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Research Article

A Novel Characteristic Expanded Mixed Method for Reaction-Convection-Diffusion Problems

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A novel characteristic expanded mixed finite element method is proposed and analyzed for reaction-convection-diffusion problems. The diffusion term $\nabla \cdot (a(\mathbf{x},t)\nabla u)$ is discretized by the novel expanded mixed method, whose gradient belongs to the square integrable space instead of the classical $\mathbf{H}(\mathrm{div};\Omega)$ space and the hyperbolic part $d(\mathbf{x})(\partial u/\partial t) + \mathbf{c}(\mathbf{x},t)\cdot\nabla u$ is handled by the characteristic method. For a priori error estimates, some important lemmas based on the novel expanded mixed projection are introduced. The fully discrete error estimates based on backward Euler scheme are obtained. Moreover, the optimal a priori error estimates in L^2- and H^1-norms for the scalar unknown u and a priori error estimates in $(L^2)^2$-norm for its gradient λ and its flux σ (the coefficients times the negative gradient) are derived. Finally, a numerical example is provided to verify our theoretical results.

1. Introduction

In this paper, we consider the following reaction-convection-diffusion problems:

$$\begin{aligned} d(\mathbf{x})\frac{\partial u}{\partial t} + \mathbf{c}(\mathbf{x},t)\cdot\nabla u - \nabla\cdot(a(\mathbf{x},t)\nabla u) + R(\mathbf{x},t)u \\ = f(\mathbf{x},t), \quad (\mathbf{x},t)\in\Omega\times J, \\ u(\mathbf{x},t) = 0, \quad (\mathbf{x},t)\in\partial\Omega\times\overline{J}, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x}\in\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded convex polygonal domain in $\mathbb{R}^2$ with *Lipschitz* continuous boundary $\partial\Omega$ and $J = (0,T]$ is the time interval with $0 < T < \infty$. The initial value $u_0(\mathbf{x})$ and the source term $f(\mathbf{x},t)$ are given functions. Throughout this paper, we assume that the accumulation, diffusion, and reaction coefficients, $d = d(\mathbf{x})$, $a = a(\mathbf{x},t)$, and $R = R(\mathbf{x},t)$, satisfy

$$\begin{aligned} &\mathbf{H}_1: \ 0 < d_0 \le d(\mathbf{x}) \le d_1 < +\infty, \\ &\mathbf{H}_2: \ 0 < a_0 \le a(\mathbf{x},t) \le a_1 < +\infty, \qquad |a_t(\mathbf{x},t)| \le \overline{a}_1, \\ &\mathbf{H}_3: \ 0 \le R_0 \le R(\mathbf{x},t) \le R_1 < +\infty, \qquad |R_t(\mathbf{x},t)| \le \overline{R}_1, \end{aligned} \tag{2}$$

and the bounded vector $\mathbf{c}(\mathbf{x},t) = (c_1(\mathbf{x},t), c_2(\mathbf{x},t))$ satisfies

$$\mathbf{H}_4: \ 0 < b_0 \le \left(\sum_{i=1}^{2} c_i^2(\mathbf{x},t)\right)^{1/2} \le b_1 < +\infty. \tag{3}$$

Reaction-convection-diffusion equations are a class of important evolution partial differential equations and have a lot of applications in many physical problems, such as the infiltration of liquid, the proliferation of gas, the conduction of heat, and the spread of impurities in semiconductor materials. In recent years, a lot of numerical methods, such as mixed finite element methods [1–3], Pod methods [4, 5], characteristic-mixed covolume methods [6], space-time discontinuous Galerkin methods [7, 8], least-squares finite element methods [9, 10], nonconforming finite element method [11], conforming rectangular element method [12], and two-grid expanded mixed methods [13–16], have been studied for reaction-convection-diffusion equations.

In 1994, Chen [17, 18] proposed an expanded mixed finite element method for second-order linear elliptic equation. Compared to standard mixed element methods, the expanded mixed method can approximate three variables simultaneously, namely, the scalar unknown, its gradient, and its flux (the tensor coefficient times the negative gradient). From then on, the expanded mixed method was applied

to many evolution equations [2, 19, 20], and some new numerical methods based on the Chen's expanded mixed method were proposed, for example, expanded mixed covolume method [21], expanded mixed hybrid methods [22], positive definite expanded mixed method [23], and so on. From the above literature on the study of expanded mixed method, we can find that all papers were studied based on Chen's expanded mixed method [17, 18].

