Research Article

Complexity of Products of Some Complete and Complete Bipartite Graphs

S. N. Daoud^{1,2}

¹ Department of Mathematics, Faculty of Science, Taibah University, Al Madinah 344, Saudi Arabia

² Department of Mathematics, Faculty of Science, El-Menoufia University, Shebeen El-Kom, Egypt

Correspondence should be addressed to S. N. Daoud; salamadaoud@gmail.com

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The number of spanning trees in graphs (networks) is an important invariant; it is also an important measure of reliability of a network. In this paper, we derive simple formulas of the complexity, number of spanning trees, of products of some complete and complete bipartite graphs such as cartesian product, normal product, composition product, tensor product, and symmetric product, using linear algebra and matrix analysis techniques.

1. Introduction

In this work we deal with simple and finite undirected graphs $G = (V, E)$, where V is the vertex set and E is the edge set. For a graph G , a spanning tree in G is a tree which has the same vertex set as G . The number of spanning trees in G , also called the complexity of the graph, denoted by $\tau(G)$, is a well-studied quantity (for long time). A classical result of Kirchhoff [1], can be used to determine the number of spanning trees for $G = (V, E)$. Let $V = \{v_1, v_2, \dots, v_n\}$; then the Kirchhoff matrix H defined as $n \times n$, characteristic matrix, $H = D - A$, where D is the diagonal matrix whose elements are the degrees of the vertices of G . While A is the adjacency matrix of G , $H = [a_{ij}]$ is defined as follows:

- (i) $a_{ij} = -1$ if v_i and v_j are adjacent and $i \neq j$,
- (ii) a_{ij} equals the degree of vertex v_i if $i = j$,
- (iii) $a_{ij} = 0$ otherwise.

All of the cofactors of H are equal to $\tau(G)$. There are other methods for calculating $\tau(G)$. Let $\mu_1 \geq \mu_2 \geq \dots \geq \mu_p$ denote the eigenvalues of H matrix of a p point graph. Then it is easily shown that $\mu_p = 0$. Furthermore, Kelmans and Chelnokov [2] have shown that $\tau(G) = (1/p) \prod_{k=1}^{p-1} \mu_k$. The formula for the number of spanning trees in a d -regular

graph G can be expressed as $\tau(G) = (1/p) \prod_{k=1}^{p-1} (d - \lambda_k)$, where $\lambda_0 = d, \lambda_1, \lambda_2, \dots, \lambda_{p-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. However, for a few special families of graphs there exist simple formulas that make it much easier to calculate and determine the number of corresponding spanning trees especially when these numbers are very large. One of the first results is due to Cayley [3] who showed that the complete graph on n vertices, K_n has n^{n-2} spanning trees, $n \geq 2$. Another result is that $\tau(K_{p,q}) = p^{q-1} q^{p-1}$, $p, q \geq 1$, where $K_{p,q}$ is the complete bipartite graph with bipartite sets containing p and q vertices, respectively. It is well known, as in, for example, [4, 5]. Another result is due to Sedláček [6] who derived a formula for the wheel on $n+1$ vertices, W_{n+1} ; he showed that $\tau(W_{n+1}) = ((3 + \sqrt{5})/2)^n + ((3 - \sqrt{5})/2)^n - 2$, for $n \geq 3$. Sedlacek [7] also later derived a formula for the number of spanning trees in a Mobius ladder, M_n , $\tau(M_n) = (n/2)[(2 + \sqrt{3})^n + (2 - \sqrt{3})^n + 2]$ for $n \geq 2$. Another class of graphs by Boesch et al., for which an explicit formula has been derived, is based on a prism [8, 9].

Now, we can introduce the following lemmas.

Lemma 1 (see [10]). Consider $\tau(G) = (1/n^2) \det(nI - \overline{D} + \overline{A})$ where $\overline{A}$ and $\overline{D}$ are the adjacency and degree matrices of $\overline{G}$ and the complement of G , respectively, and I is the $n \times n$ unit matrix.

Lemma 2. Let $E_n(x)$ be $n \times n$ matrix, $x \geq 2$ such that

$$E_n(x) = \begin{pmatrix} x & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & x \end{pmatrix}. \quad (1)$$

Then,

$$\det(E_n) = (x + n - 1)(x - 1)^{n-1}. \quad (2)$$

Proof. From the definition of the circulant determinants, we have

$$\begin{aligned} \det(E_n(x)) &= \det \begin{pmatrix} x & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & x \end{pmatrix} \\ &= \prod_{j=1}^n (x + \omega_j + \omega_j^2 + \omega_j^3 + \cdots + \omega_j^{n-1}) \\ &= (x + 1 + 1 + \cdots + 1) \\ &\quad \times \prod_{j=1, \omega_j \neq 1}^n \left(x + \underbrace{\omega_j + \omega_j^2 + \omega_j^3 + \cdots + \omega_j^{n-1}}_{=-1} \right) \\ &= (x + n - 1) \times (x - 1)^{n-1}. \end{aligned} \quad (3)$$

□

We can generalize the previous lemma as follows.

Lemma 3. Let $A, B \in F^{n \times n}$ and $F \in F^{kn \times kn}$ such that

$$F = \begin{pmatrix} A & B & \cdots & \cdots & \cdots & B \\ B & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & B \\ B & \cdots & \cdots & \cdots & B & A \end{pmatrix}. \quad (4)$$

Then,

$$\det F = [\det(A - B)]^{k-1} \det[A + (k-1)B]. \quad (5)$$

Lemma 4 (see [11]). Let $A \in F^{n \times n}$, let $B \in F^{n \times m}$, let $C \in F^{m \times n}$, and let $D \in F^{m \times m}$; assume that A, D are nonsingular matrices. Then

$$\begin{aligned} \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= (-1)^{nm} \det(A - BD^{-1}C) \det D \\ &= (-1)^{nm} \det A \det(D - CA^{-1}B). \end{aligned} \quad (6)$$

Formulas in Lemmas 2, 3, and 4 give some sort of symmetry in some matrices which facilitates our calculation of determinants.

