Multiple Positive Solutions to Multipoint Boundary Value Problem for a System of Second-Order Nonlinear Semipositone Differential Equations on Time Scales

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We study a system of second-order dynamic equations on time scales $(p_1u_1^\Delta(t) - q_1(t)u_1(t) + \lambda f_1(t, u_1(t), u_2(t)) = 0, t \in (t_1, t_n), (p_2u_2^\Delta(t) - q_2(t)u_2(t) + \lambda f_2(t, u_1(t), u_2(t)) = 0,$ satisfying four kinds of different multipoint boundary value conditions, $f_i$ is continuous and semipositone. We derive an interval of $\lambda$ such that any $\lambda$ lying in this interval, the semipositone coupled boundary value problem has multiple positive solutions. The arguments are based upon fixed-point theorems in a cone.

1. Introduction

In this paper, we consider the following dynamic equations on time scales:

\begin{align}
(p_1u_1^\Delta)(t) - q_1(t)u_1(t) + \lambda f_1(t, u_1(t), u_2(t)) &= 0, \\
&\quad t \in (t_1, t_n), \lambda > 0, \\
(p_2u_2^\Delta)(t) - q_2(t)u_2(t) + \lambda f_2(t, u_1(t), u_2(t)) &= 0,
\end{align}

satisfying one of the boundary value conditions

\begin{align}
\alpha_1u_1(t_1) - \beta_1p_1(t_1)u_1^\Delta(t_1) &= 0, \\
\gamma_1u_1(t_n) + \delta_1p_1(t_n)u_1^\Delta(t_n) &= \sum_{i=2}^{n-2} b_1u_1(t_i), \\
\alpha_2u_2(t_1) - \beta_2p_2(t_1)u_2^\Delta(t_1) &= 0, \\
\gamma_2u_2(t_n) + \delta_2p_2(t_n)u_2^\Delta(t_n) &= \sum_{i=2}^{n-2} b_2u_2(t_i),
\end{align}

\begin{align}
\alpha_1u_1(t_1) - \beta_1p_1(t_1)u_1^\Delta(t_1) &= \sum_{i=2}^{n-2} a_1u_1(t_i), \\
\gamma_1u_1(t_n) + \delta_1p_1(t_n)u_1^\Delta(t_n) &= 0, \\
\alpha_2u_2(t_1) - \beta_2p_2(t_1)u_2^\Delta(t_1) &= \sum_{i=2}^{n-2} a_2u_2(t_i), \\
\gamma_2u_2(t_n) + \delta_2p_2(t_n)u_2^\Delta(t_n) &= 0.
\end{align}
\[
gamma_1 u_1(t_n) + \delta_1 p_1(t_n) u_1'(t_n) = 0,
\]
\[
\alpha_2 u_2(t_1) - \beta_2 p_2(t_1) u_2'(t_1) = 0,
\]
\[
\gamma_2 u_2(t_n) + \delta_2 p_1(t_n) u_2'(t_n) = \sum_{i=2}^{n-2} h_i u_2(t_i),
\]
(5)

where

\[
p_i, q_i : [t_1, t_n] \to (0, +\infty)
\]

with \( p_i \in C[t_1, t_n], q_i \in C[t_1, t_n] \) for \( i = 1, 2 \);

\[
\alpha_i, \beta_i, \gamma_i, \delta_i \in [0, +\infty)
\]

with \( \alpha_i + \alpha_i \gamma_i + \beta_i \gamma_i > 0 \) for \( i = 1, 2 \),

and \( f_i \) is continuously and nonnegative functionsquad, \( a_i, b_i \in [0, +\infty) \) for \( i \in \{1, 2, \ldots, n\} \); the points \( t_i \in \mathbb{T}_k \) for \( i \in \{1, 2, \ldots, n\} \) with \( t_1 < t_2 < \cdots < t_n \).

In the past few years, the boundary value problems of dynamic equations on time scales have been studied by many authors (see [1–19] and references). Recently, multipoint boundary value problems on time scale have been studied, for instance, see [1–12].

In 2006, Anderson and Ma [1] studied the second-order multiple time-scale eigenvalue problem:

\[
\left( p y^{\nu} \right)^\Delta(t) - q(t) y(t) + \lambda h(t) f(y) = 0,
\]

\[
t \in (t_1, t_n), \quad \lambda > 0,
\]

\[
ay(t_1) - \beta p(t_1) y'(t_1) = \sum_{i=2}^{n-2} a_i y(t_i),
\]

\[
y(t_n) + \delta p(t_n) y'(t_n) = \sum_{i=2}^{n-2} b_i y(t_i),
\]
(7)

where the functions \( f : [0, +\infty) \to [0, +\infty) \) and \( h : [t_1, t_n] \to [0, +\infty) \) are continuous. The authors discuss conditions for the existence of at least one positive solution to the second-order Sturm-Liouville-type multiple eigenvalue problem on time scales.

In 2009, Feng et al. [2] studied

\[
\left( p y^{\nu} \right)^\Delta(t) - q(t) y(t) = f(t, y),
\]

\[
t \in (t_1, t_n),
\]

\[
ay(t_1) - \beta p(t_1) y'(t_1) = \sum_{i=2}^{n-2} a_i y(t_i),
\]

\[
y(t_n) + \delta p(t_n) y'(t_n) = \sum_{i=2}^{n-2} b_i y(t_i),
\]
(8)

where the functions \( f(t, y) = \sum_{j=1}^{n} c_j(t) y^{\nu_j}, c_j \in C([t_1, t_n], [0, +\infty)), \nu_j \in [0, +\infty), j = 1, 2, \ldots, n \). This paper shows the existence of multiple positive solutions for the boundary value problem on time scales.

In 2009, Topal and Yantir [3] studied the second-order nonlinear \( m \)-point boundary value problems

\[
u^\nu(t) + a(t) u^\Delta(t) + b(t) u(t) + \lambda q(t) f(t, u(t)) = 0, \quad t \in (0, 1)_T,
\]

\[
\sum_{i=1}^{m-2} \alpha_i u(\eta_i),
\]
(9)

where \( \alpha_i \geq 0, 0 < \eta_i < \eta_{i+1} < 1; \) for all \( i = 1, 2, \ldots, m - 2 \); \( a \in C([0, 1], [0, +\infty)), b \in C([0, 1], [-\infty, 0]), f, q \) are continuously and nonnegative functions. The authors deal with determining the value of \( \lambda \), and the existence of multiple positive solutions of the equation are obtained. In 2010, Yuan and Liu [4] also study the second-order \( m \)-point boundary value problems; Yuan and Liu shows the existence of multiple positive solutions if \( f \) is semipositone and superlinear.

Motivated by the above results mentioned, we study the second-order nonlinear \( m \)-point boundary value problem (1) with boundary condition (k), and nonlinear term may be singularity and semipositone.

In this paper, the nonlinear term \( f_i \) of (1) is suit to and semipositone and the superlinear case, we shall prove our two existence results for the problem (1) with (k) by using a nonlinear alternative of Leray-Schauder type and Krasnosel’skii fixed-point theorem. This paper is organized as follows. In Section 2, we start with some preliminary lemmas. In Section 3, we give the main result which state the sufficient conditions for (1) with \( m \)-point boundary value (k) to have existence of positive solutions (\( k = 2, \ldots, 5 \)).