In 2011, we developed a new expanded mixed finite element method [24] based on the mixed weak formulations [25–27]. In this paper, we develop and analyze a novel characteristic expanded mixed finite element method, which combines the novel expanded mixed method [24] applied to approximating the diffusion term and the characteristic method that handled the hyperbolic part, for reaction-convection-diffusion equations. The gradient for our method belongs to the square integrable space instead of the classical $\mathbf{H}(\mathrm{div};\Omega)$ space of Chen's expanded mixed method. We derive a priori error estimates based on backward Euler method. Moreover, we prove the optimal a priori error estimates in L^2- and H^1-norms for the scalar unknown u and a priori error estimates in $(L^2)^2$-norm for its gradient λ and its flux σ (the coefficients times the negative gradient).

Throughout this paper, C will denote a generic positive constant which does not depend on the spatial mesh parameter h and time discretization parameter Δt. At the same time, we denote the natural inner product in $L^2(\Omega)$ or $(L^2(\Omega))^2$ by $(\cdot,\cdot)$ with the corresponding norm $\|\cdot\|$. The other notations and definitions of Sobolev spaces as in [28] are used.

2. Novel Characteristic Expanded Mixed System

Introducing the two auxiliary variables $\lambda = \nabla u$, $\sigma = -(a(\mathbf{x},t)\nabla u) = -(a(\mathbf{x},t)\lambda)$, we obtain the following first-order system for (1):

$$\begin{aligned} d\frac{\partial u}{\partial t} + \mathbf{c}\cdot\nabla u + \nabla\cdot\sigma + Ru &= f(\mathbf{x},t), \\ \lambda - \nabla u &= 0, \\ \sigma + a\lambda &= 0. \end{aligned} \tag{4}$$

Let the characteristic direction corresponding to the hyperbolic part $d\partial u/\partial t + \mathbf{c}\cdot\nabla u$ be denoted by $\tau = \tau(\mathbf{x})$, which is subject to

$$\frac{\partial}{\partial\tau(\mathbf{x})} = \frac{d(\mathbf{x})}{\psi(\mathbf{x},t)}\frac{\partial}{\partial t} + \frac{\mathbf{c}(\mathbf{x})}{\psi(\mathbf{x},t)}\cdot\nabla, \tag{5}$$

where $\psi(\mathbf{x},t) = (d^2(\mathbf{x}) + |\mathbf{c}(\mathbf{x},t)|^2)^{1/2}$.

Then the first-order system (4) can be rewritten as

$$\begin{aligned} \psi\frac{\partial u}{\partial\tau} + \nabla\cdot\sigma + Ru &= f(\mathbf{x},t), \\ \lambda - \nabla u &= 0, \\ \sigma + a\lambda &= 0. \end{aligned} \tag{6}$$

Using the similar method to the one in [29], the expanded mixed weak formulation for problem (1) is to find $\{u,\lambda,\sigma\}$: $[0,T]\mapsto W\times\mathbf{V}\times\mathbf{X}$ such that

$$\left(\psi\frac{\partial u}{\partial\tau},v\right) + (\nabla\cdot\sigma,v) + (Ru,v) = (f,v), \quad \forall v\in W, \tag{7a}$$

$$(\lambda,\mathbf{w}) + (u,\nabla\cdot\mathbf{w}) = 0, \quad \forall\mathbf{w}\in\mathbf{V}, \tag{7b}$$

$$(\sigma,\mathbf{z}) + (a\lambda,\mathbf{z}) = 0, \quad \forall\mathbf{z}\in\mathbf{X}, \tag{7c}$$

where $W = L^2(\Omega)$ or $W = \{w\in L^2(\Omega)|w|_{\partial\Omega} = 0\}$, $\mathbf{V} = \mathbf{H}(\mathrm{div};\Omega) = \{\mathbf{v}\in(L^2(\Omega))^2 \mid \nabla\cdot\mathbf{v}\in L^2(\Omega)\}$, and $\mathbf{X} = (L^2(\Omega))^2$.