2. Number of Spanning Trees of Cartesian Product of Graphs

The Cartesian product, $G_1 \times G_2$, is the simple graph with vertex set $V(G_1 \times G_2) = V_1 \times V_2$ and edge set $E(G_1 \times G_2) = [(E_1 \times V_2) \cup (V_1 \times E_2)]$ such that two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \times G_2$ if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$ [12].

Theorem 5. For $n, m \geq 1$, we have

$$\begin{aligned} \tau(K_2 \times K_{m,n}) &= m^{n-1} n^{m-1} (m+2)^{n-1} \\ &\quad \times (n+2)^{m-1} (n+m+2). \end{aligned} \quad (7)$$

Proof. Applying Lemma 1, we have

$$\begin{aligned} \tau(K_2 \times K_{m,n}) &= \frac{1}{(2(m+n))^2} \det(2(m+n)I - \overline{D} + \overline{A}) \\ &= \frac{1}{4(m+n)^2} \end{aligned}$$

$$\begin{aligned}
& \times \det \begin{pmatrix} n+2 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & n+2 & 0 & \cdots & \cdots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & m+2 & 1 & \cdots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & m+2 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & n+2 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 & n+2 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & 0 & m+2 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & m+2 \end{pmatrix} \\
& = \frac{1}{4(m+n)^2} \det \begin{pmatrix} n+2 & 2 & \cdots & 2 & 1 & \cdots & \cdots & 1 \\ 2 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \vdots \\ 2 & \cdots & 2 & n+2 & 1 & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & 1 & m+2 & 2 & \cdots & 2 \\ \vdots & \ddots & \ddots & \vdots & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 2 \\ 1 & \cdots & \cdots & 1 & 2 & \cdots & 2 & m+2 \end{pmatrix} \\
& \times \det \begin{pmatrix} n+2 & 0 & \cdots & 0 & -1 & \cdots & \cdots & -1 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & n+2 & -1 & \cdots & \cdots & -1 \\ -1 & \cdots & \cdots & -1 & m+2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ -1 & \cdots & \cdots & -1 & 0 & \cdots & 0 & m+2 \end{pmatrix} \\
& = \frac{1}{4(m+n)^2} \det \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \times \det \begin{pmatrix} D & E \\ E^T & F \end{pmatrix} \\
& = \frac{1}{4(m+n)^2} \times \det A \det (C - B^T A^{-1} B) \times \det D \det (F - E^T D^{-1} E) \\
& = \frac{1}{4(m+n)^2} \det \begin{pmatrix} n+2 & 2 & \cdots & 2 \\ 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 \\ 2 & \cdots & 2 & n+2 \end{pmatrix}_{m \times m}
\end{aligned}$$

$$\begin{aligned}
& \times \det \left(\begin{array}{cccc} \frac{n(m+2)+m(2m+3)}{n+2m} & \frac{2n+3m}{n+2m} & \cdots & \frac{2n+3m}{n+2m} \\ \frac{2n+3m}{n+2m} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{2n+3m}{n+2m} \\ \frac{2n+3m}{n+2m} & \cdots & \frac{2n+3m}{n+2m} & \frac{n(m+2)+m(2m+3)}{n+2m} \end{array} \right)_{n \times n} \\
& \times \det \left(\begin{array}{cccc} n+2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n+2 \end{array} \right)_{m \times m} \\
& \times \det \left(\begin{array}{cccc} \frac{n(m+2)+(m+4)}{n+2} & \frac{-m}{n+2} & \cdots & \frac{-m}{n+2} \\ \frac{-m}{n+2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{-m}{n+2} \\ \frac{-m}{n+2} & \cdots & \frac{-m}{n+2} & \frac{n(m+2)+(m+4)}{n+2} \end{array} \right)_{n \times n} \\
& = \frac{1}{4(m+n)^2} \times 2^m \det \left(\begin{array}{cccc} \frac{n+2}{2} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{n+2}{2} \end{array} \right)_{m \times m} \\
& \times \left(\frac{2n+3m}{n+2m} \right)^n \det \left(\begin{array}{cccc} \frac{n(m+2)+m(2m+3)}{2n+3m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{n(m+2)+m(2m+3)}{2n+3m} \end{array} \right)_{n \times n} \\
& \times \det \left(\begin{array}{cccc} n+2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n+2 \end{array} \right)_{m \times m} \\
& \times \left(\frac{-m}{n+2} \right)^n \det \left(\begin{array}{cccc} \frac{n(m+2)+(m+4)}{-m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{n(m+2)+(m+4)}{-m} \end{array} \right)_{n \times n} \\
& = \frac{1}{4(m+n)^2} \times 2^m \times \left(\frac{n+2}{2} + m - 1 \right) \left(\frac{n+2}{2} - 1 \right)^{m-1} \\
& \times \left(\frac{2n+3m}{n+2m} \right)^n \times \left(\frac{n(m+2)+m(2m+3)}{2n+3m} + n - 1 \right)
\end{aligned}$$

$$\begin{aligned}
& \times \left(\frac{n(m+2) + m(2m+3)}{2n+3m} - 1 \right)^{n-1} \times (n+2)^m \times \left(-\frac{m}{n+2} \right)^n \\
& \times \left(-\frac{n(m+2) + (m+4)}{m} + n-1 \right) \times \left(-\frac{n(m+2) + (m+4)}{m} - 1 \right)^{n-1}.
\end{aligned} \tag{8}$$

