Research Article

Sign-Changing Solutions for a Fourth-Order Elliptic Equation with Hardy Singular Terms

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The existence and multiplicity of sign-changing solutions for a class of fourth elliptic equations with Hardy singular terms are established by using the minimax methods.

1. Introduction

Consider the following Navier boundary value problem:

$$\begin{aligned} \Delta^2 u(x) - \frac{N^2(N-4)^2}{16} \frac{u}{|x|^4} &= f(x, u), \quad \text{in } \Omega, \\ u = \Delta u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (1)$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ ($N \geq 5$), $0 \in \Omega$.

The conditions imposed on $f(x, t)$ are as follows:

(H₁) there exists $C > 0$ such that

$$|f(x, t)| \leq C(1 + |t|^p), \quad \forall t \in \mathbb{R}, \quad \forall x \in \Omega, \quad (2)$$

where $1 < p < (N+4)/(N-4)$;

(H₂) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $f(x, t)t \geq 0$ for all $x \in \Omega$, $t \in \mathbb{R}$;

(H₃) $\lim_{|t| \rightarrow 0} f(x, t)/t = f_0$, $\lim_{|t| \rightarrow \infty} f(x, t)/t = l$ uniformly for $x \in \Omega$, where f_0 and l are constants;

(H₄) $\lim_{|t| \rightarrow \infty} [f(x, t)t - 2F(x, t)] = -\infty$ uniformly for $x \in \Omega$, where $F(x, t) = \int_0^t f(x, s)ds$;

(H₅) there exist $\mu > 2$ and $R > 0$ such that

$$0 < \mu F(x, t) \leq f(x, t)t, \quad x \in \Omega, \quad |t| \geq R; \quad (3)$$

(H₆) $f(x, t)$ is odd in t .

In recent years, this fourth-order semilinear elliptic problem:

$$\begin{aligned} \Delta^2 u(x) + c \Delta u &= f(x, u), \quad \text{in } \Omega, \\ u = \Delta u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (4)$$

can be considered as an analogue of a class of second-order problems which have been studied by many authors. In [1], there was a survey of results obtained in this direction. In [2], Micheletti and Pistoia showed that (4) admits at least two solutions by a variation of linking if $f(x, u)$ is sublinear. And in [3], the authors proved that the problem (4) has at least three solutions by a variational reduction method and a degree argument. In [4], Zhang and Li showed that (4) admits at least two nontrivial solutions by Morse theory and local linking if $f(x, u)$ is superlinear and subcritical on u .

To the authors' knowledge, there seem few results about the sign-changing solutions on problem (1) with hardy singular terms. In this paper, motivated by [5–8], the existence and multiplicity of sign-changing solutions for problem (1) are obtained by introducing a compact embedding theorem and a maximum principle. Our results are new.

2. Preliminaries and Auxiliary Lemmas

We introduce the new working space E which is obtained by the completion of $C_0^\infty(\Omega)$ with respect to the norm (see [5])

$$\|u\| = \left(\int_{\Omega} \left(|\Delta u|^2 - \frac{N^2(N-4)^2}{16} \frac{|u|^2}{|x|^4} \right) dx \right)^{1/2} \quad (5)$$

associated with the inner product

$$\langle u, v \rangle = \int_{\Omega} \left(\Delta u \Delta v - \frac{N^2(N-4)^2}{16} \frac{uv}{|x|^4} \right) dx. \quad (6)$$

Throughout this paper, we denoted by $\|\cdot\|_p$ the $L^p(\Omega)$ norm.

At first, we here give two important lemmas.

Lemma 1. $E \hookrightarrow L^2(\Omega)$ (see [5]).

Lemma 2 (see [6, Corollary 4.1]). Assume $N \geq 5, V \in L^\infty(\Omega)$, and $V \geq 0$. Let us suppose that the operator $\Delta^2 - (V/|x|^4)$ is coercive on $H^2(\Omega) \cap H_0^1(\Omega)$. Let $f \in L^2(\Omega)$ such that $f \geq 0$. Let $u \in H^2(\Omega)$ be a solution of

$$\begin{aligned} \Delta^2 u(x) - \frac{V}{|x|^4} u &= f, \text{ in } \Omega, \\ u &= \Delta u = 0 \text{ on } \partial\Omega. \end{aligned} \quad (7)$$

Then $u \geq 0$ in Ω .

