Research Article

On the Domination Number of Cartesian Product of Two Directed Cycles

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Denote by $\gamma(G)$ the domination number of a digraph G and $C_m \square C_n$ the Cartesian product of C_m and C_n , the directed cycles of length $m, n \geq 2$. In 2010, Liu et al. determined the exact values of $\gamma(C_m \square C_n)$ for $m = 2, 3, 4, 5, 6$. In 2013, Mollard determined the exact values of $\gamma(C_m \square C_n)$ for $m = 3k + 2$. In this paper, we give lower and upper bounds of $\gamma(C_m \square C_n)$ with $m = 3k + 1$ for different cases. In particular, $\lceil (2k + 1)n/2 \rceil \leq \gamma(C_{3k+1} \square C_n) \leq \lfloor (2k + 1)n/2 \rfloor + k$. Based on the established result, the exact values of $\gamma(C_m \square C_n)$ are determined for $m = 7$ and 10 by the combination of the dynamic algorithm, and an upper bound for $\gamma(C_{13} \square C_n)$ is provided.

1. Introduction

All of the digraphs considered in this paper are finite and simple, that is, without multiple edges or loops. For a digraph $G = (V, E)$ and a vertex $v \in V$, $N_G^+(v)$ and $N_G^-(v)$ denote the set of out-neighbors and in-neighbors of v . Given two vertices u and v in G , we say u dominates v if $u = v$ or $uv \in E$. Let $N_G^+[v] = N_G^+(v) \cup \{v\}$. A vertex v dominates all vertices in $N_G^+[v]$. A set $D \subseteq V$ is a dominating set of G if D dominates $V(G)$. The domination number of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G .

The Cartesian product of graphs G and H is the graph $G \square H$ with the vertex set $G \times H$, and $(g, h)(g', h') \in E(G \square H)$ if either $gg' \in E(G)$ and $h = h'$ or $hh' \in E(H)$ and $g = g'$. The Cartesian product is commutative and associative, having the one-vertex graph as a unit. The subgraph of $G \square H$ induced by $V(G) \times h$, where $h \in V(H)$, is isomorphic to G , called a G -layer (over h) and denoted by G^h . For more information on the Cartesian product of graphs see [1].

We use a 0-1 matrix pattern $\mathcal{A}$ with m rows and n columns to represent a dominating set D of $C_m \square C_n$, where $(i, j) \in D$ if and only if the value at the entry (i, j) of $\mathcal{A}$ equals 1. For

example, the pattern in Figure 2 represents the dominating set (the set of black circles) shown in Figure 1. Let P be a pattern; we denote by $c(P)$ the column number of P . Let A and B be patterns; then the pattern AB denotes the concatenation of patterns A and B , and A^k denotes the concatenation of patterns A for k times.

Let $[n]$ denote the set $\{0, 1, \dots, n-1\}$. For an integer $n \geq 3$, the directed cycle of length n is the graph C_n whose vertices are $0, 1, \dots, n-1$ and whose edges are the pairs $i, i+1$, where the arithmetic is done modulo n . We denote $V(C_n)$ by $[n]$. In this paper, when we consider the Cartesian product graph $C_m \square C_n$ of two cycles C_m and C_n , the edge set of C_m is $\{(i, i+1) : i \in [m]\}$ and C_n is $\{(i+1, i) : i \in [n]\}$. Throughout the paper, when considering the vertex (i, j) in the graph $C_m \square C_n$, we use the arithmetic operations of the index i over modulo m and j modulo n .

The dominating set problem requires determining the domination number of a given graph. It has natural applications in numerous facility location problems. In such problems, the vertices of a graph correspond to locations, adjacency represents some notion of accessibility, and the goal is to find a subset of locations accessible from all other locations at which to install fire stations, bus stops, post

FIGURE 1: A dominating set of $C_7 \square C_{18}$.

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101001001001001010
001010010010010001
010010100100100110
100100101001001000
001001001010010011
010010010010100100
100100100100101001

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FIGURE 2: Pattern of the dominating set shown in Figure 1.

offices, or similar facilities [2]. Dominating sets have also been applied in coding theory [2] and social networks [3]. For more information on the history and applications of the dominating set problem, see [4].

In 1990, Faudree and Schelp [5] initially discussed the domination number of the Cartesian product of two undirected graphs. The domination number for Cartesian product of two graphs has attracted lots of attention, and there are many works on undirected graphs (see, e.g., [6–12]). Recently, there are also some works on the domination number of Cartesian product of two directed graphs, in particular, the Cartesian product of directed paths and cycles [13–15].

In [13, 15], Liu et al. determined the exact values of $\gamma(C_m \square C_n)$ for $m = 2, 3, 4, 5, 6$ and showed the following.

Theorem 1. $\gamma(C_m \square C_n) = mn/3$ if $m \equiv 0 \pmod{3}$ and $n \equiv 0 \pmod{3}$.