2. Preliminaries

In this section, we state the preliminary information that we need to prove the main results.

In this paper, for our constructions, we shall consider the Banach space \( E = C[\rho(t_i), t_n] \) equipped with standard norm \( \|x\| = \max_{t \in [t_i, t_n]} |x(t)|, x \in E; \) for each \( (x, y) \in E \times E \), we write \( ||(x, y)||_1 = ||x|| + ||y||. \) Clearly, \( (E \times E, ||.||_1) \) is a Banach space. Denote by \( \phi_{11} \) and \( \phi_{12} (i = 1, 2) \), the solutions of the equation

\[
\left( p_i u_i^{\nu} \right)^\Delta(t) - q_i(t) u_i(t) = 0, \quad t \in [t_1, t_n],
\]
(10)

under the initial conditions

\[
u_i(t_1) = \beta_i, \quad p_i(t_1) u_i'(t_1) = \alpha_i,
\]

\[
u_i(t_n) = \delta_i, \quad p_i(t_n) u_i'(t_n) = -\gamma_i,
\]
(11)
respectively. So that $\phi_{i1}$ and $\phi_{i2}$ $(i = 1, 2)$ satisfy

\[
\left( p_i \phi_{i1}^\Delta \right)(t) - q_i(t) \phi_{i1}(t) = 0, \quad t \in [t_1, t_n], \quad \phi_{i1}(t_1) = \beta_i, \quad p_i(t_1) \phi_{i1}(t_1) = \alpha_i, \quad (12)
\]

\[
\left( p_i \phi_{i2}^\Delta \right)(t) - q_i(t) \phi_{i2}(t) = 0, \quad t \in [t_1, t_n], \quad \phi_{i2}(t_n) = \delta_i, \quad p_i(t_n) \phi_{i2}(t_n) = -\gamma_i,
\]

respectively. For $i = 1, 2$, set $d_i = \alpha_i \phi_{i2}(t_1) - \beta_i p_i(t_1) \phi_{i1}(t_1) = \gamma_i \phi_{i2}(t_n) + \delta_i p_i(t_n) \phi_{i1}(t_n)$, the Green's function of the corresponding homogeneous boundary value problem is defined by

\[
G_i(t, s) = \frac{1}{d_i} \begin{cases} \phi_{i2}(t) \phi_{i1}(s), & \rho(t_1) \leq s \leq t \leq t_n, \\ \phi_{i1}(t) \phi_{i2}(s), & t \leq s \leq t_n, \end{cases} \quad \text{for } i = 1, 2.
\]

From Lemmas 3.1 and 3.3 in [1], we have the following lemma.

**Lemma 1.** If $d_i \neq 0, (u_1, u_2)$ is a solution of (1) with boundary value condition (k) if and only if

\[
\begin{align*}
&u_1(t) = \lambda \int_{t_1}^{t_n} H_{1k}(t, s) f_1(s, u_1(s), u_2(s)) \Delta s, \\
&u_2(t) = \lambda \int_{t_1}^{t_n} H_{2k}(t, s) f_2(s, u_1(s), u_2(s)) \Delta s,
\end{align*}
\]

where $k = 2, \ldots, 5$, and $H_{12}(t, s)$

\[
H_{12}(t, s) = G_i(t, s) + \frac{1}{d_i} \sum_{j=2}^{n-1} b_j G_i(t_j, s) \phi_{i1}(t), \quad (i = 1, 2),
\]

$H_{13}(t, s)$

\[
H_{13}(t, s) = G_i(t, s) + \frac{1}{d_i} \sum_{j=2}^{n-1} a_j G_i(t_j, s) \phi_{i2}(t), \quad (i = 1, 2),
\]

\[
H_{14}(t, s) = G_1(t, s) + \frac{1}{d_1 - \sum_{j=2}^{n-1} b_j \phi_{i1}(t)} \sum_{j=2}^{n-1} b_j G_i(t_j, s) \phi_{i1}(t),
\]

\[
H_{24}(t, s) = G_2(t, s) + \frac{1}{d_2 - \sum_{j=2}^{n-1} a_j \phi_{i2}(t)} \sum_{j=2}^{n-1} a_j G_i(t_j, s) \phi_{i2}(t),
\]

$H_{15}(t, s)$

\[
H_{15}(t, s) = G_1(t, s) + \frac{1}{d_1 - \sum_{j=2}^{n-1} b_j \phi_{i1}(t)} \sum_{j=2}^{n-1} b_j G_i(t_j, s) \phi_{i1}(t),
\]

$H_{25}(t, s)$

\[
H_{25}(t, s) = G_2(t, s) + \frac{1}{d_2 - \sum_{j=2}^{n-1} a_j \phi_{i2}(t)} \sum_{j=2}^{n-1} a_j G_i(t_j, s) \phi_{i2}(t),
\]

From $\phi_{i1}$ is nondecreasing on $[\rho(t_1), t_n], \phi_{i2}$ is nonincreasing on $[\rho(t_1), t_n]$ (see [2, Proposition 2.3]), it is easy to verify the following inequalities:

\[
d_i G_i(t, s) \leq \phi_{i1}(t) \phi_{i2}(t),
\]

\[
d_i G_i(t, s) \leq \phi_{i1}(s) \phi_{i2}(s),
\]

\[
d_i G_i(t, s) \geq \frac{1}{\| \phi_{i1} \| \| \phi_{i2} \|} \phi_{i1}(t) \phi_{i2}(t) \phi_{i1}(s) \phi_{i2}(s).
\]

**Lemma 2.** The Green's function $G_i(t, s)$ has properties

\[
G_i(t, s) \leq G_i(t, t),
\]

\[
G_i(t, s) \leq G_i(t, s),
\]

\[
G_i(t, s) \leq G_i(t, s) \leq G_i(s, s) \leq G_i(s, s) \leq G_i(s, s).
\]
Lemma 3. For $H_{ik}(t, s), k = 2, \ldots, 5$ and $i = 1, 2$, one has the conclusions $H_{ik}(t, s) \leq C^* G_i(s, s)$ and

$$c_i \phi_{1i}(t) G_i(s, s) \leq H_{12}(t, s) \leq C^* \phi_{1i}(t), \quad (i = 1, 2),$$
$$c_i \phi_{2i}(t) G_i(s, s) \leq H_{13}(t, s) \leq C^* \phi_{2i}(t), \quad (i = 1, 2),$$
$$c_i \phi_{1i}(t) G_i(s, s) \leq H_{14}(t, s) \leq C^* \phi_{1i}(t), \quad (i = 1, 2),$$
$$c_i \phi_{1i}(t) G_i(s, s) \leq H_{15}(t, s) \leq C^* \phi_{1i}(t),$$
$$c_i \phi_{2i}(t) G_i(s, s) \leq H_{25}(t, s) \leq C^* \phi_{2i}(t),$$