Now, we consider the following eigenvalue problem:

$$\Delta^2 u(x) - \frac{N^2(N-4)^2}{16|x|^4} u = \lambda u, \text{ in } \Omega, \quad (8)$$

$$u = \Delta u = 0 \text{ on } \partial\Omega. \quad (9)$$

The first eigenvalue of this problem is given by

$$\lambda_1 = \inf \left\{ \|u\|^2 : u \in E, \|u\|_2 = 1 \right\}. \quad (10)$$

By Lemma 1, $E \hookrightarrow W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ for $p \rightarrow 2^-$. The minimizing sequence is compact in $L^2(\Omega)$. By standard argument, we may assume that the first eigenfunction ϕ_1 is positive in Ω (see [9, page 167]). The second eigenvalue is given by

$$\lambda_2 = \inf \left\{ \|u\|^2 : u \in E, \int_{\Omega} u \phi_1 = 0, \|u\|_2 = 1 \right\} \quad (11)$$

which possesses a sign-changing eigenfunction ϕ_2 . Similarly, we can characterize the n th eigenvalue λ_n with a sign-changing eigenfunction. By standard elliptic theory, $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$.

It follows from (H_1) that the functional

$$\begin{aligned} I(u) &= \frac{1}{2} \int_{\Omega} |\Delta u|^2 dx - \frac{N^2(N-4)^2}{32} \\ &\quad \times \int_{\Omega} \frac{u^2}{|x|^4} - \int_{\Omega} F(x, u) dx \end{aligned} \quad (12)$$

is of C^1 on the space E . Under the condition (H_1) , the critical points of I are solutions of problem (1).

If l in the above condition (H_3) is an eigenvalue of $(\Delta^2 - (N^2(N-4)^2/16)(1/|x|^4), E)$, then the problem (1) is called resonance at infinity. Otherwise, we call it nonresonance.

For looking for sign-changing solutions of problem (1), we recall a very useful result.

Proposition 3 (see [10, Theorem 3.2]). Let X be a Hilbert space and f be a C^1 functional defined on X . Assume that f satisfies the (PS) condition on X and $f'(u)$ has the expression $f'(u) = u - Au$ for $u \in X$. Assume that D_1 and D_2 are open convex subset of X with the properties that $D_1 \cap D_2 \neq \emptyset$, $A(\partial D_1) \subset D_1$, and $A(\partial D_2) \subset D_2$. If there exists a path $h : [0, 1] \rightarrow X$ such that

$$\begin{aligned} h(0) &\in D_1 \setminus D_2, \quad h(1) \in D_2 \setminus D_1, \\ \inf_{u \in \overline{D_1 \cap D_2}} f(u) &> \sup_{t \in [0,1]} f(h(t)), \end{aligned} \quad (13)$$

then f has at least four critical points, one in $D_1 \cap D_2$, one in $D_1 \setminus \overline{D_2}$, one in $D_2 \setminus \overline{D_1}$, and one in $X \setminus (\overline{D_1} \cup \overline{D_2})$.

Remark 4. If f satisfies the $(C)_c$ condition, then this proposition still holds (see [11]).

3. Main Results

Let us now state the main results.

Theorem 5. Assume conditions (H_2) and (H_3) hold. If $f_0 < \lambda_1$ and $l \in (\lambda_k, \lambda_{k+1})$ for some $k > 2$, then problem (1) has a positive solution, a negative solution, and a sign-changing solution.

Remark 6. This result is similar to [7, Theorem 1.1]. As far as verifying the (PS) condition is concerned, our method is more simple than that in [12, 13].

Theorem 7. Assume conditions (H_2) – (H_4) hold. If $f_0 < \lambda_1$ and $l = \lambda_k$ for some $k > 2$, then problem (1) has a positive solution, a negative solution, and a sign-changing solution.