Mollard [14] determined exact values of $\gamma(C_m \square C_n)$ for $m = 3k + 2$. This paper investigates properties on the domination of Cartesian product of C_m and C_n with $m = 3k + 1$. We give lower and upper bounds of $\gamma(C_m \square C_n)$ with $m = 3k + 1$ for different cases. In particular, $\lfloor (2k + 1)n/2 \rfloor \leq \gamma(C_{3k+1} \square C_n) \leq \lfloor (2k + 1)n/2 \rfloor + k$. Based on the established result, the exact values of $\gamma(C_m \square C_n)$ are determined for $m = 7$ and 10 by the combination of the dynamic algorithm, and an upper bound for $\gamma(C_{13} \square C_n)$ is provided.

2. Main Results

2.1. Bounds on $C_{3k+1} \square C_n$

Proposition 2 (Mollard [14]). *Let S be a dominating set of $C_m \square C_n$; then for all $i \in [n]$ considered modulo n one has $|S \cap C_m^{i-1}| + 2|S \cap C_m^i| \geq m$.*

Lemma 3. *Let $n \geq 3, k \geq 1$. Then there exists a minimum dominating set D of $C_{3k+1} \square C_n$ such that $|C_{3k+1}^i \cap D| \geq k$ for every $i \in V(C_n)$.*

Proof. For a dominating set D of $C_{3k+1} \square C_n$, we define a function f as follows:

$$f(D) = \sum_{t=0}^{n-1} x_t, \quad (1)$$

where

$$x_t = \begin{cases} 0, & \text{if } |D \cap C_{3k+1}^t| \geq k + 1, \\ |D \cap C_{3k+1}^t|, & \text{if } |D \cap C_{3k+1}^t| \leq k. \end{cases} \quad (2)$$

We now assume that D is a minimum dominating set of $C_{3k+1} \square C_n$ and suppose to the contrary that $|C_{3k+1}^{i+1} \cap D| < k$ for some i such that $f(D)$ is maximized. Then there exists a set $T = \{(j, i+1), (j+1, i+1), (j+2, i+1)\}$ such that $u \notin D$ for any $u \in T$. Thus, $\{(j, i), (j+1, i)\} \subseteq D$. By Proposition 2, we have $x_t = 0$ if $0 < x_{t+1} \leq k$. Thus $D' = D \setminus \{(j+1, i)\} \cup \{(j+1, i+1)\}$ is a dominating set with $|D| = |D'|$ such that $f(D') > f(D)$, a contradiction with the fact that $f(D)$ is maximized. $\square$

Lemma 4. *Let $n \geq 3, k \geq 1$. Then there exists a minimum dominating set D of $C_{3k+1} \square C_n$ such that $|(C_{3k+1}^i \cup C_{3k+1}^{i+1}) \cap D| \geq 2k + 1$ for every $i \in V(C_n)$.*

Proof. By Lemma 3, it is sufficient to consider a minimum dominating set D of $C_{3k+1} \square C_n$ such that $|C_{3k+1}^i \cap D| \geq k$ for every $i \in V(C_n)$. If $|C_{3k+1}^i \cap D| = k$, together with Proposition 2, we have that $|C_{3k+1}^{i-1} \cap D| \geq k + 1$, which completes the proof. $\square$

By Lemmas 3 and 4, there is an obvious lower bound in terms of $\gamma(C_{3k+1} \square C_n)$ as follows.

Lemma 5. *Let $k \geq 1$ and $n \geq 3$ be two integers; then*

$$\gamma(C_{3k+1} \square C_n) \geq \left\lfloor \frac{(2k+1)n}{2} \right\rfloor. \quad (3)$$

Lemma 6. *For two integers $k \geq 1$ and $n \geq 3$, then for any $\ell \in [k]$, one has*

- (1) $\gamma(C_{3k+1} \square C_n) \leq \lfloor (2k+1)n/2 \rfloor + \ell$ if $n \equiv 6\ell \pmod{6k+2}$,
- (2) $\gamma(C_{3k+1} \square C_n) \leq \lfloor (2k+1)n/2 \rfloor + k - \ell$ if $n \equiv 2 + 6\ell \pmod{6k+2}$,
- (3) $\gamma(C_{3k+1} \square C_n) \leq \lfloor (2k+1)n/2 \rfloor + k$ if $n \equiv 4 + 6\ell \pmod{6k+2}$,
- (4) $\gamma(C_{3k+1} \square C_n) \leq \lfloor (2k+1)n/2 \rfloor + \ell + 1$ if $n \equiv 3 + 6\ell \pmod{6k+2}$,
- (5) $\gamma(C_{3k+1} \square C_n) \leq \lfloor (2k+1)n/2 \rfloor + k - \ell$ if $n \equiv 5 + 6\ell \pmod{6k+2}$,
- (6) $\gamma(C_{3k+1} \square C_n) \leq \lfloor (2k+1)n/2 \rfloor + k$ if $n \equiv 7 + 6\ell \pmod{6k+2}$,
- (7) $\gamma(C_{3k+1} \square C_n) \leq \lfloor (2k+1)n/2 \rfloor + k$ for $n \equiv 1 \pmod{6k+2}$.