where $C^* = C_1 + C_2$ and

$$C_1 = \max_{i=1,2} \left\{ 1 + \frac{\|\phi_{1i}\|}{d_i - \sum_{j=2}^{n-1} b_j \phi_{1i}(t_j)} \sum_{j=1}^{n-1} b_j \right\},$$
$$C_2 = \max_{i=1,2} \left\{ \frac{\|\phi_{1i}\| + \|\phi_{2i}\|}{d_i} + \frac{1}{d_i - \sum_{j=2}^{n-1} b_j \phi_{1i}(t_j)} \sum_{j=1}^{n-1} b_j G_i(t_j, t_j) + \frac{1}{d_i - \sum_{j=2}^{n-1} b_j \phi_{2i}(t_j)} \sum_{j=1}^{n-1} a_j G_2(t_j, t_j) \right\},$$
$$c_i = \min \left\{ \frac{d_i}{\|\phi_{1i}\| + \|\phi_{2i}\|} d_i - \sum_{j=2}^{n-1} b_j \phi_{1i}(t_j) \sum_{j=2}^{n-1} b_j G_i(t_j, t_j), \right.$$
$$\frac{d_i}{\|\phi_{1i}\| + \|\phi_{2i}\|} d_i - \sum_{j=2}^{n-1} a_j \phi_{2i}(t_j) \sum_{j=2}^{n-1} a_j G_1(t_j, t_j); \quad i = 1, 2 \right\}. $$

Proof. From Lemma 2 and

$$\frac{1}{d_i - \sum_{j=2}^{n-1} b_j \phi_{1i}(t_j)} \sum_{j=2}^{n-1} b_j G_i(t_j, s) \phi_{1i}(t) \leq \frac{1}{d_i - \sum_{j=2}^{n-1} b_j \phi_{1i}(t_j)} \sum_{j=2}^{n-1} b_j G_i(s, s) \phi_{1i}(t),$$

we have

$$H_{ik}(t, s) \leq C_i G_i(s, s) \leq C^* G_i(s, s).$$

For $k = 2$ or $3$, we have

$$H_{12}(t, s) \leq \frac{\|\phi_{12}(t)\|}{d_i} \phi_{12}(t) + \frac{1}{d_i - \sum_{j=2}^{n-1} b_j \phi_{11}(t)} \sum_{j=2}^{n-1} b_j G_i(t_j, t_j) \phi_{12}(t) \leq C^* \phi_{12}(t),$$
$$H_{13}(t, s) \leq \frac{\|\phi_{13}(t)\|}{d_i} \phi_{13}(t) + \frac{1}{d_i - \sum_{j=2}^{n-1} a_j \phi_{12}(t)} \sum_{j=2}^{n-1} a_j G_i(t_j, t_j) \phi_{12}(t) \leq C^* \phi_{13}(t),$$
$$H_{12}(t, s) \geq \frac{1}{d_i - \sum_{j=2}^{n-1} b_j \phi_{11}(t)} \sum_{j=2}^{n-1} b_j G_i(t_j, s) \phi_{11}(t) \geq C_i \phi_{11}(t),$$
$$H_{13}(t, s) \geq \frac{1}{d_i - \sum_{j=2}^{n-1} a_j \phi_{12}(t)} \sum_{j=2}^{n-1} a_j G_i(t_j, s) \phi_{12}(t) \geq C_i \phi_{12}(t).$$
\[ \sum_{j=2}^{n-1} a_j \frac{d_i}{\|\phi_i\|} G_j(t_j, t_j) G_j(s, s) \phi_2(t) \geq c_4 \phi_{12}(t) G_i(s, s). \]

So, we have
\[ c_4 \phi_{11}(t) G_i(s, s) \leq H_{12}(t, s) \leq C^* \phi_{11}(t), \quad (i = 1, 2), \]
\[ c_4 \phi_{12}(t) G_i(s, s) \leq H_{13}(t, s) \leq C^* \phi_{12}(t), \quad (i = 1, 2). \]

Since \( H_{14}(t, s) = H_{15}(t, s) = H_{23}(t, s) = H_{25}(t, s) = H_{22}(t, s) \), then we also have
\[ c_4 \phi_{11}(t) G_i(s, s) \leq H_{14}(t, s) = H_{15}(t, s) \leq C^* \phi_{11}(t), \]
\[ c_4 \phi_{22}(t) G_i(s, s) \leq H_{24}(t, s) = H_{23}(t, s) \leq C^* \phi_{22}(t), \]
\[ c_4 \phi_{12}(t) G_i(s, s) \leq H_{15}(t, s) = H_{13}(t, s) \leq C^* \phi_{12}(t), \]
\[ c_4 \phi_{21}(t) G_i(s, s) \leq H_{25}(t, s) = H_{22}(t, s) \leq C^* \phi_{21}(t). \]

The proof is complete. \( \square \)

The following theorems will play a major role in our next analysis.

**Theorem 4** (see [20]). Let \( X \) be a Banach space, and \( \Omega \subset X \) closed and convex. Assume \( U \) is a relatively open subset of \( \Omega \) with \( 0 \in U \), and let \( S : \overline{U} \to \overline{\Omega} \) be a compact, continuous map. Then either

1. \( S \) has a fixed point in \( \overline{U} \), or
2. there exists \( u \in \partial U \) and \( v \in (0, 1) \), with \( u = \nu Su \).

**Theorem 5** (see [21]). Let \( X \) be a Banach space, and let \( P \subset X \) be a cone in \( X \). Let \( \Omega_1, \Omega_2 \) be bounded open subsets of \( X \) with \( 0 \in \Omega_1 \subset \Omega \subset \Omega_2 \), and let \( S : P \to P \) be a completely continuous operator such that, either

1. \( \|Su\| \leq \|u\| \), \( u \in P \cap \partial \Omega_1 \), \( \|Su\| \geq \|u\| \), \( u \in P \cap \partial \Omega_2 \),
2. \( \|Su\| \geq \|u\| \), \( u \in P \cap \partial \Omega_1 \), \( \|Su\| \leq \|u\| \), \( u \in P \cap \partial \Omega_2 \).

Then \( S \) has a fixed point in \( P \cap \overline{\Omega_2} \setminus \Omega_1 \).

### 3. Main Results

We make the following assumptions:

\( (H_1) \) \( f_i(t, u_1, u_2) \in C([t_i, t_n] \times [0, +\infty)^2, (-\infty, +\infty)), \)

\( (H_2) \) \( f_i(t, u_1, u_2) \in C((t_i, t_n], [0, +\infty), (-\infty, +\infty)) \) and

\( (H_3) \) \( \lim_{u_i \to 0^+} \int_{|t_i|}^{\infty} \min_{t \in [0, t_i]} (f_i(t, u_1, u_2)/(u_1 + u_2)) = +\infty, \)

\( (H_4) \) \( \int_{t_i}^{\infty} G_i(s, s) g(s) \nu s < +\infty \) and \( \int_{t_i}^{\infty} G_i(s, s) f_i(s, z_1, z_2) \nu s < +\infty \) for any \( z_1 \in [0, m], m > 0 \) is any constant \( (i = 1, 2) \).

