(E_1) - r(E_4) = p
\]
or \(r(E_3) - r\left[\begin{array}{c}
B_3 \\
A_2
\end{array}\right] + i_+(E_2) - r(E_5) = p.
\]

(f) \(A_3 - B_3 X - (B_3 X)^* - C_3 Y D_3 - (C_3 Y D_3)^* < 0\) for all \(X \in G \) and \(Y \in S\) if and only if

\[
r(E_3) - r\left[\begin{array}{c}
B_3 \\
A_2
\end{array}\right] + i_-(E_1) - r(E_4) = p
\]
or \(r(E_3) - r\left[\begin{array}{c}
B_3 \\
A_2
\end{array}\right] + i_-(E_2) - r(E_5) = p.
\]

(g) \(A_3 - B_3 X - (B_3 X)^* - C_3 Y D_3 - (C_3 Y D_3)^* \geq 0\) for all \(X \in G \) and \(Y \in S\) if and only if

\[
i_+(E_1) - r(A_1) - r(A_2) \leq 0 \quad (54)
or \quad i_+(E_2) - r(B_1) - r(A_2) \leq 0,
\]

(h) \(A_3 - B_3 X - (B_3 X)^* - C_3 Y D_3 - (C_3 Y D_3)^* \leq 0\) for all \(X \in G \) and \(Y \in S\) if and only if

\[
i_-(E_1) - r(A_1) - r(A_2) \leq 0 \quad (55)
or \quad i_-(E_2) - r(B_1) - r(A_2) \leq 0,
\]

(i) there exist \(X \in G \) and \(Y \in S\) such that \(A_3 - B_3 X - (B_3 X)^* - C_3 Y D_3 - (C_3 Y D_3)^* \) is nonsingular if and only if

\[
r(E_1) - 2r(A_1) - 2r(A_2) \geq p, \quad (56)
r(E_2) - 2r(B_1) - 2r(A_2) \geq p,
r(E_3) - 2r(A_2) - r(A_1) - r(B_1) \geq p.
\]

3. Relations between the Hermitian Part of the Solutions to (13) and (11)

Now we consider the extremal ranks and inertias of the difference between the Hermitian part of the solutions to (13) and (11).

\section*{Theorem 9}

Let \(A_1 \in \mathbb{C}^{m \times p}, \ C_1 \in \mathbb{C}^{m \times p}, \ B_1 \in \mathbb{C}^{p \times l}, \ D_1 \in \mathbb{C}^{p \times l}, \) \(A_2 \in \mathbb{C}^{s \times p}, \) and \(C_2 \in \mathbb{C}^{s \times p}, \) be given. Suppose that the system of matrix equations (13) and (11) is consistent,
respectively. Denote the set of all solutions to (13) by \( S \) and (11) by \( G \). Put

\[
H_1 = \begin{bmatrix}
0 & I & C^*_1 & -I & C^*_2 \\
I & 0 & A^*_1 & 0 & 0 \\
C_1 & A_1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & A^*_2 \\
C_2 & 0 & 0 & A_2 & 0
\end{bmatrix}
\]

\[H_2 = r \begin{bmatrix}
0 & I & D_1 & -I & C^*_2 \\
I & 0 & B_1 & 0 & 0 \\
D^*_1 & B^*_1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & A^*_2 \\
C_2 & 0 & 0 & A_2 & 0
\end{bmatrix}
\]

\[
H_3 = \begin{bmatrix}
0 & -I & I & C^*_1 & C^*_1 \\
-I & 0 & 0 & A^*_1 & 0 \\
I & 0 & 0 & 0 & A^*_1 \\
0 & 0 & B^*_1 & 0 & 0 \\
C_1 & 0 & A_1 & 0 & 0 \\
C_2 & A_2 & 0 & 0 & 0
\end{bmatrix}
\]

\[
H_4 = \begin{bmatrix}
0 & -I & I & C^*_1 & D_1 \\
-I & 0 & 0 & A^*_1 & 0 \\
I & 0 & 0 & 0 & B_1 \\
D^*_1 & 0 & 0 & A^*_1 & 0 \\
0 & 0 & A_1 & 0 & 0 \\
C_2 & A_2 & 0 & 0 & 0
\end{bmatrix}
\]