Remark 8. When $l = \lambda_k$ ($k > 2$), the case is called resonance and not considered by [7]. This result is completely new.

Theorem 9. Assume conditions (H_1) , (H_5) , and (H_6) hold. If $f_0 = 0$, then problem (1) has infinitely many sign-changing solutions.

Lemma 10. Under the assumptions of Theorem 5, if $\lambda_k < l < \lambda_{k+1}$, then I satisfies the (PS) condition.

Proof. Let $\{u_n\} \subset E$ be a sequence such that $|I(u_n)| \leq c$, $I'(u_n)$, and $\phi \rightarrow 0$. Since

$$\begin{aligned} \langle I'(u_n), \phi \rangle &= \int_{\Omega} \left(\Delta u_n \Delta \phi - \frac{N^2(N-4)^2}{16} \frac{u_n \phi}{|x|^4} \right) dx \\ &\quad - \int_{\Omega} f(x, u_n) \phi dx = o(\|\phi\|) \end{aligned} \quad (14)$$

for all $\phi \in E$. If $\|u_n\|_2$ is bounded, we can take $\phi = u_n$. By (H_3) , there exists a constant $c > 0$ such that $|f(x, u_n(x))| \leq c|u_n(x)|$, a.e. $x \in \Omega$. So u_n is bounded in E . If $\|u_n\|_2 \rightarrow +\infty$, as $n \rightarrow \infty$, set $v_n = u_n/\|u_n\|_2$, then $\|v_n\|_2 = 1$. Taking $\phi = v_n$ in (14),

it follows that $\|v_n\|$ is bounded. Without loss of generality, we assume $v_n \rightarrow v$ in E , and then $v_n \rightarrow v$ in $L^2(\Omega)$. Hence, $v_n \rightarrow v$ a.e. in Ω and $|v_n| \leq q(x)$ ($q(x) \in L^2(\Omega)$). Dividing both sides of (14) by $\|u_n\|_2$, for all $\phi \in E$, we get

$$\begin{aligned} & \int_{\Omega} \left(\Delta v_n \Delta \phi - \frac{N^2(N-4)^2}{16} \frac{v_n \phi}{|x|^4} \right) dx \\ & - \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|_2} \phi dx = o \left(\frac{\|\phi\|}{\|u_n\|_2} \right). \end{aligned} \quad (15)$$

Then for a.e. $x \in \Omega$, we have $f(x, u_n)/\|u_n\|_2 \rightarrow lv$ as $n \rightarrow \infty$. In fact, if $v(x) \neq 0$, by (H_3) , we have

$$\begin{aligned} |u_n(x)| &= |v_n(x)| \|u_n\|_2 \rightarrow +\infty, \\ \frac{f(x, u_n)}{\|u_n\|_2} &= \frac{f(x, u_n)}{u_n} v_n \rightarrow lv. \end{aligned} \quad (16)$$

If $v(x) = 0$, we have

$$\frac{|f(x, u_n)|}{\|u_n\|_2} \leq c |v_n| \rightarrow 0. \quad (17)$$

Since $|f(x, u_n)|/\|u_n\|_2 \leq c|v_n| \leq cq(x)$, by (15) and the Lebesgue dominated convergence theorem, we arrive at

$$\int_{\Omega} \Delta v \Delta \phi dx - \frac{N^2(N-4)^2}{16} \frac{v\phi}{|x|^4} - \int_{\Omega} lv\phi dx = 0, \quad \forall \phi \in E. \quad (18)$$

It is easy to see that $v \neq 0$. In fact, if $v \equiv 0$, then $\|v\|_2 = 0$ contradicts $\lim_{n \rightarrow \infty} \|v_n\|_2 = \|v\|_2 = 1$. Hence, l is an eigenvalue of $(\Delta^2 - (N^2(N-4)^2/16)(1/|x|^4), E)$. This contradicts our assumption. Thus $\{u_n\}$ is bounded. By standard argument (see the proof of our Lemma 12 below), $\{u_n\} \rightarrow u$ in E . The lemma is proved. $\square$

Lemma 11. Under the assumptions of Theorem 7, if $l = \lambda_k$, then the functional I satisfies the (C) condition which is stated in [11].