In fact, we only consider the system
\[ (p_1 x_1^1(t) - q_1(t) x_1(t) + \lambda (f_1(t, x_1(t), v_k(t)) - k g(t))) = 0, \]
\[ x_1(t) = 0, \]
\[ x_2(t) = 0, \]
\[ (p_2 x_2^1(t) - q_2(t) x_2(t) + \lambda (f_2(t, x_2(t), v_k(t)) - k g(t))) = 0, \]

with one of the boundary value conditions
\[ x_1(t_1) - x_2(t_1) = 0, \]
\[ x_1(t_n) + \delta_1 x_1(t_n) = 0, \]
\[ x_2(t_n) + \delta_2 x_2(t_n) = 0, \]
\[ x_1(t_1) - x_2(t_1) = 0, \]
\[ x_1(t_n) + \delta_1 x_1(t_n) = 0, \]
\[ x_2(t_n) + \delta_2 x_2(t_n) = 0, \]
\[ x_1(t_1) - x_2(t_1) = 0, \]
\[ x_1(t_n) + \delta_1 x_1(t_n) = 0, \]
\[ x_2(t_n) + \delta_2 x_2(t_n) = 0, \]
\[ x_1(t_1) - x_2(t_1) = 0, \]
\[ x_1(t_n) + \delta_1 x_1(t_n) = 0, \]
\[ x_2(t_n) + \delta_2 x_2(t_n) = 0, \]
\[ x_1(t_1) - x_2(t_1) = 0, \]
\[ x_1(t_n) + \delta_1 x_1(t_n) = 0, \]
\[ x_2(t_n) + \delta_2 x_2(t_n) = 0, \]
\[ x_1(t_1) - x_2(t_1) = 0, \]
\[ x_1(t_n) + \delta_1 x_1(t_n) = 0, \]
\[ x_2(t_n) + \delta_2 x_2(t_n) = 0, \]
\[ x_1(t_1) - x_2(t_1) = 0, \]
\[ x_1(t_n) + \delta_1 x_1(t_n) = 0, \]
\[ x_2(t_n) + \delta_2 x_2(t_n) = 0, \]
\[ x_1(t_1) - x_2(t_1) = 0, \]
\[ x_1(t_n) + \delta_1 x_1(t_n) = 0, \]
\[ x_2(t_n) + \delta_2 x_2(t_n) = 0, \]
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\begin{align*}
\alpha_1 v_1(t_1) - \beta_1 p_1(t_1) v_1^\nu(t_1) &= \sum_{i=2}^{n-2} a_i v_1(\eta_i), \\
\alpha_1 v_1(t_n) + \delta_1 p_1(t_n) v_1^\nu(t_n) &= \sum_{i=2}^{n-2} b_i v_1(\eta_i), \\
\alpha_2 v_2(t_1) - \beta_2 p_2(t_1) v_2^\nu(t_1) &= 0, \\
\alpha_2 v_2(t_n) + \delta_2 p_1(t_n) v_2^\nu(t_n) &= 0, \\
\gamma_1 v_1(t_1) + \delta_1 p_1(t_1) v_1^\nu(t_1) &= 0, \\
\gamma_1 v_1(t_n) + \delta_1 p_1(t_n) v_1^\nu(t_n) &= \sum_{i=2}^{n-2} a_i v_1(\eta_i), \\
\gamma_2 v_2(t_1) - \beta_2 p_2(t_1) v_2^\nu(t_1) &= 0, \\
\gamma_2 v_2(t_n) + \delta_2 p_1(t_n) v_2^\nu(t_n) &= 0,
\end{align*}

\begin{equation}
(27)
\end{equation}

where

\begin{equation}
y(t)^\ast = \begin{cases} y(t), & y(t) \geq 0, \\
0, & y(t) < 0,
\end{cases}
\end{equation}

and \(v_{ik}(t) = \lambda \int_{t_1}^{t} H_{ik}(t,s) g(s) \Delta s\). For \(k = 2, \ldots, 5\), from Lemma 1, \((v_{1k}(t), v_{2k}(t))\) is the solution of the equation

\begin{align*}
\left(p_1 v_1^\nu\right)^\Delta(t) - q_1(t) v_1(t) + \lambda g(t) &= 0, \\
\lambda > 0, \quad t_1 < t < t_n,
\end{align*}

\begin{equation}
(29)
\end{equation}

respectively, satisfying the following boundary value conditions:

\begin{align*}
\alpha_1 v_1(t_1) - \beta_1 p_1(t_1) v_1^\nu(t_1) &= 0, \\
\gamma_1 v_1(t_n) + \delta_1 p_1(t_n) v_1^\nu(t_n) &= \sum_{i=2}^{n-2} b_i v_1(\eta_i), \\
\alpha_2 v_2(t_1) - \beta_2 p_2(t_1) v_2^\nu(t_1) &= 0, \\
\gamma_2 v_2(t_n) + \delta_2 p_1(t_n) v_2^\nu(t_n) &= \sum_{i=2}^{n-2} b_i v_2(\eta_i),
\end{align*}

\begin{equation}
\end{equation}

We will show that there exists a solution \((x_{1k}, x_{2k})\) to the boundary value problem \((\tilde{k})\) of the system (26) with \(x_{ik}(t) \geq v_{ik}(t), t \in [\rho(t_1), t_n]\). If this is true, then \(u_{ik}(t) = x_{ik}(t) - v_{ik}(t)\) is a nonnegative solution (positive on \((\rho(t_1), t_n)\)) of the system (1) with the boundary value problem \((\tilde{k})\), (where \(i = 1, 2; k = 2, \ldots, 5, \tilde{k} = k + 11\)). Since for any \(t \in (t_1, t_n)\), from

\begin{align*}
\left(p_1 v_1^\nu\right)^\Delta(t) - q_1(t) v_1^\nu(t) &= \left(p_1 u_{ik} + v_{ik}\right)^\Delta(t) - q_1(t) \left(u_{ik} + v_{ik}\right)^\nu(t) \\
&= -\lambda \left(f_1(t, [x_{ik}(t) - v_{ik}(t)]^\ast, [x_{2k}(t) - v_{2k}(t)]^\ast) + g(t)\right) \\
&= -\lambda \left(f_1(t, u_{ik}(t), u_{2k}(t)) + g(t)\right),
\end{align*}

\begin{equation}
(31)
\end{equation}

we have

\begin{equation}
(32)
\end{equation}

As a result, we will concentrate our study on (26) with the boundary value problem \((\tilde{k})\).
Employing Lemma 1, we note that \((x_{1k}(t), x_{2k}(t))\) is a solution of the system (26) with boundary value \((\tilde{K})\) if and only if

\[
x_{1k}(t) = \lambda \int_{t_1}^{t_n} H_{1k}(t, s) \left( f_1(s, [x_{1k}(s) - v_{1k}(s)]^*) \right.
+ g(s) \left. \right) \Delta s,
\]
\[
x_{2k}(t) = \lambda \int_{t_1}^{t_n} H_{2k}(t, s) \left( f_2(s, [x_{1k}(s) - v_{1k}(s)]^*) \right.
+ g(s) \left. \right) \Delta s.
\]

(33)

Then, we have

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(27)

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(28)

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(29)

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(30)

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(31)

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(32)

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(34)

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(35)

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(36)

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(37)

For \( k = 2, \ldots, 5 \), from (35) and Lemma 3, we have

\[
T_{ik} (x_{1k}, x_{2k}) (t) \geq 0 \quad \text{on } [0, 1], \quad \text{for } (x_{1k}, x_{2k}) \in P_i \times P_{mn}, \text{ we have}
\]

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(38)

Thus, \( T_{ij} (P_{i1} \times P_{m}) \subset P_{i1} \). Hence \( T_2 (P_{i1} \times P_{m}) \subset P_{i1} \times P_{m} \).