Thus,

$$\begin{aligned}
\tau(K_2 \times K_{m,n}) &= m^{n-1} n^{m-1} (m+2)^{m-1} (n+2)^{m-1} \\
&\times (n+m+2). \tag{9}
\end{aligned}$$

In particular,

$$\tau(K_2 \times K_{n,n}) = 2n^{2n-2} (n+1) (n+2)^{2n-2}; \quad n \geq 1. \tag{10}$$

□

Theorem 6. For $m, n \geq 1$, we have

$$\begin{aligned}
\tau(K_3 \times K_{m,n}) &= 3n^{m-1} m^{n-1} (m+3)^{2n-2} (n+3)^{2m-2} \\
&\times (n+m+3)^2. \tag{11}
\end{aligned}$$

Proof. Applying Lemma 1, we have

$$\begin{aligned}
& \tau(K_3 \times K_{m,n}) \\
&= \frac{1}{9(m+n)^2} \det(3(m+n)I - \bar{D} + \bar{A}) \\
&= \frac{1}{9(m+n)^2} \\
&\quad \times \det \begin{pmatrix}
n+3 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & n+3 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 \\
0 & \cdots & \cdots & 0 & m+3 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & m+3 & 1 & \cdots & \cdots & 1 \\
0 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & n+3 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & n+3 \\
1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 \\
1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1
\end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& \left(\begin{array}{cccccccccccc}
1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 \\
0 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 \\
0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 \\
m+3 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & m+3 & 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 \\
1 & \cdots & \cdots & 1 & n+3 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & n+3 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & m+3 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & m+3
\end{array} \right) \\
&= \frac{1}{9(m+n)^2} \det \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix} = \frac{1}{9(m+n)^2} [\det(A-B)]^2 [\det(A+2B)] \\
&= \frac{1}{9(m+n)^2} \left(\det \begin{pmatrix} n+3 & 0 & \cdots & 0 & -1 & \cdots & \cdots & -1 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & n+3 & -1 & \cdots & \cdots & -1 \\ -1 & \cdots & \cdots & -1 & m+3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ -1 & \cdots & \cdots & -1 & 0 & \cdots & 0 & m+3 \end{pmatrix} \right)^2 \\
&\quad \times \det \begin{pmatrix} n+3 & 3 & \cdots & 3 & 2 & \cdots & \cdots & 2 \\ 3 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 3 & \vdots & \ddots & \ddots & \vdots \\ 3 & \cdots & 3 & n+3 & 2 & \cdots & \cdots & 2 \\ 2 & \cdots & \cdots & 2 & m+3 & 3 & \cdots & 3 \\ \vdots & \ddots & \ddots & \vdots & 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 3 \\ 2 & \cdots & \cdots & 2 & 3 & \cdots & 3 & m+3 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{9(m+n)^2} \left(\det \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \right)^2 \times \det \begin{pmatrix} D & E \\ E^T & F \end{pmatrix} \\
&= \frac{1}{9(m+n)^2} \times (\det A)^2 (\det (C - B^T A^{-1} B))^2 \times \det D \det (F - E^T D^{-1} E).
\end{aligned}$$

(12)

Thus,

$$\begin{aligned}
\tau(K_3 \times K_{m,n}) &= \frac{1}{9(m+n)^2} \left(\det \begin{pmatrix} n+3 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n+3 \end{pmatrix}_{m \times m} \right)^2 \\
&\quad \times \left(\det \begin{pmatrix} \frac{nm+3n+2m+9}{n+3} & \frac{-m}{n+3} & \cdots & \frac{-m}{n+3} \\ \frac{-m}{n+3} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{-m}{n+3} \\ \frac{-m}{n+3} & \cdots & \frac{-m}{n+3} & \frac{nm+3n+2m+9}{n+3} \end{pmatrix}_{n \times n} \right)^2 \\
&\quad \times \det \begin{pmatrix} n+3 & 3 & \cdots & 3 \\ 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 3 \\ 3 & \cdots & 3 & n+3 \end{pmatrix}_{m \times m} \\
&\quad \times \det \begin{pmatrix} \frac{nm+3n+3m^2+5m}{n+3m} & \frac{3n+5m}{n+3m} & \cdots & \frac{3n+5m}{n+3m} \\ \frac{3n+5m}{n+3m} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{3n+5m}{n+3m} \\ \frac{3n+5m}{n+3m} & \cdots & \frac{3n+5m}{n+3m} & \frac{nm+3n+3m^2+5m}{n+3m} \end{pmatrix}_{n \times n} \\
&= \frac{1}{9(m+n)^2} (m+3)^{2m} \times \left(\frac{-m}{n+3} \right)^{2n} \\
&\quad \times \left(\det \begin{pmatrix} \frac{mn+3n+2m+9}{-m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{mn+3n+2m+9}{-m} \end{pmatrix} \right)^2 \\
&\quad \times 3^m \times \det \begin{pmatrix} \frac{n+3}{3} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{n+3}{3} \end{pmatrix}_{m \times m} \times \left(\frac{3n+5m}{n+3m} \right)^n
\end{aligned}$$

$$\times \det \begin{pmatrix} \frac{mn+3n+2m^2+5m}{3n+5m} & 1 & \dots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & \frac{mn+3n+2m^2+5m}{3n+5m} \end{pmatrix}. \quad (13)$$