Proof. For any $i \in [n]$, let

$$A_i = \begin{cases} \left\{ \left(3q + \frac{i}{2}, i \right) : 0 \leq q \leq k \right\}, & \text{if } n \text{ is even,} \\ \left\{ \left(3q + 2 + \left\lfloor \frac{i}{2} \right\rfloor, i \right) : 0 \leq q \leq k-1 \right\}, & \text{if } n \text{ is odd.} \end{cases} \quad (4)$$

It can be seen that if $i \equiv 0 \pmod{2}$, $|A_i| = k + 1$; otherwise, $|A_i| = k$. Let $S = \bigcup_{i=0}^{n-1} A_i$; then $|S| = \lceil (2k + 1)n/2 \rceil$ and there are at most k vertices, $(1, 0), (4, 0), (7, 0), \dots, (3k - 2, 0)$ of $C_{3k+1} \square C_n$ of which each is not dominated by S . Denoted by $B = \{(1, n - 1), (4, n - 1), \dots, (3k - 2, n - 1)\}$, we consider the following cases.

(1) $n \equiv 6\ell \pmod{6k + 2}$. In this case, it is easy to see that $|A_{n-1} \cap B| = k - \ell$. By adding at most ℓ vertices to S , we get a dominating set of $\gamma(C_{3k+1} \square C_n)$ as required.

(2) $n \equiv 2 + 6\ell \pmod{6k + 2}$. Since $|A_{n-1} \cap B| = \ell$, we will obtain a desired dominating set of $\gamma(C_{3k+1} \square C_n)$ by adding at most $k - \ell$ vertices to S .

(3) $n \equiv 4 + 6\ell \pmod{6k + 2}$. Since $|A_{n-1} \cap B| = \emptyset$, we have to add at most k vertices so that a dominating set of $\gamma(C_{3k+1} \square C_n)$ is constructed based on S .

(4) $n \equiv 3 + 6\ell \pmod{6k + 2}$. Since $n - 1$ is an even and $|A_{n-1} \cap B| = k - \ell$, we have to add at most ℓ vertices to S so that a dominating set of $\gamma(C_{3k+1} \square C_n)$ will be formed.

(5) $n \equiv 5 + 6\ell \pmod{6k + 2}$. Since $|A_{n-1} \cap B| = \ell$, we will obtain a dominating set of $\gamma(C_{3k+1} \square C_n)$ by adding at most $k - \ell$ vertices to S .

(6) $n \equiv 7 + 6\ell \pmod{6k + 2}$. Let $A'_{n-3} = \{(1 + 3\ell + 3k, n - 3), (2 + 3\ell, n - 3), (5 + 3\ell, n - 3), \dots, (2 + 3\ell + 3(k - 1), n - 3)\}$ when $2 + 3\ell > 2 + 3\ell + 3k \pmod{3k + 1}$; otherwise, let $A'_{n-3} = \{(1 + 3\ell, n - 3), (5 + 3\ell, n - 3), (8 + 3\ell, n - 3), \dots, (2 + 3\ell + 3k, n - 3)\}$ when $2 + 3\ell < 2 + 3\ell + 3k \pmod{3k + 1}$; let $A'_{n-2} = C_{3k+1}^{n-2} \setminus \{(1, n - 2), (4, n - 2), (7, n - 2), \dots, (3k - 2, n - 2), (1 + 3\ell + 3k, n - 2)\}$ when $2 + 3\ell > 2 + 3\ell + 3k \pmod{3k + 1}$; otherwise, let $A'_{n-2} = C_{3k+1}^{n-2} \setminus \{(1, n - 2), (4, n - 2), (7, n - 2), \dots, (3k - 2, n - 2), (1 + 3\ell, n - 2)\}$ when $2 + 3\ell < 2 + 3\ell + 3k \pmod{3k + 1}$; let $A'_{n-1} = \{(1, n - 1), (4, n - 1), \dots, (3k - 2, n - 1)\}$, $A'_i = A_i$ for $i \in [n - 4]$. In this way, if we let $D = \bigcup_{i=0}^{n-1} A'_i$, then D is a dominating set of $C_{3k+1} \square C_n$. In addition, $|A'_{n-3}| = k + 1$, $|A'_{n-2}| = 2k$, and $|A'_{n-1}| = k$. So, $D = ((2k + 1)n)/2 + ((2k - 1)/2) = \lfloor ((2k + 1)n)/2 \rfloor + k$.

(7) $n \equiv 1 \pmod{6k + 2}$. Let $D = \sum_{i=0}^{n-2} A_i \cup \{(1 + 3q, n - 1), (3 + 3q, n - 1) : 0 \leq q \leq k - 1\}$. Then D is a dominating set of $C_{3k+1} \square C_n$ with cardinality $\lfloor (2k + 1)n/2 \rfloor + k$. $\square$

The result of Lemmas 5 and 6 show the following.