Then one has the following:

\[
\begin{align*}
\max_{X \leq Y \in S} r \left( (X + X^*) - (Y + Y^*) \right) & = \min \{ p, r(H_1) - 2r(A_1) - 2r(A_2), r(H_2) - 2r(B_1) - 2r(A_2) \}, \\
\min_{X \leq Y \in S} r \left( (X + X^*) - (Y + Y^*) \right) & = 2r(H_3) - 2p \\
& + \max \{ r(H_1) - 2r(H_4), r(H_2) - 2r(H_5), i_+ (H_1) + i_- (H_2) - r(H_4) - r(H_5), i_- (H_1) + i_+ (H_2) - r(H_4) - r(H_5) \}, \\
& \max_{X \leq Y \in S} i_+ \left( (X + X^*) - (Y + Y^*) \right) \\
& = \min \{ i_+ (H_1) - r(A_1) - r(A_2), i_+ (H_2) - r(B_1) - r(A_2) \}, \\
& = r(E_4) - p + \max \{ i_+ (H_1) - r(H_4), i_+ (H_2) - r(H_5) \}.
\end{align*}
\]

**Proof.** By letting \( A_3 = 0, B_5 = -I, C_3 = I, \) and \( D_3 = I \) in Theorem 7, we can get the results.

**Corollary 10.** Let \( A_1, C_1, B_1, D_1, A_2, C_2, \) and \( H_i, (i = 1, 2, \ldots, 5) \) be as in Theorem 9, and suppose that the system of matrix equations (13) and (11) is consistent, respectively. Denote the set of all solutions to (13) by \( S \) and (11) by \( G \). Then, one has the following:

(a) there exist \( X \in G \) and \( Y \in S \) such that \( (X + X^*) > (Y + Y^*) \) if and only if

\[
i_+ (H_1) - r(A_1) - r(A_2) \geq p,
\]

(b) there exist \( X \in G \) and \( Y \in S \) such that \( (X + X^*) < (Y + Y^*) \) if and only if

\[
i_- (H_1) - r(A_1) - r(A_2) \geq p,
\]

(c) there exist \( X \in G \) and \( Y \in S \) such that \( (X + X^*) \geq (Y + Y^*) \) if and only if

\[
r(H_3) - p + i_- (H_1) - r(H_4) \leq 0,
\]

(d) there exist \( X \in G \) and \( Y \in S \) such that \( (X + X^*) \leq (Y + Y^*) \) if and only if

\[
r(H_3) - p + i_+ (H_1) - r(H_5) \leq 0,
\]

(e) \( (X + X^*) > (Y + Y^*) \) for all \( X \in G \) and \( Y \in S \) if and only if

\[
r(H_3) - p + i_+ (H_1) - r(H_4) = p,
\]

(f) \( (X + X^*) < (Y + Y^*) \) for all \( X \in G \) and \( Y \in S \) if and only if

\[
r(H_3) - p + i_- (H_1) - r(H_5) = p,
\]

(g) \( (X + X^*) \geq (Y + Y^*) \) for all \( X \in G \) and \( Y \in S \) if and only if

\[
i_- (H_1) - r(A_1) - r(A_2) \leq 0
\]

or \( i_- (H_2) - r(B_1) - r(A_2) \leq 0.\)
(h) \((X + X^*) \leq (Y + Y^*)\) for all \(X \in G\) and \(Y \in S\) if and only if
\[
i_+(H_1) - r(A_1) - r(A_2) \leq 0
\]
\[
or\quad i_+(H_2) - r(B_1) - r(A_2) \leq 0,
\]
(i) there exist \(X \in G\) and \(Y \in S\) such that \((X + X^*) - (Y + Y^*)\) is nonsingular if and only if
\[
r(H_1) - 2r(A_1) - 2r(A_2) \geq p,
\]
\[
r(H_2) - 2r(B_1) - 2r(A_2) \geq p,
\]
\[
(66)
\]
\[
(67)
\]
4. The Solvability Conditions and the General Solution to System (6)

We now turn our attention to (6). We in this section use Theorem 9 to give some necessary and sufficient conditions for the existence of a solution to (6) and present an expression of the general solution to (6). We begin with a lemma which is used in the latter part of this section.

Lemma 11 (see [14]). Let \(A_1 \in C^{m \times n_1}, B_1 \in C^{m \times n_2}, C_1 \in C^{n \times m},\) and \(E_1 \in C^{m \times m}\) be given. Let \(A = R_A B_1, B = C_1 R_A, E = R_A E_1 R_A, M = R_A B^*, N = A^* L_B,\) and \(S = B^* L_M.\) Then the following statements are equivalent:

