Research Article

A Reverse Analytic Inequality for the Elementary Symmetric Function with Applications

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We give a reverse inequality involving the elementary symmetric function by use of the Schur harmonic convexity theory. As applications, several new analytic inequalities for the n -dimensional simplex are established.

1. Introduction

Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n = (-\infty, +\infty)^n$. The elementary symmetric functions are defined by

$$E_k(\mathbf{x}) = E_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k x_{i_j}, \quad (1)$$

$$k = 1, \dots, n,$$

$E_0(\mathbf{x}) = 1$, and $E_k(\mathbf{x}) = 0$ for $k < 0$ or $k > n$.

In 1997, Leng and Tang [1] in order to achieve higher dimensional generalization of the famous Pedoe inequality [2–5] presented the following analytic inequality.

Let $\mathbf{x} \in (0, 1/2)^n$, and $E_1(\mathbf{x}) = 1$. Then

$$E_{n-1}\left(\frac{1}{\mathbf{x}} - 2\right) = E_{n-1}\left(\frac{1}{x_1} - 2, \dots, \frac{1}{x_n} - 2\right) \geq n(n-2)^{n-1}. \quad (2)$$

Chen et al. [6] gave an analytic proof which is skillful. By the method of successive adjustment, Fan [7] gave a simple elementary proof for inequality under conditions $\mathbf{x} \in (0, 1)^n$, $n \geq 2$, and $E_1(\mathbf{x}) = 1$. Using the same method, Shi [8] extended inequality (2) to the cases of $E_k((1/\mathbf{x}) - 2)$ for $k = 1, 2, \dots, n-1$, where $\mathbf{x} \in \mathbb{R}_+^n$.

Theorem A. Let $\mathbf{x} \in \mathbb{R}_+^n = (0, +\infty)^n$, $n \geq 2$ and $E_1(\mathbf{x}) = 1$. Then for $k = 1, 2, \dots, n-1$, one has

$$E_k\left(\frac{1}{\mathbf{x}} - 2\right) = E_k\left(\frac{1}{x_1} - 2, \dots, \frac{1}{x_n} - 2\right) \geq \binom{n}{k} (n-2)^k, \quad (3)$$

where $\binom{n}{k} = n!/k!(n-k)!$.

And Shi also pointed out that inequality (3) does not hold for $k = n$.

In 2002, using analytical methods, Ma and Pu [9] gave parameter extension of the inequality (3).

Theorem B. Let $\mathbf{x} \in \mathbb{R}_+^n$, $n, k \in \mathbb{N} = \{1, 2, \dots\}$, $\mu > 0$, $n \geq 2$ and $E_1(\mathbf{x}) = 1$. Then for $n \geq k + \mu - 1$, one has

$$E_k\left(\frac{1}{\mathbf{x}} - \mu\right) = E_k\left(\frac{1}{x_1} - \mu, \dots, \frac{1}{x_n} - \mu\right) \geq \binom{n}{k} (n-\mu)^k. \quad (4)$$

In 2005, Li et al. [10] gave an elementary proof for Theorem B when $k \leq n-1$.

In this paper, by studying the Schur harmonic convexity of $E_k((1/\mathbf{x}) - \mu)$, we obtain a reverse of inequality (4) and accordingly establish the reverse of the corresponding simplex inequalities.

Our main result is the following theorem.

Theorem 1. Let $\mathbf{x} \in \mathbb{R}_+^n$, $n, k \in \mathbb{N}$, $\mu > 0$, $n \geq 2$. Then for $n \geq k + \mu - 1$, $E_k((1/\mathbf{x}) - \mu)$ is Schur harmonically concave on $\Omega = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}_+^n, E_1(\mathbf{x}) \leq 1\}$. When $E_1(\mathbf{x}) \leq 1$, one has

$$E_k\left(\frac{1}{\mathbf{x}} - \mu\right) = E_k\left(\frac{1}{x_1} - \mu, \dots, \frac{1}{x_n} - \mu\right) \leq \binom{n}{k} \left(\frac{1}{H} - \mu\right)^k, \quad (5)$$

where $H = n(\sum_{i=1}^n x_i^{-1})^{-1}$ is the harmonic mean of $\mathbf{x}$.

Taking $\mu = 2$ in Theorem 1, we infer the following corollary.