Corollary 7. Let $k \geq 1$ and $n \geq 3$ be two integers; then

$$\left\lfloor \frac{(2k + 1)n}{2} \right\rfloor \leq \gamma(C_{3k+1} \square C_n) \leq \left\lfloor \frac{(2k + 1)n}{2} \right\rfloor + k. \quad (5)$$

3. On the Domination of $C_7 \square C_n$

Lemma 8. Let $k \geq 1$ and $n \geq 3$ be two integers. Suppose D is a minimum dominating set of $C_{3k+1} \square C_n$. Let $S_i = D \cap C_{3k+1}^i$ and $a_i = |S_i|$ for $i = 0, 1, \dots, n - 1$.

(a) If there is an integer i such that $a_i + a_{i+1} = 2k + 1$ (considered modulo n), then one of $\{a_i, a_{i+1}\}$ is k and the other is $k + 1$.

(b) If there is an integer i such that $a_i = k + 1$, $a_{i+1} = k$, $a_{i+2} = k + 1$, and $a_{i+3} = k$, then for any distinct vertices $(j_1, i + 1), (j_2, i + 1)$ in S_{i+1} and $(j_3, i + 3), (j_4, i + 3)$ in S_{i+3} , $j_1 - j_2 \equiv 0 \pmod{3}$, $j_3 - j_4 \equiv 0 \pmod{3}$; for any distinct vertex (j'_1, i) , (j'_2, i) in S_i and $(j'_3, i + 2), (j'_4, i + 2)$ in S_{i+2} , $|j'_1 - j'_2| \equiv 0 \pmod{3}$ or $|j'_1 - j'_2| = 1$ and $|j'_3 - j'_4| \equiv 0 \pmod{3}$ or $|j'_3 - j'_4| = 1$, where $j_1 < j_2, j_3 < j_4, j'_1 < j'_2, j'_3 < j'_4$, and they are taken modulo $3k + 1$.

Proof. It follows immediately that (a) holds by Lemmas 3 and 4. In addition, since $a_i = k + 1$, $a_{i+1} = k$, $a_{i+2} = k + 1$, and $a_{i+3} = k$, each of vertices of C_{3k+1}^{i+1} and C_{3k+1}^{i+3} is dominated by only one vertex; also, only one vertex of C_{3k+1}^{i+2} is dominated by two vertices. Now we will first show that it is impossible that there are three consecutive vertices in S_i . Otherwise, suppose that (s, i) , $(s + 1, i)$, and $(s + 2, i)$ (modulo $3k + 1$) are three consecutive vertices. We can deduce that vertices $(s + 1, i + 2)$ and $(s + 2, i + 3)$ are in S_{i+2} and S_{i+3} , respectively, so $(s + 1, i + 3)$ is dominated by both $(s + 1, i + 2)$ and $(s + 2, i + 3)$, a contradiction.

Suppose that there are $(j_1, i + 1), (j_2, i + 1)$ (or $(j_3, i + 3), (j_4, i + 3)$) in S_{i+1} (or S_{i+3}) such that $j_1 - j_2 \not\equiv 0 \pmod{3}$ (or $j_3 - j_4 \not\equiv 0 \pmod{3}$). Then it is easy to show that there exist three consecutive vertices $\in S_i$ or three consecutive vertices $(s_1, i + 2), (s_1 + 1, i + 2)$, and $(s_1 + 2, i + 2) \in S_{i+2}$ (modulo $3k + 1$), thus $(s_1, i + 1), (s_1 + 1, i + 1)$, and $(s_1 + 2, i + 1)$ (modulo $3k + 1$) can only be dominated by S_i , and $(s_1 + 4, i + 1) \in S_{i+1}$ since it is impossible that there are four consecutive vertices in S_{i+3} by $a_{i+3} = k$. So, it is an easy task to prove that either $(s_1 + 3, i + 2)$ is not dominated by D or $(s_1 + 4, i + 2)$ is dominated by two vertices, a contradiction. By using the same approach as above, we also can prove the cases of vertices in S_i and S_{i+2} . $\square$

Lemma 9. Let $k \geq 1$ and $n \geq 3$. If D is a minimum dominating set of $C_{3k+1} \square C_n$ such that $(a_i, a_{i+1}, a_{i+2}, a_{i+3}, \dots, a_{i+2\ell}) = (k + 1, k, k + 1, k, \dots, k + 1)$, where $S_i = D \cap C_{3k+1}^i$ and $a_i = |S_i|$ for some i, ℓ , then $(j, i + 1) \in S_{i+1}$ if and only if $(j + t - 1, i + 2t - 1) \in S_{i+2t-1}$ for any $j \in \{0, 1, \dots, 3k\}$ and $t \in \{1, 2, \dots, \ell\}$.