Proof. Suppose I satisfies

$$I(u_n) \rightarrow c \in \mathbb{R}, \quad (1 + \|u_n\|) \|I'(u_n)\| \rightarrow 0 \quad (19)$$

as $n \rightarrow \infty$.

In view of (H_3) , it suffices to prove that u_n is bounded in E . Similar to the proof of Lemma 10, we have

$$\begin{aligned} & \int_{\Omega} \left(\Delta v \Delta \phi - \frac{N^2(N-4)^2}{16} \frac{v\phi}{|x|^4} \right) dx \\ & - \int_{\Omega} lv\phi dx = 0, \quad \forall \phi \in E. \end{aligned} \quad (20)$$

Therefore $v \neq 0$ is an eigenfunction of λ_k , and then $|u_n(x)| \rightarrow \infty$ for a.e. $x \in \Omega$. It follows from (H_4) that

$$\lim_{n \rightarrow +\infty} [f(x, u_n(x)) u_n(x) - 2F(x, u_n(x))] = -\infty \quad (21)$$

holds uniformly in $x \in \Omega$, which implies that

$$\int_{\Omega} (f(x, u_n) u_n - 2F(x, u_n)) dx \rightarrow -\infty \text{ as } n \rightarrow \infty. \quad (22)$$

On the other hand, (19) implies that

$$2I(u_n) - \langle I'(u_n), u_n \rangle \rightarrow 2c \text{ as } n \rightarrow \infty. \quad (23)$$

Thus

$$\int_{\Omega} (f(x, u_n) u_n - 2F(x, u_n)) dx \rightarrow 2c \text{ as } n \rightarrow \infty, \quad (24)$$

which contradicts (22). Hence u_n is bounded. $\square$

Lemma 12. Assume (H_1) and (H_5) hold. Then I satisfies the (PS) condition.

Proof. Assume that $\{u_n\}$ is a (PS) sequence; $\|I'(u_n)\| \rightarrow 0$ and $\{I(u_n)\}$ is bounded. A routine argument implies that $\{\|u_n\|\}$ is bounded. By [6, Theorem A.2], we have

$$E \hookrightarrow W_0^{1,q}(\Omega), \quad (25)$$

where $1 \leq q < 2$. For p given in (H_1) , $p < (n+4)/(n-4)$, and we may choose q such that $(p+1) < qN/(N-q)$, $q < 2$. By the Sobolev embedding theorem, we have

$$W^{1,q}(\Omega) \hookrightarrow L^t(\Omega), \quad \forall t < \frac{Nq}{N-q}. \quad (26)$$

We infer from (26) that $\{u_n\}$ is compact in $L^{p+1}(\Omega)$. By (H_1) ,

$$\begin{aligned} & \|u_n - u_m\|^2 \\ &= \int_{\Omega} |f(x, u_n) - f(x, u_m)| |u_n - u_m| dx + o(1) \\ &\leq C \left(\int_{\Omega} |u_n - u_m|^{p+1} dx \right)^{1/(p+1)} + o(1) \rightarrow 0. \end{aligned} \quad (27)$$

This completes the proof of this lemma. $\square$

For the aim of using Proposition 3 that proves our main results, we prove an important lemma below.

From previous Section 1, we know that I is C^1 functional and its gradient at u is given by

$$\begin{aligned} I'(u) &= u - A(u), \quad A: E \rightarrow E, \\ A(u) &= \left(\Delta^2 - \frac{N^2(N-4)^2}{16} \frac{1}{|x|^4} \right)^{-1} f(x, u). \end{aligned} \quad (28)$$

Then $\langle A(u), \phi \rangle = \int_{\Omega} f(x, u) \phi dx$ for all $\phi \in E$. We consider the convex cones $P = \{u \in H : u \geq 0\}$ and $-P = \{u \in E : u \leq 0\}$; moreover, for $\epsilon > 0$, assume

$$\begin{aligned} P_{\epsilon} &= \{u \in E : \text{dist}(u, P) < \epsilon\}, \\ -P_{\epsilon} &= \{u \in E : \text{dist}(u, -P) < \epsilon\}. \end{aligned} \quad (29)$$

Note that P_{ϵ} and $-P_{\epsilon}$ are open convex subsets of E and $E \setminus (\overline{P_{\epsilon}} \cup \overline{-P_{\epsilon}})$ contains only sign-changing functions.