When \( k = 3 \), we have

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(39)

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(40)

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(41)

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(42)

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(43)

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(44)

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(45)

\[
\frac{\partial}{\partial t} \left( x_{1k}, x_{2k} \right) - A_1 x_{1k} - A_2 x_{2k} + B_1 \left( x_{1k} - V_{1k} \right)^* + B_2 \left( x_{2k} - V_{2k} \right)^* + G(t) \leq 0.
\]

(46)
\[\lambda \int_{t_i}^{t_n} c_\ast \phi_{12}(t) G_i(s, s) \left( f_i(s, [x_{1k}(s) - v_{1k}(s)])^* \right) + g(s) \Delta s \geq \frac{c_\ast}{C}\phi_{12}(t) \lambda \int_{\rho(t_1)}^{\sigma(t_1)} C^* G_i(s, s) \left( f_i(s, [x(s) - v(s)])^* \right) + g(s) \Delta s \]

Thus, \( T_R(P_{12} \times P_{22}) \subset P_{12} \). Hence \( T_R(P_{12} \times P_{22}) \subset P_{12} \times P_{22} \).

Similarly, we also have \( T_R(P_{12} \times P_{22}) \subset P_{12} \times P_{22} \), \( T_R(P_{22} \times P_{21}) \subset P_{21} \times P_{12} \). In addition, standard arguments show that \( T_R \) is a completely continuous operator.

For simplicity, we adopt the notation: \( P_k := P_{12} \) and \( r_k := P_{12} \), then, we can write \( T_k(P_{12} \times P_{22}) \subset P_{12} \times P_{22} \), that is, \( T_k(P_{12} \times P_{22}) \subset P_{12} \times P_{22} \). We can show that \( T_k \) is a completely continuous operator. Therefore, we have

\[\lambda \int_{t_i}^{t_n} H_{ik}(t, s) \left( f_i(t, [x_{1k}(t) - v_{1k}(t)])^* \right) + g(t) \Delta s \geq \frac{c_\ast}{C}\phi_{12}(t) \lambda \int_{\rho(t_1)}^{\sigma(t_1)} C^* G_i(s, s) \left( f_i(s, [x(s) - v(s)])^* \right) + g(s) \Delta s \]

\[\lambda \int_{t_i}^{t_n} C^* G_i(s, s) \left( f_i(s, [x(s) - v(s)])^* \right) + g(s) \Delta s \geq \frac{c_\ast}{C}\phi_{12}(t) \lambda \int_{\rho(t_1)}^{\sigma(t_1)} C^* G_i(s, s) \left( f_i(s, [x(s) - v(s)])^* \right) + g(s) \Delta s \]

\[\lambda \int_{t_i}^{t_n} C^* G_i(s, s) \left( f_i(s, [x(s) - v(s)])^* \right) + g(s) \Delta s \leq \frac{c_\ast}{C}\phi_{12}(t) \lambda \int_{\rho(t_1)}^{\sigma(t_1)} C^* G_i(s, s) \left( f_i(s, [x(s) - v(s)])^* \right) + g(s) \Delta s \]

\[\lambda \int_{t_i}^{t_n} C^* G_i(s, s) \left( f_i(s, [x(s) - v(s)])^* \right) + g(s) \Delta s \leq \frac{c_\ast}{C}\phi_{12}(t) \lambda \int_{\rho(t_1)}^{\sigma(t_1)} C^* G_i(s, s) \left( f_i(s, [x(s) - v(s)])^* \right) + g(s) \Delta s \]

It follows that

\[R_0 \leq \frac{\|x_{1k}, x_{2k}\|_1}{2 \lambda c \bar{f}(R_0)}, \]

that is,

\[\bar{f}(R_0) \leq \frac{1}{2 \lambda} \leq \frac{1}{4 \lambda} = \frac{\bar{f}(R_0)}{R_0}, \]

which implies that \( \|x_{1k}, x_{2k}\|_1 \neq R_0 \). By the nonlinear alternative of Leray-Schauder type, \( T_k \) has a fixed point \( (x_{1k}, x_{2k}) \in \mathcal{U}_k \). Moreover combining (40) and the fact that \( R_0 < \varepsilon \), we obtain

\[x_{1k} = \lambda \int_{t_i}^{t_n} H_{ik}(t, s) \left( f_i(t, [x_{1k}(t) - v_{1k}(t)])^* \right) + g(t) \Delta s \]

\[x_{1k} = \lambda \int_{t_i}^{t_n} H_{ik}(t, s) \left( f_i(t, [x_{1k}(t) - v_{1k}(t)])^* \right) + g(t) \Delta s \]

\[x_{1k} = \lambda \int_{t_i}^{t_n} H_{ik}(t, s) \left( f_i(t, [x_{1k}(t) - v_{1k}(t)])^* \right) + g(t) \Delta s \]

\[x_{1k} = \lambda \int_{t_i}^{t_n} H_{ik}(t, s) \left( f_i(t, [x_{1k}(t) - v_{1k}(t)])^* \right) + g(t) \Delta s \]

\[x_{1k} = \lambda \int_{t_i}^{t_n} H_{ik}(t, s) \left( f_i(t, [x_{1k}(t) - v_{1k}(t)])^* \right) + g(t) \Delta s \]

Then \( T_k \) has a positive fixed point \( (x_{1k}, x_{2k}) \) and \( \|x_{1k}, x_{2k}\|_1 \neq R_0 < 1 \); that is, \( (x_{1k}, x_{2k}) \) is a positive solution of the boundary value problem (26) with \( x_{1k} > x_{2k} \) for \( t \in (t_1, t_n) \).

Let \( u_{1k} = u_{2k} \) with \( (\rho(t_1), t_n) \) of the boundary value problem (1).

\[\text{Theorem 7. Suppose that (H') and (H) hold. Then there exists a constant } \lambda^* > 0 \text{ such that, for any } 0 < \lambda \leq \lambda^*, (1) \text{ with boundary value condition (k) has at least one positive solution } (k = 2, \ldots, 5).\]
Proof. We fix \( k = 2, \ldots, 5 \). Let \( \Omega_1 = \{ (x_{1k}, x_{2k}) \in E \times E : \|x_{ik}\| < R_1, i = 1, 2 \} \), where \( R_1 = \max\{1, r\} \) and \( r = (C^* / c_*) \int_{t_1}^{t_2} g(s) \Delta s \). Choose

\[
\lambda^* = \min \left\{ \frac{1}{2} \left( R_1 + 1 \right)^{-1}, \frac{R_1}{2r} \right\},
\]

(48)

where \( R = \int_{t_1}^{t_2} C^* G_i(s, s) (\max_{0 \leq z \leq R} f_i(s, z_1, z_2) + g(s)) \Delta s \) and \( R \geq 0 \).