Proof. By Lemma 8 and the symmetrical structure of $C_m \square C_n$, we assume without loss of generality $S_{i+2\ell-1} = \{(3q, i + 2\ell - 1) : 0 \leq q \leq k - 1\}$. Since each vertex in $\{(3q - 1, i + 2\ell - 1) : 0 \leq q \leq k - 1\} \cup \{-2, i + 2\ell - 1\}$ is not dominated by $C_{3k+1}^{i+2\ell-1}$, we have $S_{i+2\ell-2} = \{(3q - 1, i + 2\ell - 2) : 0 \leq q \leq k - 1\} \cup \{-2, i + 2\ell - 2\}$. By observing that each vertex in $S_{i+2\ell-2}$ needs to be dominated, we have the desired set $S_{i+2\ell-3}$. Repeating this process until S_{i+1} is determined, we have the desired result. $\square$

Theorem 10. Let $n \geq 7$. Then

$$\gamma(C_7 \square C_n) = \begin{cases} \left\lfloor \frac{5n}{2} \right\rfloor, & n \equiv 0 \pmod{14}, \\ \left\lfloor \frac{5n}{2} \right\rfloor + 1, & n \equiv 3, 4, 6, 8, 10, 11 \pmod{14}, \\ \left\lfloor \frac{5n}{2} \right\rfloor + 2, & n \equiv 1, 2, 5, 7, 9, 12, 13 \pmod{14}. \end{cases} \quad (6)$$

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101001010010100100100100
001010100100101001001001
010010001001001010010010
100100110010010010100100
001001000100100100101001
010010011001001001001010
100100101010010010010010

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FIGURE 3: A dominating set of $C_7 \square C_{24}$.

Proof. Consider the following.

Case 1. $n \equiv 0 \pmod{14}$.

The lower bound follows from Lemma 5, and the upper bound follows from Lemma 6 with $\ell = 0$ in Case 1.

Case 2. $n \equiv 3, 4, 6, 8, 10, 11 \pmod{14}$.

By using Lemmas 3 and 4, we obtain that $\gamma(C_7 \square C_n) = \lfloor 5n/2 \rfloor$ if and only if $n \equiv 0 \pmod{14}$. Therefore, the desired lower bound is established. Upper bound can be obtained by applying Lemma 6 with $k = 2$ in the following cases:

- (i) $n \equiv 3 \pmod{14}$, $\ell = 0$ in case 4;
- (ii) $n \equiv 6 \pmod{14}$, $\ell = 1$ in case 1;
- (iii) $n \equiv 8 \pmod{14}$, $\ell = 1$ in case 2;
- (iv) $n \equiv 11 \pmod{14}$, $\ell = 1$ in case 5.

We give the upper bound for the case $n \equiv 4, 10 \pmod{14}$ by constructions.

Let $n = 14k + 4$. The pattern depicted in Figure 2 is a dominating set of $C_7 \square C_{18}$ with 46 vertices. Moreover, the leftmost 14 columns induce a dominating set of $C_7 \square C_{14}$ with 35 vertices. By repeating the leftmost 14 columns for k times, we get a dominating set of $C_7 \square C_{14k+4}$ with $35k + 11$ vertices. Therefore, $\gamma(C_7 \square C_{14k+4}) \leq 35k + 11$, and thus, $\gamma(C_7 \square C_n) \leq \lfloor 5n/2 \rfloor + 1$.

Let $n = 14k + 10$. The pattern depicted in Figure 3 is a dominating set of $C_7 \square C_{24}$ with 61 vertices. Moreover, the leftmost 14 columns induce a dominating set of $C_7 \square C_{14}$ with 35 vertices. By repeating the leftmost 14 columns for k times, we get a dominating set of $C_7 \square C_{14k+10}$ with $35k + 26$ vertices. Therefore, $\gamma(C_7 \square C_{14k+10}) \leq 35k + 26$, and thus, $\gamma(C_7 \square C_n) \leq \lfloor 5n/2 \rfloor + 1$.

Case 3. $n \equiv 1, 2, 5, 7, 9, 12, 13 \pmod{14}$.

The upper bound follows from Lemma 6. We then show the lower bound. Let $n \equiv 1 \pmod{14}$. If there exists a minimum dominating set of $C_{3k+1} \square C_n$ with cardinality $\lfloor 5n/2 \rfloor + 1$, then by Lemmas 3 and 4, we have such a dominating set D with $(a_0, a_1, a_2, a_3, \dots, a_{n-2}, a_{n-1}) = (3, 2, 3, 2, \dots, 2, 3)$, where $S_i = D \cap C_7^i$ and $a_i = |S_i|$. We without loss of generality assume $S_{n-2} = \{(0, n-2), (3, n-2)\}$. Then $S_1 = \{(1, 1), (4, 1)\}$ and so $S_0 = \{(0, 0), (3, 0), (6, 0)\}$. We now consider S_{n-2} , S_{n-1} , and S_0 with S_{n-2} and S_0 known. Then there are four vertices in S_{n-1} not dominated by $S_{n-2} \cup S_0$, a contradiction. Therefore, $\gamma(C_7 \square C_n) \geq \lfloor 5n/2 \rfloor + 2$ if $n \equiv 1 \pmod{14}$. By using the same approach, we can obtain the desired lower bound if $n \equiv 2, 5, 7, 9, 12, 13 \pmod{14}$ and the proof is omitted. $\square$

FIGURE 4: Two vertices of D_7 .