Corollary 2. Let $\mathbf{x} \in \mathbb{R}_+^n$, $n, k \in \mathbb{N}$, and $E_1(\mathbf{x}) \leq 1$. Then for $n \geq k + 1$, one has

$$E_k\left(\frac{1}{\mathbf{x}} - 2\right) = E_k\left(\frac{1}{x_1} - 2, \dots, \frac{1}{x_n} - 2\right) \leq \binom{n}{k} \left(\frac{1}{H} - 2\right)^k, \quad (6)$$

where $H = n(\sum_{i=1}^n x_i^{-1})^{-1}$ is the harmonic mean of $\mathbf{x}$.

Taking $k = n - 1$ in Corollary 2, we obtain the following inequality.

Corollary 3. Let $\mathbf{x} \in \mathbb{R}_+^n$, $n \in \mathbb{N}$, and $E_1(\mathbf{x}) \leq 1$. Then for $n \geq 2$, one has

$$\sum_{i=1}^n \prod_{j=1, j \neq i}^n \left(\frac{1}{x_j} - 2\right) \leq n \left(\frac{1}{H} - 2\right)^{n-1}, \quad (7)$$

where $H = n(\sum_{i=1}^n x_i^{-1})^{-1}$ is the harmonic mean of $\mathbf{x}$.

2. Definitions and Lemmas

Definition 4 (see [11, 12]). Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$.

(i) $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ (in symbols $\mathbf{x} < \mathbf{y}$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \dots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq \dots \geq x_{[n]}$ and $y_{[1]} \geq \dots \geq y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a descending order.

(ii) $\Omega \subseteq \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \dots, \alpha x_n + \beta y_n) \in \Omega$ for every $\mathbf{x}$ and $\mathbf{y} \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.

(iii) Let $\Omega \subseteq \mathbb{R}^n$. The function $\varphi: \Omega \rightarrow \mathbb{R}$ is said to be a Schur convex function on Ω if $\mathbf{x} < \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. φ is said to be a Schur concave function on Ω if and only if $-\varphi$ is Schur convex.

Lemma 5 (see [11, 12]). Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Then

$$(A, \dots, A) < (x_1, x_2, \dots, x_n), \quad (8)$$

where $A = (1/n) \sum_{i=1}^n x_i$ is the arithmetic mean of $\mathbf{x}$.

Definition 6 (see [11, 12]). Let $\Omega \subseteq \mathbb{R}^n$.

(i) Ω is called a symmetric set, if $\mathbf{x} \in \Omega$ implies $\mathbf{x}P \in \Omega$ for every $n \times n$ permutation matrix P .

(ii) The function $\varphi: \Omega \rightarrow \mathbb{R}$ is called symmetric if for every permutation matrix P , $\varphi(\mathbf{x}P) = \varphi(\mathbf{x})$ for all $\mathbf{x} \in \Omega$.

Lemma 7 (see [11, p. 84]). Let $\Omega \subset \mathbb{R}^n$ be symmetric and have a nonempty interior convex set Ω^0 , and let $\varphi: \Omega \rightarrow \mathbb{R}$ be continuous on Ω and differentiable in Ω^0 . Then φ is the Schur-concave (Schur-convex) function if and only if φ is symmetric on Ω and

$$(x_1 - x_2) \left(\frac{\partial \varphi(\mathbf{x})}{\partial x_1} - \frac{\partial \varphi(\mathbf{x})}{\partial x_2} \right) \geq 0 (\leq 0) \quad (9)$$

holds for any $\mathbf{x} \in \Omega^0$.

The Schur-convexity described the ordering of majorization; the order-preserving functions were first comprehensively studied by Issai Schur in 1923. It has important applications in analytic inequalities, combinatorial optimization, quantum physics, information theory, and so on. See [11].

In 2009, Chu and Lv [13] introduced the notion of harmonically Schur convex function and some interesting inequalities were obtained. The Schur harmonic convexity involving some special functions has been investigated; see, for example, [14–20].

Definition 8 (see [13]). Let $\Omega \subset \mathbb{R}_+^n$.

(i) A set Ω is said to be harmonically convex if $\mathbf{xy}/(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \in \Omega$ for every $\mathbf{x}, \mathbf{y} \in \Omega$ and $\lambda \in [0, 1]$, where $\mathbf{xy} = \sum_{i=1}^n x_i y_i$ and $(1/\mathbf{x}) = (1/x_1, \dots, 1/x_n)$.