Lemma 13. Assume (H_2) and (H_3) hold. Then, there exists $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$ there holds

$$A(\partial(\pm P_\epsilon)) \subset \pm P_\epsilon. \quad (30)$$

Moreover, if $u \in \pm P_\epsilon$ is a nontrivial solution of problem (1), then u is positive (negative) in the sense that $u > 0$ ($u < 0$) in Ω .

Proof. Indeed, if $u \in E$ and $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$, then

$$\begin{aligned} \text{dist}(A(u), P) &\leq \inf_{w \in P} \|A(u) - w\| \\ &= \inf_{w \in P} \|A(u)^+ + A(u)^- - w\| \leq \|A(u)^-\|. \end{aligned} \quad (31)$$

For every $s \in (2, 2N/(N-4))$, there exists $C_s > 0$ such that

$$\|u^\pm\|_s \leq \inf_{w \in \mp P} \|u - w\|_s \leq C_s \text{dist}(u, \mp P). \quad (32)$$

Choose $\epsilon' > 0$ such that $(f_0 + \epsilon') < \lambda_1$. Using (32), the Hölder inequality, the Poincaré inequality, and the Sobolev embedding theorem, we have

$$\begin{aligned} \text{dist}(A(u), P) \|A(u)^-\| &\leq \|A(u)^-\|^2 \\ &= \int_{\Omega} f(x, u) A(u)^- dx \\ &\leq \int_{\Omega} f(x, u^-) A(u)^- dx \\ &\leq \int_{\Omega} ((f_0 + \epsilon') |u^-| + C'_\epsilon |u^-|^p) A(u)^- dx \\ &\leq (f_0 + \epsilon') \|u^-\|_2 \|A(u)^-\|_2 \\ &\quad + C'_\epsilon \|u^-\|_{p+1}^p \|A(u)^-\|_{p+1} \\ &\leq (f_0 + \epsilon') \inf_{w \in P} \|u - w\|_2 \|A(u)^-\|_2 \\ &\quad + C \inf_{w \in P} \|u - w\|_{p+1}^p \|A(u)^-\|_{p+1} \\ &\leq \frac{f_0 + \epsilon'}{\lambda_1} \text{dist}(u, P) \|A(u)^-\| \\ &\quad + C \text{dist}(u, P)^p \|A(u)^-\|, \end{aligned} \quad (33)$$

where $C'_\epsilon, C > 0$ are constants. Hence

$$\text{dist}(A(u), P) \leq (\delta + C \text{dist}(u, P)^{p-1}) \text{dist}(u, P), \quad (34)$$

where $\delta = (f_0 + \epsilon')/\lambda_1 < 1$. Take ϵ_0 such that $\delta_1 = \delta + C\epsilon_0^{p-1} < 1$. Now if $\text{dist}(u, P) < \epsilon < \epsilon_0$, then we have

$$\text{dist}(A(u), P) \leq \delta_1 \text{dist}(u, P). \quad (35)$$

Thus for every $u \in \partial P_\epsilon$, by (35) we have

$$\text{dist}(A(u), P) \leq \delta_1 \epsilon; \quad (36)$$

thus $A(u) \in P_\epsilon$. Hence $A(\partial P_\epsilon) \subset P_\epsilon$. In a similar way, $A(\partial(-P_\epsilon)) \subset (-P_\epsilon)$. If $0 < \epsilon \leq \epsilon_0$, and $u \in P_\epsilon$ (resp., $-P_\epsilon$) is a nontrivial solution of problem (1), then $I'(u) = 0$. By (35) we have $\text{dist}(u, P) = 0$; that is, $u \in P$ (resp., $u \in -P$). By Lemma 2, we imply that $u > 0$ ($u < 0$) in Ω . $\square$

Lemma 14. Assume (H_1) , (H_2) , and (H_5) hold. Then, there exists $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$ there holds

$$A(\partial(\pm P_\epsilon)) \subset \pm P_\epsilon. \quad (37)$$

Proof. The proof is quite similar to that of Lemma 4.2 in [8]. We omit it here. $\square$

Lemma 15. Assume (H_5) holds. Then

$$I(u) \longrightarrow -\infty, \quad \forall u \in E_k, \quad (38)$$

where the definition of E_k introduced in our proof of Theorem 9.