Using Lemma 2, we have

$$\begin{aligned} \tau(K_3 \times K_{m,n}) &= \frac{1}{9(m+n)^2} \times (n+3)^{2m} \times \left(\frac{-m}{n+3}\right)^{2n} \\ &\quad \times \left[-\frac{nm+3n+2m+9}{m} + n-1\right]^2 \\ &\quad \times \left[-\frac{nm+3n+2m+9}{m} - 1\right]^{2n-2} \times 3^m \left(\frac{n+3}{3} + m-1\right) \\ &\quad \times \left(\frac{n+3}{3} - 1\right)^{m-1} \times \left(\frac{3n+5m}{n+3m}\right)^n \\ &\quad \times \left[\frac{nm+3n+3m^2+5m}{3n+5m} + n-1\right] \\ &\quad \times \left[\frac{nm+3n+3m^2+5m}{3n+5m} - 1\right]^{n-1} \\ &= \frac{1}{9(m+n)^2} (n+3)^{2m} \\ &\quad \times \left[\frac{1}{(n+3)^{2n}} \times (3n+3m+9)^2 \right. \\ &\quad \left. \times (nm+3n+3m+9)^{2n-2}\right] \end{aligned}$$

$$\begin{aligned} &\times \left[(n+3m) \times n^{m-1} \times \frac{1}{(n+3m)^n} \right. \\ &\quad \times (6nm+3n^2+3m^2) \\ &\quad \left. \times (nm+3m^2)^{n-1} \right] \\ &= 3n^{m-1} m^{n-1} (m+3)^{2n-2} (n+3)^{2m-2} (n+m+3)^2. \end{aligned} \quad (14)$$

In particular,

$$\tau(K_3 \times K_{n,n}) = 3n^{2n-2} (2n+3)^2 (n+3)^{4n-4}; \quad n \geq 1. \quad (15)$$

□

3. Number of Spanning Trees of Normal Product of Graphs

The normal product, or the strong product, $G_1 \circ G_2$, is the simple graph with $V(G_1 \circ G_2) = V_1 \times V_2$, where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \circ G_2$ if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 , u_1 is adjacent to v_1 and $u_2 = v_2$, or u_1 is adjacent to v_1 and u_2 is adjacent to v_2 [13].

Theorem 7. For $n, m \geq 1$, we have

$$\begin{aligned} \tau(K_2 \circ K_{m,n}) &= 2^{2m+2n-2} \times n^{m-1} \\ &\quad \times m^{n-1} \times (n+1)^m \times (m+1)^n. \end{aligned} \quad (16)$$

Proof. Applying Lemma 1, we have

$$\begin{aligned} \tau(K_2 \circ K_{m,n}) &= \frac{1}{4(m+n)^2} \det(2(m+n)I - \overline{D} + \overline{A}) \\ &= \frac{1}{4(m+n)^2} \end{aligned}$$

[illegible]

$$= \frac{1}{4(m+n)^2} \det \begin{pmatrix} 2n+2 & 2 & \cdots & 2 & 0 & \cdots & \cdots & 0 \\ 2 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \vdots \\ 2 & \cdots & 2 & 2n+2 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 2m+2 & 2 & \cdots & 2 \\ \vdots & \ddots & \ddots & \vdots & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 2 \\ 0 & \cdots & \cdots & 0 & 2 & \cdots & 2 & 2m+2 \end{pmatrix}$$

$$\times \det \begin{pmatrix} 2n+2 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 2n+2 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 2m+2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 2m+2 \end{pmatrix}$$

$$= \frac{1}{4(m+n)^2} \det \begin{pmatrix} 2n+2 & 2 & \cdots & 2 \\ & 2 & \ddots & \vdots \\ & \vdots & \ddots & 2 \\ 2 & \cdots & 2 & 2n+2 \end{pmatrix}_{m \times m} \times \det \begin{pmatrix} 2m+2 & 2 & \cdots & 2 \\ & 2 & \ddots & \vdots \\ & \vdots & \ddots & 2 \\ 2 & \cdots & 2 & 2m+2 \end{pmatrix}_{n \times n}$$

$$\times \det \begin{pmatrix} 2n+2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2n+2 \end{pmatrix}_{m \times m} \times \det \begin{pmatrix} 2m+2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2m+2 \end{pmatrix}_{n \times n}$$

$$\begin{aligned}
&= \frac{1}{4(m+n)^2} \times 2^m (n+m) n^{m-1} \times 2^n (n+m) m^{n-1} \times 2^m (n+1)^m \times 2^n (m+1)^n \\
&= 2^{2m+2n-2} \times n^{m-1} \times m^{n-1} \times (n+1)^m \times (m+1)^n.
\end{aligned} \tag{17}$$

In particular,

$$\tau(K_2 \circ K_{n,n}) = 2^{4n-2} \times n^{2n-2} \times (n+1)^{2n}; \quad n \geq 1. \tag{18}$$