(ii) A function $\varphi: \Omega \rightarrow \mathbb{R}_+$ is said to be Schur harmonically convex on Ω if $1/\mathbf{x} < 1/\mathbf{y}$ implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. A function φ is said to be a Schur harmonically concave function on Ω if and only if $-\varphi$ is a Schur harmonically convex function.

Lemma 9 (see [13]). Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric and harmonically convex set with inner points and let $\varphi: \Omega \rightarrow \mathbb{R}_+$ be a continuously symmetric function which is differentiable on Ω^0 . Then, φ is Schur harmonically convex (Schur harmonically concave) on Ω if and only if

$$(x_1 - x_2) \left(x_1^2 \frac{\partial \varphi(\mathbf{x})}{\partial x_1} - x_2^2 \frac{\partial \varphi(\mathbf{x})}{\partial x_2} \right) \geq 0 (\leq 0), \quad \mathbf{x} \in \Omega^0. \quad (10)$$

Lemma 10. Let $\mathbf{x} \in \mathbb{R}_+^n$, $n, k \in \mathbb{N}$, $\mu > 0$, $n \geq 2$, and $E_1(\mathbf{x}) \leq 1$. Then for $n \geq k + \mu - 1$, one has

$$E_k\left(\frac{1}{\mathbf{x}} - \mu\right) \geq 0. \quad (11)$$

Proof. Let $E_1(\mathbf{x}) = \sigma \leq 1$. Then $E_1(\mathbf{x}/\sigma) = (x_1/\sigma) + \dots + (x_n/\sigma) = 1$; and $n \geq k + \mu - 1 \geq k + \sigma\mu - 1$, from Theorem B, we have

$$E_k\left(\frac{\sigma}{\mathbf{x}} - \sigma\mu\right) = E_k\left(\frac{\sigma}{x_1} - \sigma\mu, \dots, \frac{\sigma}{x_n} - \sigma\mu\right) \geq \binom{n}{k} (n - \sigma\mu)^k, \quad (12)$$

and then

$$\begin{aligned} E_k\left(\frac{1}{\mathbf{x}} - \mu\right) &= E_k\left(\frac{1}{x_1} - \mu, \dots, \frac{1}{x_n} - \mu\right) \\ &\geq \binom{n}{k} \left(\frac{n - \sigma\mu}{\sigma}\right)^k \geq 0. \end{aligned} \quad (13)$$

Lemma 10 is proved. $\square$

3. Proof of Main Result

Proof of Theorem 1. When $n \geq 2, k = 1$, obviously, inequality (5) holds. When $n \geq 2, k \geq 2$, noting that

$$\begin{aligned} \varphi(\mathbf{x}) &:= E_k\left(\frac{1}{\mathbf{x}} - \mu\right) \\ &= \left(\frac{1}{x_1} - \mu\right) \left(\frac{1}{x_2} - \mu\right) E_{k-2}\left(\frac{1}{x_3} - \mu, \dots, \frac{1}{x_n} - \mu\right) \\ &\quad + \left(\frac{1}{x_1} - \mu + \frac{1}{x_2} - \mu\right) E_{k-1}\left(\frac{1}{x_3} - \mu, \dots, \frac{1}{x_n} - \mu\right) \\ &\quad + E_k\left(\frac{1}{x_3} - \mu, \dots, \frac{1}{x_n} - \mu\right), \end{aligned} \quad (14)$$

we have

$$\begin{aligned} \frac{\partial \varphi(\mathbf{x})}{\partial x_1} &= -\frac{1}{x_1^2} \left(\frac{1}{x_2} - \mu\right) E_{k-2}\left(\frac{1}{x_3} - \mu, \dots, \frac{1}{x_n} - \mu\right) \\ &\quad - \frac{1}{x_1^2} E_{k-1}\left(\frac{1}{x_3} - \mu, \dots, \frac{1}{x_n} - \mu\right), \end{aligned} \quad (15)$$

$$\begin{aligned} \frac{\partial \varphi(\mathbf{x})}{\partial x_2} &= -\frac{1}{x_2^2} \left(\frac{1}{x_1} - \mu\right) E_{k-2}\left(\frac{1}{x_3} - \mu, \dots, \frac{1}{x_n} - \mu\right) \\ &\quad - \frac{1}{x_2^2} E_{k-1}\left(\frac{1}{x_3} - \mu, \dots, \frac{1}{x_n} - \mu\right). \end{aligned} \quad (16)$$