Then for any \( (x_{1k}, x_{2k}) \in (P_{1(k-1)} \times P_{2(k-1)}) \cap \partial \Omega_1 \), \( x_{ik}(s) \leq x_{ik} \leq \|x_{ik}\| \leq R_1 \) (\( i = 1, 2 \)) and for \( 0 < \lambda \leq \lambda^* \), we have

\[
\|T_{ik}(x_{1k}, x_{2k})(t)\| \leq \lambda \int_{t_1}^{t_2} C^* G_i(s, s) \left( f_i(s, [x_{ik}(s) - v_{ik}(s)]^*) + [x_{2k}(s) - v_{2k}(s)]^* + g(s) \right) \Delta s \\
\leq \lambda \int_{t_1}^{t_2} C^* G_i(s, s) \left( \max_{0 \leq z \leq R_1} f_i(s, z_1, z_2) + g(s) \right) \Delta s \\
\leq \lambda R \\
\leq \frac{R_1}{2}.
\]

(49)

This implies

\[
\|T_{ik}(x_{1k}, x_{2k})(t)\| \leq R_1 \leq \|(x_{1k}, x_{2k})(t)\|, \\
(x_{1k}, x_{2k}) \in (P_{1(k-1)} \times P_{2(k-1)}) \cap \partial \Omega_1.
\]

(50)

Choose a constant \( N > 1 \) such that

\[
\lambda N^2 y \left( \frac{c_*}{2(\|\phi_1\| + \|\phi_2\|)} \int_{\theta_1}^{\theta_2} G_1(s, s) \phi_1(s) \phi_2(s) \Delta s \right) \geq 1,
\]

(51)

where \( y = \min_{\theta_1 \leq \theta_2} \int_{\theta_1}^{\theta_2} \phi_1(t) \phi_2(t) \Delta t \).

By assumption (H_3) and (H_4), there exists a constant \( B > R_1 \) such that

\[
\frac{f_i(t, z_1, z_2)}{z_1 + z_2} > N, \quad \text{that is,} \quad f_i(t, z_1, z_2) > N (z_1 + z_2),
\]

for \( t \in [\theta_1, \theta_2] \), \( z_1 + z_2 > B \) (\( i = 1, 2 \)).

(52)

Choose \( R_3 = \max\{R_1 + 1, 2B, 2C^*(B + 1)/c_* y\} \) and let \( \Omega_2 = \{ (x_{1k}, x_{2k}) \in E \times E : \|x_{ik}\| < R_3, i = 1, 2 \} \). We note that \( x(t) \geq (c_* / C^*) \phi_1(t) \|x\| \) for all \( x \in P_{ij} \), by Lemma 3, we have \( H_{ik}(t, s) \leq (C^* / c_*) (x(t) / \|x\|) \). Then for any \( (x_{1k}, x_{2k}) \in (P_{1(k-1)} \times P_{2(k-1)}) \cap \partial \Omega_2 \), we have \( \|x_{ik}\| = R_3 \) or \( \|x_{ik}\| = R_3 \).

Without loss of generality let \( \|x_{ik}\| = R_3 \), so we have

\[
x_{ik}(t) - v_{ik}(t) = x_{ik}(t) - \lambda \int_{t_1}^{t_2} H_{ik}(t, s) g(s) \Delta s \\
\geq x_{ik}(t) - \lambda \int_{t_1}^{t_2} C^* \frac{x_{ik}(t)}{\|x_{ik}\|} g(s) \Delta s \\
= x_{ik}(t) - \lambda \int_{t_1}^{t_2} C^* \frac{x_{ik}(t)}{\|x_{ik}\|} g(s) \Delta s \\
\geq x_{ik}(t) - \frac{x_{ik}(t)}{\|x_{ik}\|} \lambda r \\
\geq \frac{1 - \lambda r}{R_3} x_{ik}(t) \\
\geq \frac{1}{2} x_{ik}(t) \geq 0, \quad t \in [\rho(t_1), t_2].
\]

(53)

Thus

\[
\min_{\theta_1 \leq s \leq \theta_2} \left\{ \|x_{ik}(t) - v_{ik}(t)\|^* + \|x_{2k}(t) - v_{2k}(t)\|^* \right\} \\
\geq \min_{\theta_1 \leq s \leq \theta_2} \|x_{ik}(t) - v_{ik}(t)\| \\
\geq \min_{\theta_1 \leq s \leq \theta_2} \left\{ \frac{c_*}{2C^*} \phi_1(t) \|x_{ik}\|, \frac{c_*}{2C^*} \phi_2(t) \|x_{ik}\| \right\} \\
= \frac{c_*}{2C^*} R_3 \min_{\theta_1 \leq s \leq \theta_2} \{ \phi_1(t) \phi_2(t) \} \geq B + 1 > B.
\]

(54)

Now since \( B > R_1 \), it follows that

\[
T_{ik}(x_{1k}, x_{2k})(t) = \lambda \int_{t_1}^{t_2} H_{ik}(t, s) f_i(s, [x_{ik}(s) - v_{ik}(s)]^*) + [x_{2k}(s) - v_{2k}(s)]^* + g(s) \Delta s \\
\geq \lambda \int_{t_1}^{t_2} H_{ik}(t, s) f_i(s, [x_{ik}(s) - v_{ik}(s)]^*), \\
\geq \lambda \int_{t_1}^{t_2} H_{ik}(t, s) f_i(s, [x_{ik}(s) - v_{ik}(s)]^*), \\
\geq \lambda \int_{t_1}^{t_2} H_{ik}(t, s) f_i(s, [x_{ik}(s) - v_{ik}(s)]^*) \Delta s \\
\geq \lambda \int_{t_1}^{t_2} H_{ik}(t, s) f_i(s, [x_{ik}(s) - v_{ik}(s)]^*) \Delta s.
\]
\[ \geq \lambda \int_{t_1}^{t_2} H_{ik}(t,s) N \left( x_{ik}(s) - v_{ik}(s) \right) \Delta s + \left[ x_{ik}(s) - v_{ik}(s) \right] \Delta s \]

\[ \geq \lambda \int_{t_1}^{t_2} H_{ik}(t,s) N \left( x_{ik}(s) - v_{ik}(s) \right) \Delta s \]

\[ \geq \lambda \int_{t_1}^{t_2} c_s \min_{\theta_1 \leq s \leq \theta_2} \left\{ \phi_{ik}(t), \phi_{2k}(t) \right\} G_i(s,s) N \left( x_{ik}(s) - v_{ik}(s) \right) \Delta s \]

\[ \geq \lambda \int_{t_1}^{t_2} c_s \min_{\theta_1 \leq s \leq \theta_2} \left\{ \phi_{ik}(t), \phi_{2k}(t) \right\} G_i(s,s) N \frac{1}{2} \left( x_{ik}(s) \right) ds \]

\[ \geq \lambda \min_{\theta_1 \leq s \leq \theta_2} \left\{ \phi_{ik}(t), \phi_{2k}(t) \right\} \]

\[ \times \int_{t_1}^{t_2} c_s G_i(s,s) \frac{\theta_1}{\left( \phi_{i1} + \phi_{i2} \right)} \phi_{i1}(s) \phi_{i2}(s) \left\| x_{ik} \right\| \Delta s \]

\[ \geq \lambda N \frac{\theta_1}{\left( \phi_{i1} + \phi_{i2} \right)} \int_{t_1}^{t_2} G_i(s,s) \phi_{i1}(s) \phi_{i2}(s) \Delta s \]

\[ R_2, \quad t \in \left[ \theta_1, \theta_2 \right]. \]