In this paper, we propose and discuss a novel characteristic expanded mixed method. The new characteristic expanded mixed weak formulation is to find $\{u,\lambda,\sigma\}$: $[0,T]\mapsto H_0^1\times(L^2(\Omega))^2\times(L^2(\Omega))^2$ such that

$$\left(\psi\frac{\partial u}{\partial\tau},v\right) - (\sigma,\nabla v) + (Ru,v) = (f,v), \quad \forall v\in H_0^1, \tag{8a}$$

$$(\lambda,\mathbf{w}) - (\nabla u,\mathbf{w}) = 0, \quad \forall\mathbf{w}\in\left(L^2(\Omega)\right)^2, \tag{8b}$$

$$(\sigma,\mathbf{z}) + (a\lambda,\mathbf{z}) = 0, \quad \forall\mathbf{z}\in\left(L^2(\Omega)\right)^2. \tag{8c}$$

For approximating the solution at time $t_n = n\Delta t$, the characteristic derivative will be approximated by

$$\hat{\mathbf{x}}_n = \mathbf{x} - \frac{\Delta t}{d(\mathbf{x})}c(\mathbf{x},t_n), \tag{9}$$

and then we have the following approximation:

$$\begin{aligned} \psi(\mathbf{x},t_n)\left.\frac{\partial u}{\partial\tau}\right|_{t_n} &\approx \psi(\mathbf{x},t_n)\frac{u(\mathbf{x},t_n) - u(\hat{\mathbf{x}}_n,t_{n-1})}{\sqrt{(\mathbf{x}-\hat{\mathbf{x}}_n)^2 + (\Delta t)^2}} \\ &= d(\mathbf{x})\frac{u(\mathbf{x},t_n) - u(\hat{\mathbf{x}}_n,t_{n-1})}{\Delta t}. \end{aligned} \tag{10}$$

Let $(V_h,\mathbf{W}_h)\subset(H_0^1,(L^2(\Omega))^2)$ be defined by the following finite element pair $P_1 - P_0^2$ [25, 26]:

$$\begin{aligned} V_h &= \left\{v_h\in C^0(\Omega)\cap H_0^1 \mid v_h\in P_1(K), \forall K\in\mathscr{K}_h\right\}, \\ \mathbf{W}_h &= \left\{\mathbf{w}_h = (w_{1h},w_{2h})\in\left(L^2(\Omega)\right)^2 \mid w_{1h},w_{2h}\in P_0(K), \right. \\ &\qquad \left.\forall K\in\mathscr{K}_h\right\}. \end{aligned} \tag{11}$$

Now the novel characteristic expanded mixed finite element procedure for (8a), (8b), and (8c) is to find $\{u_h^n, \lambda_h^n, \sigma_h^n\} : [0,T] \mapsto V_h \times \mathbf{W}_h \times \mathbf{W}_h$ satisfying

$$\left(d\frac{u_h^n - \widehat{u}_h^{n-1}}{\Delta t}, v_h\right) - \left(\sigma_h^n, \nabla v_h\right) + \left(Ru_h^n, v_h\right) = \left(f^n, v_h\right), \quad \forall v_h \in V_h, \tag{12a}$$

$$\left(\lambda_h^n, \mathbf{w}_h\right) - \left(\nabla u_h^n, \mathbf{w}_h\right) = 0, \quad \forall \mathbf{w}_h \in \mathbf{W}_h, \tag{12b}$$

$$\left(\sigma_h^n, \mathbf{z}_h\right) + \left(a^n \lambda_h^n, \mathbf{z}_h\right) = 0, \quad \forall \mathbf{z}_h \in \mathbf{W}_h, \tag{12c}$$

where $u_h^n = u_h(t_n), \widehat{u}_h^{n-1} = u_h(\widehat{\mathbf{x}}_n, t_{n-1}) = u_h(\mathbf{x} - (\Delta t/d(\mathbf{x}))c(\mathbf{x}, t_n), t_{n-1})$.

Remark 1. From [25, 26], we find that $(V_h, \mathbf{W}_h)$ satisfies the so-called discrete Ladyzhenskaya-Babuska-Brezzi condition.