1. equation (5) is consistent,
2. \(R_M R_A E = 0, R_A E R_A = 0, L_B E L_B = 0,\)
3. \(r \left[ \begin{array}{c} E_1 \\ A_1 \\ B_1 \\ C_1 \\ A_{1}^{*} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \end{array} \right] = r \left[ \begin{array}{c} B_1 \\ C_1 \\ A_1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \end{array} \right] + r(A_1),
\]
\[
(69)
\]
In this case, the general solution of (5) can be expressed as
\[
Y = \frac{1}{2} \left[ A^t E B^t - A^t B^t M E B^t + A^t S(B^*) S E B^t + A^t E(M^*)^t + (N^*)^t E B^t S E S + L_A V_1 + V_2 R_B \\
+ U_1 L_A S L_M + R_N U_2 S L_M - A^t S U_2 R_N A^* B^t, X = A_1^t \left[ E_1 - B_1 Y C_1 - (B_1 Y C_1)^* \right] \\
- \frac{1}{2} A_1^t \left[ E_1 - B_1 Y C_1 - (B_1 Y C_1)^* \right] A_1^t A_1^t \\
- A_1^t W_1 A_1^* + W_1^* A_1 A_1^* + L_A W_2, \right.
\]
where \(U_1, U_2, V_1, V_2, W_1,\) and \(W_2\) are arbitrary matrices over \(C\) with appropriate sizes.

Now we give the main theorem of this section.

**Theorem 12.** Let \(A_i, C_i, (i = 1, 2, 3), B_j,\) and \(D_j, (j = 1, 3)\) be given. Set
\[
A = B_2 L_{A_2}, B = C_3 L_{A_3}, C = R_B D_3, F = R_A B_3, G = C R_A, M = R_3 G^*, N = F^* L_G, S = G^* L_M, D = A_3 - B_3 A_1 C_2 - (B_3 A_1^* C_2)^*, E = R_A D_R A_1. \]

Then the following statements are equivalent:

1. system (6) is consistent,
2. the equalities in (14) and (17) hold, and
\[
R_M R_A E = 0, R_A E R_A = 0, L_B E L_B = 0,
\]
3. the equalities in (15) and (18) hold, and
\[
r \left[ \begin{array}{c} A_3 \\ C_3 \\ D_3^* \\ B_3 \\ C_2 \\ B_2 \\ A_2 \\ 0 \\ 0 \\ 0 \\ \end{array} \right] = r \left[ \begin{array}{c} C_3 \\ D_3^* \\ B_3 \\ C_2 \\ B_2 \\ A_2 \\ 0 \\ 0 \\ 0 \\ \end{array} \right] + r \left[ \begin{array}{c} A_3 \\ C_3 \\ D_3^* \\ B_3 \\ C_2 \\ B_2 \\ A_2 \\ 0 \\ 0 \\ 0 \\ \end{array} \right],
\]
\[
r \left[ \begin{array}{c} A_3 \\ C_3 \\ D_3^* \\ B_3 \\ C_2 \\ B_2 \\ A_2 \\ 0 \\ 0 \\ 0 \\ \end{array} \right] = 2r \left[ \begin{array}{c} A_3 \\ C_3 \\ D_3^* \\ B_3 \\ C_2 \\ B_2 \\ A_2 \\ 0 \\ 0 \\ 0 \\ \end{array} \right],
\]
\[
r \left[ \begin{array}{c} A_3 \\ C_3 \\ D_3^* \\ B_3 \\ C_2 \\ B_2 \\ A_2 \\ 0 \\ 0 \\ 0 \\ \end{array} \right] = 2r \left[ \begin{array}{c} A_3 \\ C_3 \\ D_3^* \\ B_3 \\ C_2 \\ B_2 \\ A_2 \\ 0 \\ 0 \\ 0 \\ \end{array} \right].
\]

In this case, the general solution of system (6) can be expressed as
\[
X = A_3^* C_2 + L_{A_2} U_1, \quad Y = A_3^* C_1 + L_{A_3} D_2 U_1 + L_{A_3} V_R B_1.
\]
where
\[
V = \frac{1}{2} \left[ F^t G^t - F^t G^* M^t E G^t - F^t S(G^t)^* E N^t F^* G^t \right].
\]

\[
+ F^t (M^t)^* + (N^t)^* E G^t S^t S + L_E V_1
\]
\[
+ V_2 R_G + U_1 L_S L_M + R_N U_2^* L_M - F^t S U_2 R_N F^* G^t,
\]
\[
U = A^t \left[ D - B V C - (B V C)^* \right]
\]
\[
- \frac{1}{2} A^t \left[ D - B V C - (B V C)^* \right] A^t
\]
\[
- A^t W_1 A^* + W_1^* A A^t + L_A W_2,
\]
(77)

where \( U_1, U_2, V_1, V_2, W_1, \) and \( W_2 \) are arbitrary matrices over \( C \) with appropriate sizes.