Proof. Because $\dim E_k < \infty$, then by (H_5) ,

$$\frac{I(u)}{\|u\|^2} \leq \frac{1}{2} - \int_{\Omega} \frac{F(x, u)}{\|u\|^2} dx \longrightarrow -\infty \quad (39)$$

as $\|u\| \rightarrow \infty$, $u \in E_k$. This lemma follows immediately. $\square$

Lemma 16. Assume (H_2) and (H_3) hold. Let $0 < \epsilon \leq \epsilon_0$, and then there exists $C_0 > -\infty$ such that $\inf_{\overline{P_\epsilon} \cap (-\overline{P_\epsilon})} I(u) = C_0$.

Proof. By the conditions (H_2) and (H_3) , we know that, for any $\epsilon' > 0$, there exists $C > 0$, such that

$$|f(x, t)| \leq (f_0 + \epsilon') |t| + C |t|^p \quad \left(1 < p < \frac{N+4}{N-4}\right). \quad (40)$$

Using (40) and the Sobolev embedding theorem, we have

$$\begin{aligned} I(u) &= \frac{1}{2} \|u\|^2 - \int_{\Omega} F(x, u) dx \\ &\geq \frac{1}{2} \|u\|^2 - \frac{1}{2} (f_0 + \epsilon') \int_{\Omega} u^2 dx - C \|u\|_{p+1}^{p+1} \\ &\geq -\frac{1}{2} (f_0 + \epsilon') \|u\|_2^2 - C \|u\|_{p+1}^{p+1}. \end{aligned} \quad (41)$$

By (32) we have $\|u^\pm\|_s \leq C_s \epsilon_0$ for every $u \in P_\epsilon \cap (-P_\epsilon)$. So there exists $C_0 > -\infty$ such that

$$\inf_{\overline{P_\epsilon} \cap (-\overline{P_\epsilon})} I(u) = C_0. \quad (42)$$

Hence this lemma is proved. $\square$

4. Proof of the Main Results

Proof of Theorem 5 and Theorem 7. Motivated by the Proof of Theorem 4.2 in [10], we still define a path $h_R : [0, 1] \rightarrow E$ as

$$h_R(t) = R\phi_1 \cos \pi t + R\phi_2 \sin \pi t, \quad 0 \leq t \leq 1. \quad (43)$$

Obviously, $h_R(0) \in P_\epsilon \setminus (-P_\epsilon)$ and $h_R(1) \in (-P_\epsilon) \setminus P_\epsilon$. By the Fatou's lemma, the condition (H_3) with $l > \lambda_2$ and a direct computation shows that

$$\lim_{R \rightarrow +\infty} \sup_{t \in [0,1]} I(h_R(t)) = -\infty. \quad (44)$$

So, it yields that there exists R_0 such that $I(h_{R_0}(t)) < C_0 - c^*$ ($c^* > 0$). Hence we obtain

$$\inf_{\overline{P_\epsilon} \cap (-\overline{P_\epsilon})} I(u) > \sup_{t \in [0,1]} I(h(t)). \quad (45)$$

By using Lemmas 10, 11, and 13, Proposition 3, and Lemma 16, we can find a critical point in $P_\epsilon \setminus (-\overline{P_\epsilon})$ which is a positive solution, a critical point in $(-\overline{P_\epsilon}) \setminus P_\epsilon$ which is a negative solution, and a critical point in $E \setminus (\overline{P_\epsilon} \cup (-\overline{P_\epsilon}))$ which is a sign-changing solution. $\square$

Before beginning our proof of Theorem 9, we need the following important proposition.