This implies

\[ \left\| T_k \left( x_{1k}, x_{2k} \right) \right\|_1 \geq \left\| \left( x_{1k}, x_{2k} \right) \right\|_1, \]

\[ \left( x_{1k}, x_{2k} \right) \in (P_{1(k-1)} \times P_{2(k-1)}) \cap \partial \Omega_2. \]

For the Krasnosel’skiǐ's fixed point theorem, one deduces that

\[ T_k \text{ has a fixed point } (x_{1k}, x_{2k}) \text{ with } R_1 < \left( \left\| x_{1k} \right\| + \left\| x_{2k} \right\| \right) < R_2 \iff R_1 < \left\| x_{1k} \right\| + \left\| x_{2k} \right\| < R_2. \]

Since \( r \leq R_1 < \left\| x_{1k} \right\| < R_2 \) for \( i = 1, 2 \), then

\[ x_{ik}(t) - v_{ik}(t) = x_{ik}(t) - \lambda \int_{t_1}^{t_2} H_{ik}(t,s) g(s) \Delta s \]

\[ \geq x_{ik}(t) - \lambda \int_{t_1}^{t_2} C_s^2 x_{ik}(t) \frac{1}{c_s} \left\| x_{ik} \right\| g(s) \Delta s \]

\[ = \frac{1}{1 + \lambda} x_{ik}(t) \]

\[ \geq x_{ik}(t) - \lambda x_{ik}(t) \]

\[ = (1 - \lambda) x_{ik}(t) \]

\[ \geq (1 - \lambda) \frac{\phi_{i1}(t) \phi_{i2}(t)}{\phi_{i1} + \phi_{i2}} \left\| x_{ik} \right\| \]

\[ > 0, \quad t \in (\rho(t_1), t_2). \]

Thus \( (x_{1k}, x_{2k}) \) is a positive solution of the boundary value problem \( (26) \) with \( x_{ik}(t) > v_{ik}(t) \) for all \( (\rho(t_1), t_2) \).

Let \( u_{ik}(t) = x_{ik}(t) - v_{ik}(t) \geq 0 \) for \( i = 1, 2 \), then \((u_{1k}, u_{2k})\) is a nonnegative solution (positive on \( (\rho(t_1), t_2) \)) of the boundary value problem \( (1) \).

Since condition \( (H_4) \) implies conditions \( (H_1^*) \) and \( (H_4) \) then from the proof of Theorems 6 and 7, we immediately have the following theorem.

**Theorem 8.** Suppose that \((H_4)-(H_9)\) hold. Then \( (1) \) with boundary value condition \((k)\) has at least two positive solutions for \( \lambda > 0 \) sufficiently small \((k = 2, \ldots, 5)\).

In fact with \( 0 < \lambda < \min \{ \lambda^*, \lambda^* \} \) then \( (1) \) with boundary value condition \((k)\) has at least two positive solutions.

**Remark 9.** In Theorems 6–8, we use the assumption condition 16. If we have not the condition 16, that is, \( a_i = b_i = 0 \), then the system \((1) \) and boundary condition \((k)\) are

\[ \begin{aligned}
(p_1 u_1^\Delta (t) - q_1 (t) u_1(t) + \lambda f_1 (t, u_1(t), u_2(t))) &= 0, \\
(p_2 u_2^\Delta (t) - q_2 (t) u_2(t) + \lambda f_2 (t, u_1(t), u_2(t))) &= 0,
\end{aligned} \]

\[ t \in (t_1, t_n), \quad \lambda > 0, \]

\[ \begin{aligned}
\alpha_1 u_1 (t_1) - \beta_1 p_1 (t_1) u_1^\Delta (t_1) &= 0, \\
\gamma_1 u_1 (t_n) + \delta_1 p_1 (t_n) u_1^\Delta (t_n) &= 0, \\
\alpha_2 u_2 (t_1) - \beta_2 p_2 (t_1) u_2^\Delta (t_1) &= 0, \\
\gamma_2 u_2 (t_n) + \delta_2 p_2 (t_n) u_2^\Delta (t_n) &= 0.
\end{aligned} \]

From Lemma 2, an argument similar to those in Theorems 6–8 yields the following theorems.

**Theorem 10.** Suppose that \((H_4) \) and \((H_5) \) hold. Then there exists a constant \( \lambda^* > 0 \) such that, for any \( 0 < \lambda < \lambda^* \), the boundary value problem \( (58) \) has at least one positive solution.

**Theorem 11.** Suppose that \((H_4^*) \) and \((H_5)-(H_9) \) hold. Then there exists a constant \( \lambda^* > 0 \) such that, for any \( 0 < \lambda < \lambda^* \), the boundary value problem \( (58) \) has at least one positive solution.

**Theorem 12.** Suppose that \((H_1)-(H_9) \) hold. Then the boundary value problem \( (58) \) has at least two positive solutions for \( \lambda > 0 \) sufficiently small.

### 4. Example

To illustrate the usefulness of the results, we give some examples.

**Example 13.** Consider the boundary value problem

\[ \begin{aligned}
u'' - u &= - \lambda \left( (u + v)^2 + \frac{1}{(t - t^2)^{3/2}} \cos (2 \pi (u + v)) \right), \\
-1 < t < 1, \quad \lambda > 0, \\
v'' - v &= - \lambda \left( (t - 1)^2 + v^2 + \frac{1}{(t - t^2)^{3/2}} \sin (2 \pi u) \right), \\
u (1) &= v (1) = 0, \quad u (1) = au (0), \quad v (1) = bv (0),
\end{aligned} \]

where \( a > 1 \). Then if \( \lambda > 0 \) is sufficiently small, \( (59) \) has a positive solution \( u \) with \( u(t) > 0 \) for \( t \in (0, 1) \).
To see this, we will apply Theorem 7 with
\[
f_1(t, u, v) = (u + v)^a + \frac{1}{(t^2 - t_4)^{1/4}} \cos(2\pi (u + v)),
\]
\[
f_2(t, u, v) = (u - 1)^2 + v^2 + \frac{1}{(t^2 - t_4)^{1/4}} \sin(2\pi u),
\]
\[
g_1(t) = g_2(t) = g(t) = \frac{1}{(t^2 - t_4)^{1/4}}.
\]

Clearly for \( t \in (0, 1) \),
\[
f_i(t, u, v) + g(t) > 0, \quad \text{for } t \in (0, 1) \quad i = 1, 2,
\]
\[
\lim_{u+v \to +\infty} \inf_{u+v} \frac{f_i(t, u, v)}{u + v} = +\infty \quad \text{for all } t \in [\theta_1, \theta_2] \subset (0, 1).
\]