**Proof.** (2) \( \iff \) (3): Applying Lemma 3 and Lemma 11 gives
\[
R_M R_E E = 0 \iff r \left( R_M R_E E \right)
\]
\[
= 0 \iff r \left[ D \ B \ C^* \ A \right]
\]
\[
= r \left[ B \ C^* \ A \right] + r (A)
\]
(78)

(1) \( \iff \) (2): We separate the four equations in system (6) into three groups:

\[
A_1 Y = C_1, \quad Y B_1 = D_1.
\]
(80)

\[
A_2 X = C_2,
\]
(81)

\[
B_3 X + (B_3 X)^* + C_3 Y D_3 + (C_3 Y D_3)^* = A_3.
\]
(82)

By Lemma 1, we obtain that system (80) is solvable if and only if (14), (81) is consistent if and only if (17). The general solutions to systems (80) and (81) can be expressed as (16) and (19), respectively. Substituting (16) and (19) into (82) yields

\[
A U + (A U)^* + B V C + (B V C)^* = D.
\]
(83)

Hence, the system (5) is consistent if and only if (80), (81), and (83) are consistent, respectively. It follows from Lemma 11 that (83) is solvable if and only if

\[
R_M R_E E = 0, \quad R_E E R_E = 0, \quad L_G E L_G = 0.
\]
(84)

We know by Lemma 11 that the general solution of (83) can be expressed as (77). \( \square \)

In Theorem 12, let \( A_1 \) and \( D_1 \) vanish. Then we can obtain the general solution to the following system:

\[
A_2 X = C_2, \quad Y B_1 = D_1.
\]
(85)

\[
B_3 X + (B_3 X)^* + C_3 Y D_3 + (C_3 Y D_3)^* = A_3.
\]

**Corollary 13.** Let \( A_2, C_2, B_1, D_1, B_3, C_3, D_3, \) and \( A_3 = A_3^* \) be given. Set

\[
A = B_3 L A_2, \quad C = R_R D_3,
\]
\[
F = R_A C_2, \quad G = C R_A,
\]
\[
M = R_E G^*, \quad N = F^* L_G,
\]
\[
S = G^* L_M,
\]
(86)

\[
D = A_3 - B_3 A_2 C_2 - (B_3 A_2 C_2)^* - C_3 D_1 B_1^* D_3 - (C_3 D_1 B_1^* D_3)^*,
\]
\[
E = R_A D R_A.
\]
Then the following statements are equivalent:

(1) system (85) is consistent

(2) $R_{A_1}C_2 = 0$, $D_1L_{B_1} = 0$, $R_MR_E E = 0$, $R_{E}E_1 = 0$, $L_{G}E_{L_{G}} = 0$,

(3) $r [A_2 C_2] = r (A_2)$, $r [D_1 B_1] = r (B_1)$,

$$
\begin{bmatrix}
A_3 & C_3 & D_3^* & B_3 & C_3^* \\
0 & 0 & 0 & A_3 & 0 \\
D_3^* & C_3 & B_3 & 0 & 0 \\
C_2 & 0 & 0 & A_2 & 0
\end{bmatrix}
= 2r
\begin{bmatrix}
C_3 & B_3 \\
0 & A_2
\end{bmatrix},
$$

$$
\begin{bmatrix}
A_3 & C_3 & D_3^* & B_3 & C_3^* \\
0 & 0 & 0 & A_3 & 0 \\
D_3^* & C_3 & B_3 & 0 & 0 \\
C_2 & 0 & 0 & A_2 & 0
\end{bmatrix}
= 2r
\begin{bmatrix}
D_3^* & B_3 \\
B_1 & 0 \\
0 & A_2
\end{bmatrix}.
$$

In this case, the general solution of system (6) can be expressed as

$$
X = A^*_0 C_2 + L_{A_1} U_1,
$$

$$
Y = D_1 B_1^* + V_{R_{B_1}},
$$

where

$$
V = \frac{1}{2} \left[ F^t E G - F^t G M^t E G^t - F^t S (G^t) \right] E N^t F^t G^t
$$

$$
+ F^t E (M^t) + (N^t)^t E G S \right] L_E V_1
$$

$$
+ V_2 R_G + U_1 L_S L_M + R_N U_2^* L_M - F^t S U_2 R_N F^t G^t,
$$

$$
U = A^t \left[ D - C_3 V C - (C_3 V C)^* \right]
$$

$$
- \frac{1}{2} A^t \left[ D - C_3 V C - (C_3 V C)^* \right] A A^t
$$

$$
- A^t W_1 A^* + W_1^* A A^t + L_A W_2,
$$

and comments, which resulted in a great improvement of the original paper. This research was supported by the Grants from the National Natural Science Foundation of China (NSFC (11611008)) and Doctoral Program Fund of Ministry of Education of P.R.China (20115201110002).

**References**


Submit your manuscripts at http://www.hindawi.com