4. On the Domination of $C_{10} \square C_n$

We use the *dynamic algorithm* to provide the result on $\gamma(C_m \square C_n)$ for $m = 10$ and 13. We first describe the concept needed to describe our computer checking. The idea is introduced in [16] in a more general framework and extended in [17], but for our purpose the following description will be sufficient.

For a graph G and $D \subseteq V(G)$, we define a function $f : V(G) \rightarrow \{0, 1\}$ such that $f(v) = 0$ if $v \notin D$ and $f(v) = 1$ if $v \in D$. We call such a function the 2-coloring of G . If for $f(v) = 0$ there exists a vertex $u \in N_G^-(v)$ such that $f(u) = 1$, then we say v is dominated. We can see that this notation is equivalent to the dominating set and we will use it to describe the dynamic algorithm for this problem.

Let $C = \{u_i u_{i+1} : i \in [n]\}$ be a directed cycle in a weighted digraph G . We say that the sum of the weights of all the edges in C to be the *weight* of C .

Let u be a 2-coloring of $C_m \square P_2$; u is said to be an (a, b) -vertex if the number of ones in the first copy of C_m equals a and the second equals b (see, e.g., the left side of the vertex in Figure 4 is a $(3, 2)$ -vertex). An (a, b) -vertex is said to be *redundant* if it is *dominated* and any removal of one vertex in the first column destroys the domination.

Let $T = \{(a_i, b_i) : 1 \leq i \leq t\}$. We define a weighted digraph D_m^T (when no confusion can arise, the set T is omitted and the digraph is denoted by D_m) as follows.

The vertices of D_m^T consist of all the *nonredundant* 2-colorings of $C_m \square P_2$. Let $u = u_1 u_2$ be a vertex of D_m^T . Then u_1 and u_2 represent the 2-colorings of $C_m \square P_2$ restricted to the first and second copies of C_m . Let u and v be two vertices of D_m^T . Then $\widehat{uv}$ denotes the 2-coloring of $C_m \square P_3$ obtained by applying u_2 and v_1 to the consecutive copies of C_m . For each vertex u of D_m^T , we make an arc from u to v in D_m^T if and only if the following conditions are fulfilled:

- (i) u_2 equals v_1 ,
- (ii) each of the vertices of $C_m^1 \cup C_m^2$ is dominated (by $C_m^0 \cup C_m^1$),
- (iii) for each (a, b) -vertex of D_m^T , $(a, b) \in T$.

Finally, the weight of uv is the number of ones in C_m^1 of v .

Let $T = \{(2, 3), (3, 2)\}$. Figure 4 shows two vertices of D_7^T . We can see that the 2-coloring (0010010) of the second copy of C_7 of the first vertex equals the first copy of C_7 of the second vertex. Moreover, the weight of the second copy of C_7 of the first vertex is 2. It follows that $A(D_7^T)$ has an arc from the first (left) vertex to the second (right) with weight 2.

Lemma 11. Let $m, n \geq 3$. Then D_m possesses a directed cycle of length n with weight k if and only if $C_m \square C_n$ contains a minimal dominating set S with $|S| = k$.

Proof. If S is a minimal dominating set of $C_m \square C_n$, then S restricted to $C_m^i \cup C_m^{i+1}$ is a 2-coloring $\mathcal{C}$ of $C_m \square P_2$. Since S is minimal, we have that $\mathcal{C}$ is nonredundant.

If D_m possesses a directed cycle of length n , we concatenate the second cycle C_m of each vertex of the cycle and then obtain a dominating set S of $C_m \square C_n$. Since each vertex in the cycle is nonredundant, we have that S is minimal. $\square$

We define a weighted adjacency matrix $A(D_m)$ of D_m as follows. Let $V(D_m) = \{v_1, v_2, \dots, v_K\}$. Then

$$A(D_m)_{i,j} = \begin{cases} w(v_i v_j), & v_i v_j \in D_m, \\ +\infty, & v_i v_j \notin D_m, \end{cases} \quad (7)$$

where $w(v_i v_j)$ is the weight of the edge $v_i v_j$.