□

Theorem 8. For $m, n \geq 1$, we have

Proof. Applying Lemma 1, we have

$$\begin{aligned}
\tau(K_3 \circ K_{m,n}) &= 3^{3m+3n-2} \times n^{m-1} \times m^{n-1} \\
&\quad \times (n+1)^{2m} \times (m+1)^{2n}.
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\tau(K_3 \circ K_{m,n}) \\
&= \frac{1}{9(m+n)^2} \det(3(m+n)I - \bar{D} + \bar{A}) \\
&= \frac{1}{9(m+n)^2} \\
&\quad \times \det \begin{pmatrix}
3n+3 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 3n+3 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 \\
0 & \cdots & \cdots & 0 & 3m+3 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 3m+3 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 3n+3 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 3n+3 \\
0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 \\
0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0
\end{pmatrix}
\end{aligned}$$

$$= \frac{1}{9(m+n)^2} \det \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix} = \frac{1}{9(m+n)^2} [\det(A-B)]^2 [\det(A+2B)]$$
$$= \frac{1}{9(m+n)^2} \left(\det \begin{pmatrix} 3n+3 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 3n+3 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 3m+3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 3m+3 \end{pmatrix} \right)^2$$
$$\times \det \begin{pmatrix} 3n+3 & 3 & \cdots & 3 & 0 & \cdots & 0 \\ 3 & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 3 & \vdots & \ddots & \vdots \\ 3 & \cdots & 3 & 3n+3 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 3m+3 & 3 & \cdots & 3 \\ \vdots & \ddots & \ddots & \vdots & 3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & 3 \\ 0 & \cdots & \cdots & 0 & 3 & \cdots & 3 & 3m+3 \end{pmatrix}$$

$$\begin{aligned}
&= \frac{1}{9(m+n)^2} \left(\det \begin{pmatrix} 3n+3 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 3n+3 \end{pmatrix}_{m \times m} \right)^2 \\
&\quad \times \left(\det \begin{pmatrix} 3m+3 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 3m+3 \end{pmatrix}_{n \times n} \right)^2 \\
&\quad \times \det \begin{pmatrix} 3n+3 & 3 & \cdots & 3 \\ 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 3 \\ 3 & \cdots & 3 & 3n+3 \end{pmatrix}_{m \times m} \times \det \begin{pmatrix} 3m+3 & 3 & \cdots & 3 \\ 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 3 \\ 3 & \cdots & 3 & 3m+3 \end{pmatrix}_{n \times n}.
\end{aligned} \tag{20}$$

Using Lemma 2, we have

$$\begin{aligned}
\tau(K_3 \circ K_{m,n}) &= \frac{1}{9(m+n)^2} \times (3n+3)^{2m} \times (3m+3)^{2n} \\
&\quad \times (3^m \times (n+m) \times n^{m-1}) \\
&\quad \times (3^n \times (n+m) \times m^{n-1}) \\
&= 3^{3m+3n-2} \times n^{m-1} \times m^{n-1} \\
&\quad \times (n+1)^{2m} \times (m+1)^{2n}.
\end{aligned} \tag{21}$$

In particular,

$$\tau(K_3 \circ K_{n,n}) = 3^{6n-2} \times n^{2n-2} \times (n+1)^{4n}; \quad n \geq 1. \tag{22}$$

4. Number of Spanning Trees of Composition Product of Graphs

The composition, or lexicographic product, $G_1[G_2]$, is the simple graph with $V_1 \times V_2$ as the vertex set in which the vertices (u_1, u_2) and (v_1, v_2) are adjacent if either u_1 is adjacent to v_1 or $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 [13].

Theorem 9. For $n, m \geq 1$, we have

$$\begin{aligned}
\tau(K_2[K_{m,n}]) &= 4(m+n)^2 \\
&\quad \times (m+2n)^{2m-2} (n+2m)^{2n-2}.
\end{aligned} \tag{23}$$

Proof. Applying Lemma 1, we have

$$\begin{aligned}
&\tau(K_2[K_{m,n}]) \\
&= \frac{1}{4(m+n)^2} \det(2(m+n)I - \overline{D} + \overline{A}) \\
&= \frac{1}{4(m+n)^2}
\end{aligned}$$

$$\begin{aligned}
& \times \det \left(\begin{array}{ccccccc} m+2n+1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & m+2n+1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & n+2m+1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & n+2m+1 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \end{array} \right) \\
& \left(\begin{array}{ccccccc} 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ m+2n+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & m+2n+1 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & n+2m+1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & n+2m+1 \end{array} \right) \\
& = \frac{1}{4(m+n)^2} \left(\det \left(\begin{array}{cccc} m+2n+1 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & m+2n+1 \end{array} \right)_{m \times m} \right)^2 \\
& \times \left(\det \left(\begin{array}{cccc} n+2m+1 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & n+2m+1 \end{array} \right)_{n \times n} \right)^2
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4(m+n)^2} (2n+2m)^2 (m+2n)^{2m-2} \times (2n+2m)^2 (n+2m)^{2n-2} \\
&= 4(m+n)^2 (m+2n)^{2m-2} (n+2m)^{2n-2}.
\end{aligned} \tag{24}$$

In particular,

$$\tau(K_2[K_{n,n}]) = 16 \times 3^{4n-4} \times n^{4n-4}; \quad n \geq 1. \tag{25}$$

Theorem 10. For $m, n \geq 1$, we have

$$\tau(K_3[K_{m,n}]) = 3^4 (m+n)^4 (3m+2n)^{3n-3} (3n+2m)^{3m-3}. \tag{26}$$

□

Proof. Applying Lemma 1, we have

$$\begin{aligned}
&\tau(K_3[K_{m,n}]) \\
&= \frac{1}{9(m+n)^2} \det(3(m+n)I - \overline{D} + \overline{A}) \\
&= \frac{1}{9(m+n)^2} \\
&\quad \times \det \begin{pmatrix}
3n+2m+1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & \cdots & 1 & 3n+2m+1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 3m+2n+1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 3m+2n+1 & 0 & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 3n+2m+1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 3n+2m+1 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0
\end{pmatrix}
\end{aligned}$$

(27)

Using Lemma 2, we have

$$\tau(K_3[K_{m,n}]) = 3^4(m+n)^4(3m+2n)^{3n-3}(3n+2m)^{3m-3}. \quad (28)$$

In particular,

$$\tau(K_3[K_{n,n}]) = 6^4 \times 5^{6n-6} \times n^{6n-2}; \quad n \geq 1. \quad (29)$$

□

5. Complexity of Tensor Product of Graphs

The tensor product, or Kronecker product, $G_1 \otimes G_2$, is the simple graph with $V(G_1 \otimes G_2) = V_1 \times V_2$, where (u_1, u_2) and

(v_1, v_2) are adjacent in $G_1 \otimes G_2$ if and only if u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2 [13].