Remark 2. Compared to the scheme (7a), (7b), and (7c) based on Chen's expanded mixed element method, the gradient in the scheme (8a), (8b), and (8c) belongs to the simple square integrable space instead of the classical $\mathbf{H}(\mathrm{div};\Omega)$ space. For $\mathbf{H}(\mathrm{div};\Omega) \subset (L^2(\Omega))^2$, we easily find that our method reduces the regularity requirement on the gradient solution $\lambda = \nabla u$.

Remark 3. Based on finite element space $(V_h, \mathbf{W}_h)$ in (11), the number of total degrees of freedom for our scheme ((12a), (12b), and (12c)) is less than that for the scheme in [29]. By the same discussion as Remark 1 in [30], we can obtain the detailed analysis for degrees of freedom.

3. Some Lemmas and Error Estimates

3.1. Novel Expanded Mixed Projection and Lemmas. We first introduce the novel expanded mixed elliptic projection [24] associated with our equations to derive a priori error estimates for the proposed method.

Let $(\widetilde{u}_h, \widetilde{\lambda}_h, \widetilde{\sigma}_h) : [0,T] \to V_h \times \mathbf{W}_h \times \mathbf{W}_h$ be given by the following mixed relations:

$$\left(\sigma - \widetilde{\sigma}_h, \nabla v_h\right) = 0, \quad \forall v_h \in V_h, \tag{13a}$$

$$\left(\lambda - \widetilde{\lambda}_h, \mathbf{w}_h\right) - \left(\nabla\left(u - \widetilde{u}_h\right), \mathbf{w}_h\right) = 0, \quad \forall \mathbf{w}_h \in \mathbf{W}_h, \tag{13b}$$

$$\left(\sigma - \widetilde{\sigma}_h, \mathbf{z}_h\right) + \left(a\left(\lambda - \widetilde{\lambda}_h\right), \mathbf{z}_h\right) = 0, \quad \forall \mathbf{z}_h \in \mathbf{W}_h. \tag{13c}$$

In the following discussion, we will give some important lemmas based on the novel expanded mixed projection.

Lemma 4 (see [24]). *There is a constant C independent of h such that*

$$\|\delta\| \le Ch\left(\|\lambda\|_{(H^1(\Omega))^2} + \|u\|_{H^2}\right), \tag{14}$$

$$\|\rho\| \le Ch\left(\|u\|_{H^2} + \|\sigma\|_{(H^1(\Omega))^2} + \|\lambda\|_{(H^1(\Omega))^2}\right), \tag{15}$$

$$\|\nabla\eta\| \le Ch\left(\|u\|_{H^2} + \|\lambda\|_{(H^1(\Omega))^2}\right), \tag{16}$$

where $\eta = u - \widetilde{u}_h$, $\delta = \lambda - \widetilde{\lambda}_h$, and $\rho = \sigma - \widetilde{\sigma}_h$.

Lemma 5 (see [24]). *There is a constant C independent of h such that*

$$\begin{aligned} \|\eta\| &\le Ch^2\|u\|_{H^2}, \\ \|\eta_t\| &\le Ch^2\|u_t\|_{H^2}, \\ \|\eta\|_1 &\le Ch\left(\|u\|_{H^2} + \|\lambda\|_{(H^1(\Omega))^2}\right). \end{aligned} \tag{17}$$

Lemma 6 (see [6]). *For each n one has*

$$(d\widehat{v}, \widehat{v}) - (dv, v) \le C\Delta t\,(dv, v), \quad \forall v \in L^2(\Omega), \tag{18}$$

where $\widehat{v} = v(\mathbf{x} - \mathbf{c}(\mathbf{x}, t_n)\Delta t/d(\mathbf{x}))$.

Remark 7. For proving Lemmas 4 and 5, we introduce two linear operators [25, 26] $\Pi_h : (L^2(\Omega))^2 \to \mathbf{W}_h$ and $P_h : H_0^1(\Omega) \to V_h$ and consider the following auxiliary elliptic problem:

$$\begin{aligned} -\nabla\cdot(a\nabla\chi) &= \eta, \quad \text{in } \Omega, \\ \chi &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{19}$$

The detailed proofs for Lemmas 4 and 5 are shown in [24].