Let $A = (a)_{p \times q}$ (p rows, q columns) and $B = (b)_{q \times r}$ (q rows, r columns) be two weighted adjacency matrices. We define A multiplied by B as follows

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,q} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{p,1} & a_{p,2} & \cdots & a_{p,q} \end{pmatrix} \times \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,r} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,r} \\ \vdots & \vdots & \vdots & \vdots \\ b_{q,1} & b_{q,2} & \cdots & b_{q,r} \end{pmatrix} \\ = \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,r} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,r} \\ \vdots & \vdots & \vdots & \vdots \\ c_{p,1} & c_{p,2} & \cdots & c_{p,r} \end{pmatrix}, \quad (8)$$

where $c_{s,t} = \min_{1 \leq i \leq q} \{a_{s,i} + b_{i,t}\}$ for $s \in \{1, 2, \dots, p\}$ and $t \in \{1, 2, \dots, r\}$.

We denote by A^n the n th power of A ; we have the following result.

Theorem 12. Let $m, n \geq 3$ and $A(D_m)$ be the weighted adjacency matrix of D_m with K vertices. Then one has

$$\gamma(C_m \square C_n) = \min \left\{ \left((A(D_m))^n \right)_{i,i} : 1 \leq i \leq K \right\}. \quad (9)$$

Lemma 13. Let $n \geq 10$, $\gamma(C_{10} \square C_n) \leq \lfloor 7n/2 \rfloor + 2$, and D a dominating set with cardinality $\lfloor 7n/2 \rfloor + 2$.

- (a) If n is odd, then $|D \cap (C_{10}^i \cup C_{10}^{i+1})| \leq 8$ for any $i \in V(C_n)$.
- (b) If n is even, then $|D \cap (C_{10}^i \cup C_{10}^{i+1})| \leq 9$ for any $i \in V(C_n)$ but at most one integer i_0 such that $|D \cap (C_{10}^{i_0} \cup C_{10}^{i_0+1})| = 9$.

Proof. (a) Suppose that there exists $i_0 \in [n-1]$ such that $|D \cap (C_{10}^{i_0} \cup C_{10}^{i_0+1})| \geq 9$. We can deduce that

$$\left| D \cap \bigcup_{\substack{j \in [n-1], \\ j \notin \{i_0, i_0+1\}}} V(C_{10}^j) \right| \leq \frac{7}{2}(n-2) - \frac{1}{2}. \quad (10)$$

Thus, by the pigeonhole principle it can be seen that there exist some $i \in [n-1]$ such that $|D \cap (C_{10}^i \cup C_{10}^{i+1})| < 7$, which contradicts with Lemma 4.

(b) The proof is similar to (a). $\square$

Theorem 14. Let $n \geq 10$. Then

$$\begin{aligned} \gamma(C_{10} \square C_n) &= \begin{cases} \left\lfloor \frac{7n}{2} \right\rfloor, & n \equiv 0 \pmod{20}, \\ \left\lfloor \frac{7n}{2} \right\rfloor + 1, & n \equiv 3, 6, 14, 17 \pmod{20}, \\ \left\lfloor \frac{7n}{2} \right\rfloor + 2, & n \equiv 4, 7, 8, 9, 10, 11, 12, 13, 16 \pmod{20}, \\ \left\lfloor \frac{7n}{2} \right\rfloor + 3, & n \equiv 1, 2, 5, 15, 18, 19 \pmod{20}. \end{cases} \end{aligned} \quad (11)$$

Proof. When $n \equiv 0, 3, 6, 14, 17 \pmod{20}$, the result follows from Lemmas 6, 8, and 9. Otherwise, let $T = \{(3, 4), (4, 3), (3, 5), (5, 3), (4, 4)\}$. We implemented the algorithm as described above, and the digraph D_{10}^T was created and it has 4905 vertices. By Lemma 13, it can be seen that there may exist at most one (x, y) -vertex such that $x + y = 9$ in the auxiliary graph D_{10} . We consider each possible auxiliary graph D_{10} , whose vertex set is the union of that of D_{10}^T and one (x, y) -vertex with $x + y = 9$. Note that D_{10} has only 4906 vertices; the matrix multiplication can be done easily. By using matrix multiplication, we found a formula as stated in Theorem 14. (Such parameters a and b exist; see [17].) If $\gamma(C_{10} \square C_n)$ (for some n) is greater than $\lfloor 7n/2 \rfloor + 2$, then $\gamma(C_{10} \square C_n) = \lfloor 7n/2 \rfloor + 3$. $\square$

Remark 15. Note that when Lemmas 4 and 13 are applied, we can operate matrix multiplication on the graph $D_{10}^{T_1}$ whose row number is 4906. It can be clearly seen that $|D \cap (C_{10}^i \cup C_{10}^{i+1})| \leq 9$ for any $i \in V(C_n)$ with the condition of Lemma 13, and it is natural to consider the graph $D_{10}^{T_2}$, where $T_2 = T \cup \{(3, 6), (4, 5), (5, 4), (6, 3)\}$. However, $D_{10}^{T_2}$ has 24845 vertices, whose vertex number is much more than that of D_{10} . Therefore, Lemmas 4 and 13 play the key role in computing the exact values of $\gamma(C_{10} \square C_n)$, and this helps to make many hard instances become easy to solve.