Lemma 11. For $m, n \geq 1$, we have

$$\tau(K_2 \otimes K_{m,n}) = 0. \quad (30)$$

Theorem 12. For $m, n \geq 1$, we have

$$\tau(K_3 \otimes K_{m,n}) = 3 \times 2^{3m+3n-5} \times n^{3m-1} \times m^{3n-1}. \quad (31)$$

Proof. Applying Lemma 1, we have

$$\begin{aligned} & \tau(K_3 \otimes K_{m,n}) \\ &= \frac{1}{9(m+n)^2} \det(3(m+n)I - \overline{D} + \overline{A}) \\ &= \frac{1}{9(m+n)^2} \\ & \quad \times \det \begin{pmatrix} 2n+1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 2n+1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & \cdots & \cdots & 1 & 2m+1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 2m+1 & 0 & \cdots & 0 \\ 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 2n+1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 2n+1 \\ 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \end{aligned}$$

$$= \frac{1}{9(m+n)^2} \det \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix} = \frac{1}{9(m+n)^2} [\det(A-B)]^2 [\det(A+2B)]$$
$$= \frac{1}{9(m+n)^2} \left(\det \begin{pmatrix} 2n & 0 & \cdots & 0 & 1 & \cdots & \cdots & 1 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 2n & 1 & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & 1 & 2m & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & \cdots & 1 & 0 & \cdots & 0 & 2m \end{pmatrix} \right)^2$$
$$\times \det \begin{pmatrix} 2n+3 & 3 & \cdots & 3 & 1 & \cdots & \cdots & 1 \\ 3 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 3 & \vdots & \ddots & \ddots & \vdots \\ 3 & \cdots & 3 & 2n+3 & 1 & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & 1 & 2m+3 & 3 & \cdots & 3 \\ \vdots & \ddots & \ddots & \vdots & 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 3 \\ 1 & \cdots & \cdots & 1 & 3 & \cdots & 3 & 2m+3 \end{pmatrix}$$

$$\begin{aligned}
&= \frac{1}{9(m+n)^2} \det \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \times \det \begin{pmatrix} D & E \\ E^T & F \end{pmatrix} = \frac{1}{9(m+n)^2} \\
&\times (\det A)^2 (\det (C - B^T A^{-1} B))^2 \times \det D \det (F - E^T D^{-1} E).
\end{aligned} \tag{32}$$

Thus,

$$\begin{aligned}
\tau(K_3 \otimes K_{m,n}) &= \frac{1}{9(m+n)^2} \left(\det \begin{pmatrix} 2n & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2n \end{pmatrix}_{m \times m} \right)^2 \\
&\times \left(\det \begin{pmatrix} \frac{m(4n-1)}{2n} & \frac{-m}{2n} & \cdots & \frac{-m}{2n} \\ \frac{-m}{2n} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{-m}{2n} \\ \frac{-m}{2n} & \cdots & \frac{-m}{2n} & \frac{m(4n-1)}{2n} \end{pmatrix}_{n \times n} \right)^2 \\
&\times \det \begin{pmatrix} 2n+3 & 3 & \cdots & 3 \\ 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 3 \\ 3 & \cdots & 3 & 2n+3 \end{pmatrix}_{m \times m} \\
&\times \det \begin{pmatrix} \frac{n(4m+6)+6m^2+8m}{2n+3m} & \frac{6n+8m}{2n+3m} & \cdots & \frac{6n+8m}{2n+3m} \\ \frac{6n+8m}{2n+3m} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{6n+8m}{2n+3m} \\ \frac{6n+8m}{2n+3m} & \cdots & \frac{6n+8m}{2n+3m} & \frac{n(4m+6)+6m^2+8m}{2n+3m} \end{pmatrix}_{n \times n} \\
&= \frac{1}{9(m+n)^2} (2n)^{2m} \times \left(\frac{-m}{2n} \right)^{2n} \times \left(\det \begin{pmatrix} \frac{m(4n-1)}{-m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{m(4n-1)}{-m} \end{pmatrix}_{n \times n} \right)^2 \\
&\times 3^m \times \det \begin{pmatrix} \frac{2n+3}{3} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{2n+3}{3} \end{pmatrix} \times \left(\frac{6n+8m}{2n+3m} \right)^n
\end{aligned}$$

$$\times \det \begin{pmatrix} \frac{n(4m+6)+6m^2+8m}{6n+8m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{n(4m+6)+6m^2+8m}{6n+8m} \end{pmatrix}_{n \times n}.$$
(33)