3.2. A Priori Error Estimates. In the following discussion, we will derive a priori error estimates based on fully discrete backward Euler method. Let $0 = t_0 < t_1 < t_2 < \cdots < t_M = T$ be a given partition of the time interval $[0,T]$ with step length $\Delta t = T/M$ and nodes $t_n = n\Delta t$, for some positive integer M. For a smooth function ϕ on $[0,T]$, define $\phi^n = \phi(t_n)$.

In order to derive a priori error estimates, we now write

$$\begin{aligned} u(t_n) - u_h^n &= \left(u(t_n) - \widetilde{u}_h^n\right) + \left(\widetilde{u}_h^n - u_h^n\right) = \eta^n + \varsigma^n, \\ \lambda(t_n) - \lambda_h^n &= \left(\lambda(t_n) - \widetilde{\lambda}_h^n\right) + \left(\widetilde{\lambda}_h^n - \lambda_h^n\right) = \delta^n + \theta^n, \\ \sigma(t_n) - \sigma_h^n &= \left(\sigma(t_n) - \widetilde{\sigma}_h^n\right) + \left(\widetilde{\sigma}_h^n - \sigma_h^n\right) = \rho^n + \xi^n. \end{aligned} \tag{20}$$

Combining (8a), (8b), and (8c), (12a), (12b), and (12c) and (13a), (13b), and (13c) at $t = t_n$, we get the error equations

$$\begin{aligned} &\left(d\frac{\varsigma^n - \widehat{\varsigma}^{n-1}}{\Delta t}, v_h\right) - \left(\xi^n, \nabla v_h\right) + \left(R\varsigma^n, v_h\right) \\ &\quad = \left(\psi^n\frac{\partial u^n}{\partial\tau} - d\frac{u^n - \widehat{u}^{n-1}}{\Delta t}, v_h\right) \\ &\qquad - \left(d\frac{\eta^n - \widehat{\eta}^{n-1}}{\Delta t}, v_h\right) - \left(R\eta^n, v_h\right), \quad \forall v_h \in V_h, \end{aligned} \tag{21a}$$

$$\left(\theta^n, \mathbf{w}_h\right) - \left(\nabla\varsigma^n, \mathbf{w}_h\right) = 0, \quad \forall \mathbf{w}_h \in \mathbf{W}_h, \tag{21b}$$

$$\left(\xi^n, \mathbf{z}_h\right) + \left(a^n\theta^n, \mathbf{z}_h\right) = 0, \quad \forall \mathbf{z}_h \in \mathbf{W}_h. \tag{21c}$$

Based on the error equations (21a), (21b), and (21c), we obtain the following theorem for the fully discrete error estimates.

Theorem 8. *Assume that $u_h^0 = \widetilde{u}_h(0)$; then there exist some positive constants independent of $h = O(\Delta t)$ such that*

$$\begin{aligned}
\left\|u(t_J) - u_h^J\right\| &\le C(a_0, d_1)\left|\left\|O\left(h^2, \Delta t\right)\right\|\right| \\
&\quad + C(a_0, d_1, R_1)\, h^2 \|u\|_{L^\infty(H^2)}, \\
\left\|u(t_J) - u_h^J\right\|_1 &\le C(a_0, d_1)\left|\left\|O\left(h^1, \Delta t\right)\right\|\right| \\
&\quad + h\left[C(a_0, d_1, R_1)\|u\|_{L^\infty(H^2)}\right. \\
&\quad \left. + C\|\lambda\|_{L^\infty((H^1)^2)}\right], \\
\left\|\lambda(t_J) - \lambda_h^J\right\| &\le C(a_0, d_1, R_1)\left|\left\|O\left(h^1, \Delta t\right)\right\|\right| \\
&\quad + h\left[C\left(a_0, d_1, R_1, \overline{R}_1\right)\|u\|_{L^\infty(H^2)}\right. \\
&\quad \left. + C\|\lambda\|_{L^\infty((H^1)^2)}\right], \\
\left\|\sigma(t_J) - \sigma_h^J\right\| &\le C(a_0, d_1, R_1)\left|\left\|O\left(h^1, \Delta t\right)\right\|\right| \\
&\quad + h\left[C\left(\|\lambda\|_{L^\infty((H^1)^2)} + \|\sigma\|_{L^\infty((H^1)^2)}\right)\right. \\
&\quad \left. + C\left(a_0, d_1, R_1, \overline{R}_1\right)\|u\|_{L^\infty(H^2)}\right],
\end{aligned} \tag{22}$$