And then

$$\begin{aligned} \Delta &:= (x_1 - x_2) \left(x_1^2 \frac{\partial \varphi(\mathbf{x})}{\partial x_1} - x_2^2 \frac{\partial \varphi(\mathbf{x})}{\partial x_2} \right) \\ &= -\frac{(x_1 - x_2)^2}{x_1 x_2} E_{k-2}\left(\frac{1}{x_3} - \mu, \dots, \frac{1}{x_n} - \mu\right). \end{aligned} \quad (17)$$

Now we distinguish two cases to prove $\Delta \leq 0$.

Case 1. If $n \geq 2, k = 2$, then from the definition of elementary symmetric functions, it follows that $\Delta \leq 0$.

If $n = 3, k = 3$, then from the condition $n \geq k + \mu - 1$, it follows that $\mu \leq n - k + 1 = 1$, and by $x_i \leq E_1(\mathbf{x}) \leq 1$, it follows that $1/x_i \geq 1, i = 3, \dots, n$; therefore,

$$\begin{aligned} E_{k-2}\left(\frac{1}{x_3} - \mu, \dots, \frac{1}{x_n} - \mu\right) &= E_1\left(\frac{1}{x_3} - \mu, \dots, \frac{1}{x_n} - \mu\right) \\ &= \sum_{i=3}^n \frac{1}{x_i} - (n-2)\mu \geq 0. \end{aligned} \quad (18)$$

Case 2. If $n \geq 4$; then $n - 2 \geq 2$, from $n \geq k + \mu - 1$, it follows that

$$(n-2) \geq (k-2) + \mu - 1 \quad (19)$$

and $\sum_{i=3}^n x_i \leq 1$. So by Lemma 10, it follows that $E_{k-2}((1/x_3) - \mu, \dots, (1/x_n) - \mu) \geq 0$.

Thus, for all $n \geq 2, \Delta \leq 0$. By Lemma 9, we can derive that $E_k((1/\mathbf{x}) - \mu)$ is Schur harmonically concave on Ω .

From Lemma 5, it is seen that

$$\left(\frac{1}{H}, \dots, \frac{1}{H}\right) < \left(\frac{1}{x_1}, \dots, \frac{1}{x_n}\right). \quad (20)$$

According to Definition 8(ii), inequality (5) follows.

The proof of Theorem 1 is completed. $\square$

4. Applications in Geometry

Let Ω_n be an n -dimensional simplex in the n -dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 2$) with vertices $A_0, A_1, \dots, A_n$ whose volume is V . For $i = 0, 1, \dots, n$, let r_i be the radius of i th escribed sphere of Ω_n , and F_i the area of the i th face $f_i = \{A_0, A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n\}$ of Ω_n , $F = \sum_{i=0}^n F_i$. R and r are circumradius and inradius of Ω_n , respectively. Let P be an arbitrary interior point of the simplex Ω_n , d_i the distance from the point P to the i th face f_i of Ω_n , and h_i the altitude of Ω_n from vertex A_i for $i = 0, 1, \dots, n$.

In [9], by Theorem B, Ma and Pu obtained the following two theorems.

Theorem C. In the simplex Ω_n ($n \geq 2$), for $k = 1, 2, \dots, n$, one has

$$E_k\left(\frac{h_0}{r_0}, \frac{h_1}{r_1}, \dots, \frac{h_n}{r_n}\right) \geq \binom{n+1}{k} (n-1)^k \quad (21)$$

with equality if and only if Ω_n is an orthocentric simplex.

Theorem D. Let P be an arbitrary point in the interior of n -dimensional simplex Ω_n . Let B_i be the intersection of the line $A_i P$ with the hyperplane f_i ; further let $R_i = |A_i P|$, $d_i' = |PB_i|$, $M_i = |A_i B_i|$, $i = 0, 1, \dots, n$. Then for $k = 1, 2, \dots, n+1$, one has

$$E_k\left(\frac{R_0}{d_0'}, \frac{R_1}{d_1'}, \dots, \frac{R_n}{d_n'}\right) \geq \binom{n+1}{k} n^k \quad (22)$$

with equality if and only if Ω_n is an orthocentric simplex, and P is its orthocentre.

Now, by Theorem 1, we give the reverse of inequalities (21) and (22).