Using Lemma 2, we have

$$\begin{aligned} \tau(K_3 \otimes K_{m,n}) &= \frac{1}{9(m+n)^2} (2n)^{2m} \\ &\times \left[\left(\frac{m}{2n} \right)^{2n} \times (-3n)^2 \times (-4n)^{2n-2} \right] \\ &\times \left[3^m \left(\frac{2n+3m}{3} \right) \times \left(\frac{2n}{3} \right)^{m-1} \right] \\ &\times \left(\frac{6n+8m}{2n+3m} \right)^n \\ &\times \left[\left(\frac{4nm+6n+6m^2+8m}{6n+8m} + n - 1 \right) \right. \\ &\quad \left. \times \left(\frac{4nm+6n+6m^2+8m}{6n+8m} - 1 \right)^{n-1} \right] \\ &= 3 \times (2n)^{2m-2n} \times m^{3n-1} \\ &\quad \times n^{2n+m-1} \times 2^{5n+m-5} \end{aligned}$$

$$= 3 \times 2^{3m+3n-5} \times n^{3m-1} \times m^{3n-1}.$$
(34)

In particular,

$$\tau(K_3 \otimes K_{n,n}) = 3 \times 2^{6n-5} \times n^{6n-2}; \quad n \geq 1. \quad (35)$$

□

6. Number of Spanning Trees of Symmetric Product of Graphs

The symmetric product, $G_1 \oplus G_2$, is the simple graph with $V(G_1 \oplus G_2) = V_1 \times V_2$, where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \oplus G_2$ if and only if either u_1 is adjacent to v_1 in G_1 and u_2 is not adjacent to v_2 in G_2 , or u_1 is not adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2 [13].

Theorem 13. For $n, m \geq 1$, we have

$$\tau(K_2 \oplus K_{m,n}) = (m+n)^{2(m+n-1)}. \quad (36)$$

Proof. Applying Lemma 1, we have

$$\begin{aligned} \tau(K_2 \oplus K_{m,n}) &= \frac{1}{4(m+n)^2} \det(2(m+n)I - \overline{D} + \overline{A}) \\ &= \frac{1}{4(m+n)^2} \end{aligned}$$

$$= \frac{1}{4(m+n)^2} \det \left(\begin{array}{ccccccc} m+n+1 & 1 & \cdots & 1 & 0 & \cdots \cdots & 0 \\ 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & m+n+1 & 0 & \cdots \cdots & 0 \\ 0 & \cdots \cdots & 0 & m+n+1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & 1 \\ 0 & \cdots \cdots & 0 & 1 & \cdots & 1 & m+n+1 \\ 0 & \cdots \cdots & 0 & 1 & \cdots \cdots & 1 & \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots \cdots & 0 & 1 & \cdots \cdots & 1 & \\ 1 & \cdots \cdots & 1 & 0 & \cdots \cdots & 0 & \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots \cdots & 1 & 0 & \cdots \cdots & 0 & \\ 0 & \cdots \cdots & 0 & 1 & \cdots \cdots & 1 & \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots \cdots & 0 & 1 & \cdots & 1 & m+n+1 \end{array} \right)$$

$$\begin{aligned}
& \times \det \begin{pmatrix} m+n+1 & 1 & \cdots & 1 & -1 & \cdots & \cdots & -1 \\ 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & m+n+1 & -1 & \cdots & \cdots & -1 \\ -1 & \cdots & \cdots & -1 & m+n+1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\ -1 & \cdots & \cdots & -1 & 1 & \cdots & 1 & m+n+1 \end{pmatrix} \\
&= \frac{1}{4(m+n)^2} (m+n+1+m+n-1)(m+n+1-1)^{m+n-1} \times \det \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \\
&= \frac{1}{2} (m+n)^{m+n-2} \times \det A \det (C - B^T A^{-1} B) \\
&= \frac{1}{2} (m+n)^{m+n-2} \times (2m+n)(m+n)^{m-1} \\
& \times \det \begin{pmatrix} \frac{n^2 + (3m+1)n + 2m^2 + m}{(n+2m)} & \frac{n+m}{n+2m} & \cdots & \frac{n+m}{n+2m} \\ \frac{n+m}{n+2m} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{n+m}{n+2m} \\ \frac{n+m}{n+2m} & \cdots & \frac{n+m}{n+2m} & \frac{n^2 + (3m+1)n + 2m^2 + m}{(n+2m)} \end{pmatrix}_{n \times n} \\
&= \frac{1}{2} (m+n)^{2m+n-3} \times (2m+n) \times \left(\frac{n+m}{n+2m} \right)^n \\
& \times \det \begin{pmatrix} \frac{n^2 + (3m+1)n + 2m^2 + m}{n+m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{n^2 + (3m+1)n + 2m^2 + m}{n+m} \end{pmatrix}_{n \times n}.
\end{aligned} \tag{37}$$

Thus,

$$\begin{aligned}
\tau(K_2 \oplus K_{m,n}) &= \frac{1}{2} (m+n)^{2m+n-3} \times (2m+n) \\
&\times \left(\frac{n+m}{n+2m} \right)^n \times (2n+2m)(n+2m)^{n-1} \\
&= (m+n)^{2(m+n-1)}.
\end{aligned} \tag{38}$$

In particular,

$$\tau(K_2 \oplus K_{n,n}) = (2n)^{2(2n-1)}; \quad n \geq 1. \tag{39}$$

Theorem 14. For $m, n \geq 1$, we have

$$\begin{aligned}
\tau(K_3 \oplus K_{m,n}) &= 3(2m+n)^{3m-3} \\
&\times (2n+m)^{3n-3} (m^2 + n^2 + 3mn)^2.
\end{aligned} \tag{40}$$

Proof. Applying Lemma 1, we have

$$\begin{aligned}
&\tau(K_3 \oplus K_{m,n}) \\
&= \frac{1}{9(m+n)^2} \det(3(m+n)I - \overline{D} + \overline{A}) \\
&= \frac{1}{9(m+n)^2}
\end{aligned}$$