where $|\|O(h^{2-j}, \Delta t)\|| = h^{2-j}(||u||_{L^\infty(H^2)} + ||u_t||_{L^2(H^2)}) + \Delta t||u_{tt}||_{L^2(L^2)}$, $j = 0, 1$.

Proof. Set $v_h = \varsigma^n$ in (21a), $\mathbf{w}_h = \xi^n$ in (21b), and $\mathbf{z}_h = \theta^n$ in (21c) to obtain

$$\begin{aligned}
&\left(d\frac{\varsigma^n - \widehat{\varsigma}^{n-1}}{\Delta t}, \varsigma^n\right) - (\xi^n, \nabla\varsigma^n) + (R^n\varsigma^n, \varsigma^n) \\
&\quad = \left(\psi^n\frac{\partial u^n}{\partial\tau} - d\frac{u^n - \widehat{u}^{n-1}}{\Delta t}, \varsigma^n\right) \\
&\quad - \left(d\frac{\eta^n - \widehat{\eta}^{n-1}}{\Delta t}, \varsigma^n\right) - (R^n\eta^n, \varsigma^n),
\end{aligned} \tag{23a}$$

$$(\nabla\varsigma^n, \xi^n) - (\theta^n, \xi^n) = 0, \tag{23b}$$

$$(\xi^n, \theta^n) + \left\|(a^n)^{1/2}\theta^n\right\|^2 = 0. \tag{23c}$$

Adding the above three equations, we obtain

$$\begin{aligned}
&\left(d\frac{\varsigma^n - \widehat{\varsigma}^{n-1}}{\Delta t}, \varsigma^n\right) + \left\|(a^n)^{1/2}\theta^n\right\|^2 + \left\|(R^n)^{1/2}\varsigma^n\right\|^2 \\
&\quad = \left(\psi^n\frac{\partial u^n}{\partial\tau} - d\frac{u^n - \widehat{u}^{n-1}}{\Delta t}, \varsigma^n\right) \\
&\quad - \left(d\frac{\eta^n - \widehat{\eta}^{n-1}}{\Delta t}, \varsigma^n\right) - (R^n\eta^n, \varsigma^n).
\end{aligned} \tag{24}$$

Noting that $-d\widehat{\varsigma}^{n-1}\varsigma^n/\Delta t = (||d^{1/2}(\varsigma^n - \widehat{\varsigma}^{n-1})||^2 - (||d^{1/2}\widehat{\varsigma}^{n-1}||^2 + ||d^{1/2}\varsigma^n||^2))/2\Delta t$, (24) may be written as

$$\begin{aligned}
&\frac{\left\|d^{1/2}\varsigma^n\right\|^2 - \left\|d^{1/2}\widehat{\varsigma}^{n-1}\right\|^2}{2\Delta t} + a_0\|\theta^n\|^2 + \left\|(R^n)^{1/2}\varsigma^n\right\|^2 \\
&\quad \le \frac{\left\|d^{1/2}\varsigma^n\right\|^2 - \left\|d^{1/2}\widehat{\varsigma}^{n-1}\right\|^2 + \left\|d^{1/2}\left(\varsigma^n - \widehat{\varsigma}^{n-1}\right)\right\|^2}{2\Delta t} \\
&\quad + \left\|(a^n)^{1/2}\theta^n\right\|^2 + \left\|(R^n)^{1/2}\varsigma^n\right\|^2 \\
&\quad = \left(\psi^n\frac{\partial u^n}{\partial\tau} - d\frac{u^n - \widehat{u}^{n-1}}{\Delta t}, \varsigma^n\right) \\
&\quad - \left(d\frac{\eta^n - \widehat{\eta}^{n-1}}{\Delta t}, \varsigma^n\right) - (R^n\eta^n, \varsigma^n).
\end{aligned} \tag{25}$$