Theorem 11. In the simplex Ω_n , for $k = 1, 2, \dots, n$, one has

$$E_k\left(\frac{h_0}{r_0}, \frac{h_1}{r_1}, \dots, \frac{h_n}{r_n}\right) \leq \binom{n+1}{k} \left(\frac{1}{H} - 2\right)^k, \quad (23)$$

where $H = (n+1)(\sum_{i=0}^n ((h_i/r_i) + 2))^{-1}$.

Proof. In the n -dimensional simplex, for $k = 0, 1, \dots, n$, it is well known that

$$\begin{aligned} r_i &= \frac{nV}{\sum_{j=0}^n F_j - 2F_i}, \\ h_i &= \frac{nV}{F_i}. \end{aligned} \quad (24)$$

Let $x_i = F_i / \sum_{j=0}^n F_j$, $i = 0, 1, \dots, n$. Then, $E_1(\mathbf{x}) = 1$. From equalities (24), we have

$$\frac{1}{x_i} - 2 = \frac{\sum_{j=0}^n F_j - 2F_i}{F_i} = \frac{h_i}{r_i}, \quad i = 0, 1, \dots, n. \quad (25)$$

Thus from the inequality (6), it is deduced that the inequality (23) holds.

The proof of Theorem 11 is completed. $\square$

Theorem 12. Let P be an arbitrary point in the interior of n -dimensional simplex Ω_n . Let B_i be the intersection of the line A_iP with the hyperplane f_i ; further let $R_i = |A_iP|$, $d'_i = |PB_i|$, $M_i = |A_iB_i|$, $i = 0, 1, \dots, n$. Then, for $k = 1, 2, \dots, n+1$,

$$E_k \left(\frac{R_0}{d'_0}, \frac{R_1}{d'_1}, \dots, \frac{R_n}{d'_n} \right) \leq \binom{n+1}{k} \left(\frac{1}{H} - 1 \right)^k, \quad (26)$$

where $H = (n+1)(\sum_{i=0}^n (M_i/d'_i))^{-1}$.

Proof. It is easy to see that

$$\sum_{i=0}^n \frac{d'_i}{M_i} = 1, \quad \frac{R_i}{M_i} = 1 - \frac{d'_i}{M_i}, \quad i = 0, 1, \dots, n. \quad (27)$$

In the inequality (5), let $\mu = 1$, $x_i = d'_i/M_i$. Then, $1 \leq k \leq n+1$, $(1/x_i) - 1 = R_i/d'_i$, $i = 0, 1, \dots, n$. Thus, inequality (26) holds.

The proof of Theorem 12 is completed. $\square$

When $k = n+1$, inequality (26) is reduced to the following.

Corollary 13. Under the conditions of Theorem 12, one has

$$\prod_{i=0}^n R_i \leq \left(\frac{1}{H} - 1 \right)^{n+1} \prod_{i=0}^n d'_i. \quad (28)$$

Theorem 14. In the simplex Ω_n , let

$$\begin{aligned} \lambda_i &= \frac{F - 2F_i}{F_i}, \quad \mu_i = \frac{F_0^2 + \dots + F_n^2 - 2F_i^2}{F_i^2}, \\ (i &= 0, 1, \dots, n). \end{aligned} \quad (29)$$

Then,

$$\sum_{i=0}^n \prod_{j=0, j \neq i}^n \lambda_j \leq (n+1) \left(\frac{1}{H} - 2 \right)^n, \quad (30)$$

where $H = n(\sum_{i=0}^n (\lambda_i + 2))^{-1}$;

$$\sum_{i=0}^n \prod_{j=0, j \neq i}^n \mu_j \leq (n+1) \left(\frac{1}{H} - 2 \right)^n, \quad (31)$$

where $H = n(\sum_{i=0}^n (\mu_i + 2))^{-1}$.

Proof. Taking $x_i = (F_i/F)$ ($i = 0, 1, \dots, n$) and applying Corollary 3 for these $n+1$ positive real numbers $x_0, x_1, \dots, x_n$, inequality (30) follows.

Taking $x_i = F_i^2/(F_0^2 + F_1^2 + \dots + F_n^2)$ and applying Corollary 3, the desired inequality (31) follows.

The proof of Theorem 14 is completed. $\square$

Remark 15. Inequalities (30) and (31) give the reverse of the inequalities (2.10) and (2.11) in [10], respectively.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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