[illegible]

$$\begin{aligned}
&= \frac{1}{9(m+n)^2} \det \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix} = \frac{1}{9(m+n)^2} [\det(A-B)]^2 [\det(A+2B)] \\
&= \frac{1}{9(m+n)^2} \left(\det \begin{pmatrix} 2m+n+1 & 1 & \cdots & 1 & -1 & \cdots & -1 \\ 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 2m+n+1 & -1 & \cdots & -1 \\ -1 & \cdots & \cdots & -1 & 2n+m+1 & 1 & 1 \\ \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & 1 \\ -1 & \cdots & \cdots & -1 & 1 & \cdots & 2n+m+1 \end{pmatrix} \right)^2 \\
&\quad \times \det \begin{pmatrix} 2m+n+1 & 1 & \cdots & 1 & 2 & \cdots & 2 \\ 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 2m+n+1 & 2 & \cdots & 2 \\ 2 & \cdots & \cdots & 2 & 2n+m+1 & 1 & 1 \\ \vdots & \ddots & \ddots & \vdots & 3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & 1 \\ 2 & \cdots & \cdots & 2 & 1 & \cdots & 2n+m+1 \end{pmatrix} \\
&= \frac{1}{9(m+n)^2} \left(\det \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \right)^2 \times \det \begin{pmatrix} D & E \\ E^T & F \end{pmatrix} = \frac{1}{9(m+n)^2} \times (\det A)^2 (\det(C - B^T A^{-1} B))^2 \\
&\quad \times \det D \det(F - E^T D^{-1} E) \\
&= \frac{(3m+n)^2 (2m+n)^{2m-2}}{9(m+n)^2} \\
&\quad \times \left(\det \begin{pmatrix} \frac{2n^2 + n(7m+1) + 3m^2 + 2m}{n+3m} & \frac{n+2m}{n+3m} & \cdots & \frac{n+2m}{n+3m} \\ \frac{n+3m}{n+3m} & \ddots & \ddots & \vdots \\ \frac{n+2m}{n+3m} & \ddots & \ddots & \frac{n+2m}{n+3m} \\ \vdots & \ddots & \ddots & \frac{n+2m}{n+3m} \\ \frac{n+2m}{n+3m} & \cdots & \frac{n+2m}{n+3m} & \frac{2n^2 + n(7m+1) + 3m^2 + 2m}{n+3m} \end{pmatrix}_{n \times n} \right)^2 \\
&\quad \times (3m+n)(2m+n)^{m-1} \\
&\quad \times \det \begin{pmatrix} \frac{2n^2 + n(7m+1) + 3m^2 - m}{n+3m} & \frac{n-m}{n+3m} & \cdots & \frac{n-m}{n+3m} \\ \frac{n+3m}{n+3m} & \ddots & \ddots & \vdots \\ \frac{n-m}{n+3m} & \ddots & \ddots & \frac{n-m}{n+3m} \\ \vdots & \ddots & \ddots & \frac{n-m}{n+3m} \\ \frac{n-m}{n+3m} & \cdots & \frac{n-m}{n+3m} & \frac{2n^2 + n(7m+1) + 3m^2 - m}{n+3m} \end{pmatrix}_{n \times n}
\end{aligned}$$

$$\begin{aligned}
&= \frac{(3m+n)^2(2m+n)^{2m-2}}{9(m+n)^2} \\
&\times \left(\frac{n+2m}{n+3m} \right)^{2n} \det \left(\begin{array}{cccc} \frac{2n^2+n(7m+1)+3m^2+2m}{n+2m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{2n^2+n(7m+1)+3m^2+2m}{n+2m} \end{array} \right)_{n \times n}^2 \\
&\times (3m+n)(2m+n)^{m-1} \\
&\times \left(\frac{n-m}{n+3m} \right)^n \det \left(\begin{array}{cccc} \frac{2n^2+n(7m+1)+3m^2-m}{n-m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{2n^2+n(7m+1)+3m^2-m}{n-m} \end{array} \right)_{n \times n}.
\end{aligned} \tag{41}$$

Using Lemma 2, we have

$$\begin{aligned}
\tau(K_3 \oplus K_{m,n}) &= \frac{1}{9(m+n)^2} (3m+n)^2 (2m+n)^{2m-2} \\
&\times \left(\frac{n+2m}{n+3m} \right)^{2n} \frac{1}{(n+3m)^{2n}} \\
&\times (3n^2 + 3m^2 + 9nm)^2 \\
&\times (2n^2 + 3m^2 + 7nm)^{2n-2} \\
&\times (3m+n)(2m+n)^{m-1} \\
&\times \left(\frac{n-m}{n+3m} \right)^n \times \frac{1}{(n-m)^n} \\
&\times (3n^2 + 3m^2 + 6nm) \\
&\times (2n^2 + 3m^2 + 7nm)^{n-1} \\
&= 3(2m+n)^{3m-3} (2n+m)^{3n-3} \\
&\times (m^2 + n^2 + 3mn)^2.
\end{aligned} \tag{42}$$

In particular,

$$\tau(K_3 \oplus K_{n,n}) = 25 \times 3^{6n-5} \times n^{6n-2}; \quad n \geq 1. \tag{43}$$

□

7. Conclusion

The number of spanning trees $\tau(G)$ in graphs (networks) is an important invariant. The evaluation of this number is not only

interesting from a mathematical (computational) perspective but is also an important measure of reliability of a network and designing electrical circuits. Some computationally hard problems such as the travelling salesman problem can be solved approximately by using spanning trees. Due to the high dependence of the network design and reliability on the graph theory, we introduced the above important theorems and lemmas and their proofs.

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