Using the same analysis as that in [31], the right-hand side of (25) can be rewritten as

$$\begin{aligned}
&\left(\psi^n\frac{\partial u^n}{\partial\tau} - d\frac{u^n - \widehat{u}^{n-1}}{\Delta t}, \varsigma^n\right) - \left(d\frac{\eta^n - \widehat{\eta}^{n-1}}{\Delta t}, \varsigma^n\right) - (R^n\eta^n, \varsigma^n) \\
&\quad = \left(\psi^n\frac{\partial u^n}{\partial\tau} - d\frac{u^n - \widehat{u}^{n-1}}{\Delta t}, \varsigma^n\right) \\
&\quad - \left(d\frac{\eta^n - \eta^{n-1} + \eta^{n-1} - \widehat{\eta}^{n-1}}{\Delta t}, \varsigma^n\right) - (R^n\eta^n, \varsigma^n) \\
&\quad \le \left(\left\|\psi^n\frac{\partial u^n}{\partial\tau} - d\frac{u^n - \widehat{u}^{n-1}}{\Delta t}\right\| + \left\|d\frac{\eta^n - \eta^{n-1}}{\Delta t}\right\|\right. \\
&\quad \left. + \left\|d\frac{\eta^{n-1} - \widehat{\eta}^{n-1}}{\Delta t}\right\| + \|R^n\eta^n\|\right)\|\varsigma^n\| \\
&\quad \le C(d_1)\left[\Delta t\int_{t_{n-1}}^{t_n}\left\|\frac{\partial^2 u}{\partial\tau^2}\right\|^2 ds + \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\left\|\frac{\partial\eta}{\partial\tau}\right\|^2 ds + \|\varsigma^n\|^2\right] \\
&\quad + \|R^n\eta^n\|^2 + \left\|d\frac{\eta^{n-1} - \widehat{\eta}^{n-1}}{\Delta t}\right\|\|\varsigma^n\|.
\end{aligned} \tag{26}$$

Using the assumption $h = O(\Delta t)$, we obtain

$$\begin{aligned}
\left\|d\frac{\eta^{n-1} - \widehat{\eta}^{n-1}}{\Delta t}\right\|\|\varsigma^n\| &\le C\left(\frac{h + \Delta t}{\Delta t}\right)\left\|\eta^{n-1}\right\|\|\theta^n\| \\
&\le C(a_0)\|\eta\|_{L^\infty(L^2)}^2 + \frac{a_0}{2}\|\theta\|^2.
\end{aligned} \tag{27}$$

By Lemma 6, we have

$$\left\|d^{1/2}\varsigma^{n-1}\right\|^2 - \left\|d^{1/2}\widehat{\varsigma}^{n-1}\right\|^2 \ge -C\Delta t\left\|d^{1/2}\varsigma^{n-1}\right\|^2. \tag{28}$$

Using (28), we obtain

$$\begin{aligned}
&\frac{\left\|d^{1/2}\varsigma^{n}\right\|^{2}-\left\|d^{1/2}\widetilde{\varsigma}^{n-1}\right\|^{2}}{2\Delta t}+a_{0}\left\|\theta^{n}\right\|^{2}+\left\|\left(R^{n}\right)^{1/2}\varsigma^{n}\right\|^{2}\\
&\quad\geq\frac{\left\|d^{1/2}\varsigma^{n}\right\|^{2}-(1+C\Delta t)\left\|d^{1/2}\varsigma^{n-1}\right\|^{2}}{2\Delta t}\\
&\qquad+a_{0}\left\|\theta^{n}\right\|^{2}+\left\|\left(R^{n}\right)^{1/2}\varsigma^{n}\right\|^{2}\\
&\quad=\frac{\left\|d^{1/2}\varsigma^{n}\right\|^{2}-\left\|d^{1/2}\varsigma^{n-1}\right\|^{2}-C\Delta t\left\|d^{1/2}\varsigma^{n-1}\right\|^{2}}{2\Delta t}\\
&\qquad+a_{0}\left\|\theta^{n}\right\|^{2}+\left\|\left(R^{n}\right)^{1/2}\varsigma^{n}\right\|^{2}.
\end{aligned}\tag{29}$$