TABLE 1: Constructions for upper bounds of $C_{13} \square C_n$.

r	0	1	2	3	4	5	6	7	8
(c, ℓ)	(1, 0)	(7, 0)	(2, 0)	(4, 0)	$A^k B$	(5, 0)	(1, 1)	$A^k C$	(2, 1)
r	9	10	11	12	13	14	15	16	17
(c, ℓ)	(4, 1)	$A^k D$	(5, 1)	(1, 2)	(6, 1)	(2, 2)	(4, 2)	$A^k E$	(5, 2)
r	18	19	20	21	22	23	24	25	
(c, ℓ)	(1, 3)	$A^k F$	(2, 3)	(4, 3)	$A^k G$	(5, 3)	(1, 4)	(6, 3)	

5. Bounds on the Domination

Number of $C_{13} \square C_n$

We will conclude this work with upper bounds for the domination number of $C_{13} \square C_n$.

Theorem 16. *Let $n \geq 13$. Then*

$$\gamma(C_{13} \square C_n) \leq \begin{cases} \left\lfloor \frac{9n}{2} \right\rfloor, & n \equiv 0 \pmod{26}, \\ \left\lfloor \frac{9n}{2} \right\rfloor + 1, & n \equiv 3, 6, 20, 23 \pmod{26}, \\ \left\lfloor \frac{9n}{2} \right\rfloor + 2, & n \equiv 9, 10, 12, 14, 16, 17 \pmod{26}, \\ \left\lfloor \frac{9n}{2} \right\rfloor + 3, & n \equiv 4, 7, 8, 11, 13, 15, 18, 19, 22 \pmod{26}, \\ \left\lfloor \frac{9n}{2} \right\rfloor + 4, & n \equiv 1, 2, 5, 21, 24, 25 \pmod{26}. \end{cases} \quad (12)$$

Proof. We show the desired weight for all cases in Table 1, where $r \equiv n \pmod{26}$ and (c, ℓ) stands for case (c) with ℓ in Lemma 6. For example, the result of the case $n \equiv 0 \pmod{26}$ follows from case 1 with $\ell = 0$ in Lemma 6.

Also, patterns A–G with 13 rows are presented (see Figures 5 and 6). By repeating the pattern of A for k times and concatenating $H \in \{B, C, D, E, F, G\}$, we will obtain a pattern $A^k H$ of $C_{13} \square C_n$, where $n = 26k + c(H)$. From the above, it can be seen that all the desired upper bounds are established. $\square$

Disclosure

The authors confirm that they have given due consideration to the protection of intellectual property associated with this work and that there are no impediments to publication, including the timing of publication, with respect to intellectual property. In so doing, the authors confirm that they have followed the regulations of their institutions concerning intellectual property. The authors understand that the corresponding author is the sole contact for the editorial process (including editorial manager and direct communications with the office). He/she is responsible for communicating with the other authors about progress, submissions of revisions, and final approval of proofs. The authors confirm that they have provided current, correct

01001001001010010010010010	0010	0100010	0100100010
10010010010010100100100100	1100	1001100	1001001100
00100100100100101001001001	0001	0010001	0010010001
01001001001001001010010010	0110	0100110	0100100110
10010010010010010010100100	1000	1001000	1001001000
00100100100100100100101001	0011	0010011	0010010011
01001001001001001001001010	0100	0100100	0100100100
10010010010010010010010010	1001	1001001	1001001001
10100100100100100100100100	1010	1010010	1010010010
00101001001001001001001001	1001	0000101	0010100101
01001010010010010010010010	0010	0111010	0100101010
10010010100100100100100100	1100	1000100	1001000100
00100100101001001001001001	1001	0011001	0010011001
A		B	C
		D	

FIGURE 5: AB: dominating set of $C_{13} \square C_{30}$; AC: dominating set of $C_{13} \square C_{33}$; AD: dominating set of $C_{13} \square C_{36}$.

0100100010010010	0100100010010010	0100100100010010
1001001100100100	1001001100100100	1001001001100100
0010010001001001	0010010001001001	0010010010101001
0100100110010010	0100100110010010	0100100101001001
1001001010100100	1001001010100100	1001001010010100
0010010100101001	0010010100101001	0010010100100101
0100101001001010	0100101001001010	0100101001001010
1001010010010010	1001010010010010	1001010010010010
1010100100100100	1010100100100100	1010101001001001
0010001001001001	0010001001001001	0010010001001001
0100110010010010	0100110010010010	0100100110010010
1001000100100100	1001000100100100	1001001000100100
0010011001001001	0010011001001001	0010010011001001
E		F
		G

FIGURE 6: AE: dominating set of $C_{13} \square C_{42}$; AF: dominating set of $C_{13} \square C_{45}$; AG: dominating set of $C_{13} \square C_{48}$.

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Conflict of Interests

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