Proposition 17 (see [9, Theorem 5.6]). *Assume E is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|\cdot\|$, $I \in C^1(E, \mathbb{R})$ and $I(u) = (1/2)\|u\|^2 - G(u)$, $u \in E$, where $G \in C^1(E, \mathbb{R})$. P denotes a positive closed convex cone of E .*

(A) *Assume that $A(\pm D_0) \subset \pm D_0$, where $D_0 := \{u \in E : \text{dist}(u, P) < \mu_0\}$, $\mu_0 > 0$, and $A = G'$.*

(A₁^{*}) *Assume that, for any $a, b > 0$, there is a constant $C > 0$ such that*

$$G(u) \leq a, \quad \|u\|_* \leq b \implies \|u\| \leq C, \quad (46)$$

where $\|\cdot\|_*$ denotes another norm of E such that $\|u\|_* \leq C\|u\|$ for all $u \in E$.

(A₂^{*}) *Assume that $\lim_{u \in Y, \|u\| \rightarrow \infty} I(u) = -\infty$, $\sup_Y I := \beta$.*

If the even functional I satisfies (PS) condition at level c for each $c \in [\gamma, \beta]$, then

$$\mathcal{K}[\gamma - \epsilon, \beta + \epsilon] \cap (E \setminus (-P \cup P)) \neq \emptyset \quad (47)$$

for all $\epsilon > 0$ small, where $(\sup_Y I := \beta$ (Y and M are two subspaces of E with $\dim Y < \infty$, $\dim Y - \text{codim} M \geq 1$), $\inf_{Q^{**}} I := \gamma$, and $Q^{**} := Q^*(\rho) \cap I^\beta$ ($Q^*(\rho) := \{u \in M : (\|u\|_*^p / \|u\|^2) + ((\|u\| \|u\|_*) / (\|u\| + D_* \|u\|_*)) = \rho\}$, where $\rho > 0$, $D_* > 0$, and $p > 2$ are fixed constants).

Now, we give an outline proof for our Theorem 9.

Proof of Theorem 9. Let N_k denote the eigenspace of λ_k . We fix k and let $E_k := N_1 \oplus \cdots \oplus N_k$. Consider another norm

$\|\cdot\|_* := \|\cdot\|_{p+1}$ of E , $p \in (1, ((N+4)/(N-4)))$. Write $E = E_{k-1} \oplus E_{k-1}^\perp$.
Let

$$Q^*(\rho) := \left\{ u \in E_{k-1}^\perp : \frac{\|u\|_{p+1}^{p+1}}{\|u\|^2} + \frac{\|u\| \|u\|_{p+1}}{\|u\| + D_* \|u\|_{p+1}} = \rho \right\}, \quad (48)$$

where ρ, D_* are fixed constants. By our assumptions, we may find a constant $C > 0$ such that

$$F(x, t) \leq \frac{1}{4} \lambda_1 t^2 + C|t|^{p+1}, \quad \forall x \in \Omega, \quad t \in \mathbb{R}, \quad (49)$$

where $1 < p < (N+4)/(N-4)$. For any $a, b > 0$, there is a constant $C > 0$ such that

$$I(u) \leq a, \quad \|u\|_{p+1} \leq b \implies \|u\| \leq C. \quad (50)$$

By Lemma 15,

$$\lim_{u \in Y, \|u\| \rightarrow \infty} I(u) = -\infty, \quad (51)$$

where $Y = E_k$. Then (A_1^*) and (A_2^*) are satisfied. By Lemma 14, the condition (A) holds.

Now, we define

$$\sup_Y I := \beta. \quad (52)$$

Let

$$Q^{**} := Q^*(\rho) \cap I^\beta, \quad \inf_{Q^{**}} I := \gamma. \quad (53)$$

By Lemma 12, I satisfies the (PS) condition. Thus, by Proposition 17 and the Proof of Theorem 5.7 in [9], we know that the functional I possesses a sequence sign-changing solution $\{u_k\}$. $\square$

Conflict of Interests

The authors declare that they have no competing interests.

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