Now (H_1^*, H_2), and (H_4) hold. We note that the boundary condition of (59) is in accord with (4), and from [1], we have
\[
\phi_{11} = \phi_{21} = \frac{e^{t+1} - e^{-t-1}}{2}, \quad \phi_{12} = \phi_{22} = \frac{e^{t+1} - e^{-t-1}}{2},
\]
\[
d_1 = d_2 = \sinh(2).
\]

Then
\[
G_i(t, s) = G_i(t, s)
\]
\[
= \frac{1}{d_1} \left( \phi_{12}(t) \phi_{12}(s), \quad \rho(t_1) \leq s \leq t \leq t_n, \right.
\]
\[
H_{14}(t, s) = G_i(t, s) + \frac{1}{d_1 - a\phi_{11}(0)} a G_i(0, s) \phi_{11}(t),
\]
\[
H_{24}(t, s) = G_i(t, s) + \frac{1}{d_2 - b\phi_{22}(0)} b G_i(0, s) \phi_{22}(t).
\]

Note \( r = \frac{1}{t} (C^{1/2} / c) g(s) ) \Delta s. \) Let \( R_1 = r + 1 \) and we have
\[
R = \left[ \int_{R_1}^{t} C^* G_i(s, s) \left( \max_{0 \leq z_1, z_2 \leq R_1} f_i(s, z_1, z_2) + g(s) \right) \Delta s \right.
\]
\[
\leq \left[ \int_{R_1}^{t} C^* G_i(s, s) \left( 2^{a+2} R_1^{a+2} + \frac{2}{(s^2 - s^2)^{1/4}} \right) \Delta s \right.
\]
\[
\leq \left[ \int_{0}^{1} C^* \frac{e^4}{4} \left( 2^{a+2} R_1^{a+2} + \frac{2}{(s^2 - s^2)^{1/4}} \right) \Delta s \right.
\]
\[
= \left[ \int_{0}^{1} C^* \frac{e^4}{2} \left( 2^{a+2} R_1^{a+2} + \frac{2}{(s^2 - s^2)^{1/4}} \right) \Delta s \right.
\]
\[
\leq \left[ \int_{0}^{1} C^* \frac{e^4}{2} \left( R_1^{a+2} + \frac{2}{(s - s^2)^{1/2}} \right) \Delta s \right.
\]
\[
\leq 2^{a+1} C^* \frac{e^4}{2} \left( R_1^{a+2} + \pi \right).
\]

Also let
\[
\lambda^* = \min \left\{ 1, \frac{R_1}{2^{a+2} C^* e^4 \left( R_1^{a+2} + \pi \right)^{-1}}, \frac{R_1}{2^{a+1} C^* e^4 \left( R_1^{a+2} + \pi \right)} \right\}.
\]

Now, if \( \lambda < \lambda^* \), Theorem 7 guarantees that (59) has a positive solutions \((u, v)\) with \(|u| \geq 1\) and \(|v| \geq 1\).

**Example 14.** Consider the boundary value problem:
\[
\left( p_i u_i \right) \Delta (t) - q_i (t) u_i (t)
\]
\[
= -\lambda \left( e^{u_i + u_i^2 + 7 \cos(2\pi u_i)} \right),
\]
\[
t_1 < t < t_n, \quad \lambda > 0,
\]
\[
\left( p_i u_i \right) \Delta (t) - q_i (t) u_i (t)
\]
\[
= -\lambda \left( (u_i - 1)^2 + u_i^2 + 5 \sin(2\pi u_i) \right)
\]
satisfying one of the boundary value conditions \((k), (k = 2, \ldots, 5)\).

Then if \( \lambda > 0 \) is sufficiently small, (66) has two solutions \((u_{i1}, u_{i2})\), \((u_{i3}, u_{i4})\) with \(u_i(t) > 0\) for \( t \in (0, 1), i, j = 1, 2\).

To see this, we will apply Theorem 8 with
\[
f_1(t, u, v) = e^{u_i + u_i^2 + 7 \cos(2\pi u_i)},
\]
\[
f_2(t, u, v) = (u_i - 1)^2 + u_i^2 + 5 \sin(2\pi u_i),
\]
\[
g_1(t) = g_2(t) = g(t) = 8.
\]

Clearly, for \( t \in (0, 1) \),
\[
f_i(t, u, v) + g(t) \geq 0, \quad \text{for } 0 \leq t \leq 1, \quad 0 \leq u_i \leq e, \quad i = 1, 2,
\]
\[
\lim_{u+v \to +\infty} \inf_{u+v} \frac{f_i(t, u, v)}{u + v} = +\infty, \quad i = 1, 2.
\]

Now (H_1^*, H_2) hold. Let \( \delta = 1/100, \varepsilon = 1/8 \), and we have
\[
f_i(t, u, v) \geq \delta f_i(t, 0, 0), \quad \text{for } 0 \leq t \leq 1, \quad 0 \leq u_i \leq \varepsilon, \quad i = 1, 2.
\]

Furthermore let
\[
\int f_i(t, u_1, u_2) + g(t) \Delta t = \max_{u_1, u_2} \int f_i(t, u_1, u_2) + g(t) \Delta t,
\]
\[
= \frac{1}{4c \int \left( \frac{1}{3 \varepsilon + 1/3} > \frac{1}{352c} \right).
\]

Let \( \lambda = 1/352c \). Now, if \( 0 < \lambda < \lambda \) then \( 0 < \lambda < e/4c \) and Theorem 6 guarantees that (66) has positive solutions \((u_{i1}, u_{i2})\) with \(\|u_i\| \leq (1/8) \) \((j = 1, 2)\).
Next note $r = 8C^*\Delta(t_n - t_1)/c_*$ and let $R_1 = r + 2$ so we have

$$R = \int_0^1 C^*G_i(s, s)\left(\max_{0 \leq z_1, z_1} f_i(s, z_1, z_2) + g(s)\right)\Delta s$$

$$\leq \int_0^1 C^*G_i(s, s)\left(e^{R_1} + 2R_1^2 + 7 + 8\right)\Delta s$$

$$\leq \int_0^1 C^*G_i(s, s)\Delta \left(e^{R_1} + 2R_1^2 + 15\right)$$

$$\leq \left(e^{R_1} + 2R_1^2 + 15\right)c.$$

Also let

$$\lambda' = \min\left\{1, \frac{R_1}{2(e^{R_1} + 2R_1^2 + 15)c}, \frac{R_1}{2r}\right\}.$$  \hspace{1cm} (72)

Now, if $\lambda < \lambda'$, Theorem 7 guarantees that (59) has a positive solutions $(u_{21}, u_{22})$ with $\|u_{2j}\| \geq 2, j = 1, 2$.

Thus, if $\lambda < \min(\lambda, \lambda')$, Theorem 8 guarantees that (66) has two solutions $(u_{11}, u_{12})$ and $(u_{21}, u_{22})$ with $u_{ij} > 0$ for $t \in (0, 1), i, j = 1, 2$.

References


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