Multiplying by $2\Delta t$, summing (29) from $n=1$ to J, and using (25)–(29), the resulting inequality becomes

$$\begin{aligned}
&d_{0}\left\|\varsigma^{J}\right\|^{2}+\Delta t\sum_{n=1}^{J}\left(a_{0}\left\|\theta^{n}\right\|^{2}+\left\|\left(R^{n}\right)^{1/2}\varsigma^{n}\right\|^{2}\right)\\
&\quad\leq d_{1}\left\|\varsigma^{0}\right\|^{2}+C\left(a_{0},d_{1}\right)\\
&\qquad\times\left[\Delta t\sum_{n=1}^{J}\left(\Delta t\int_{t_{n-1}}^{t_{n}}\left\|\frac{\partial^{2}u}{\partial\tau^{2}}\right\|^{2}ds+\frac{1}{\Delta t}\int_{t_{n-1}}^{t_{n}}\left\|\frac{\partial\eta}{\partial\tau}\right\|^{2}ds\right)\right]\\
&\qquad+C\left(a_{0},d_{1},R_{1}\right)\|\eta\|_{L^{\infty}(L^{2})}^{2}+C\left(d_{1}\right)\Delta t\sum_{n=1}^{J}\left\|\varsigma^{n-1}\right\|^{2}\\
&\quad=d_{1}\left\|\varsigma^{0}\right\|^{2}+C\left(a_{0},d_{1}\right)\\
&\qquad\times\left[(\Delta t)^{2}\int_{t_{0}}^{t_{J}}\left\|\frac{\partial^{2}u}{\partial\tau^{2}}\right\|^{2}ds+\int_{t_{0}}^{t_{J}}\left\|\frac{\partial\eta}{\partial\tau}\right\|^{2}ds\right]\\
&\qquad+C\left(a_{0},d_{1},R_{1}\right)\|\eta\|_{L^{\infty}(L^{2})}^{2}+C\left(d_{1}\right)\Delta t\sum_{n=1}^{J}\left\|\varsigma^{n-1}\right\|^{2}.
\end{aligned}\tag{30}$$

Using Gronwall lemma, we have

$$\begin{aligned}
&d_{0}\left\|\varsigma^{J}\right\|^{2}+\Delta t\sum_{n=1}^{J}\left(a_{0}\left\|\theta^{n}\right\|^{2}+\left\|\left(R^{n}\right)^{1/2}\varsigma^{n}\right\|^{2}\right)\\
&\quad\leq C\left(a_{0},d_{1}\right)\left[(\Delta t)^{2}\int_{t_{0}}^{t_{J}}\left\|\frac{\partial^{2}u}{\partial\tau^{2}}\right\|^{2}ds+\int_{t_{0}}^{t_{J}}\left\|\frac{\partial\eta}{\partial\tau}\right\|^{2}ds\right]\\
&\qquad+C\left(a_{0},d_{1},R_{1}\right)\|\eta\|_{L^{\infty}(L^{2})}^{2}.
\end{aligned}\tag{31}$$

From (21b), we get

$$\left(\frac{\theta^{n}-\theta^{n-1}}{\Delta t},\mathbf{w}_{h}\right)-\left(\frac{\nabla\varsigma^{n}-\nabla\varsigma^{n-1}}{\Delta t},\mathbf{w}_{h}\right)=0.\tag{32}$$

Choose $\mathbf{w}_h=\xi^n$ in (32), $v_h=(\varsigma^n-\varsigma^{n-1})/\Delta t$ in (21a), and $\mathbf{z}_h=(\theta^n-\theta^{n-1})/\Delta t$ in (21c) to obtain

$$\begin{aligned}
&\left(d\frac{\varsigma^{n}-\widetilde{\varsigma}^{n-1}}{\Delta t},\frac{\varsigma^{n}-\varsigma^{n-1}}{\Delta t}\right)-\left(\xi^{n},\frac{\nabla\varsigma^{n}-\nabla\varsigma^{n-1}}{\Delta t}\right)\\
&\qquad+\left(R^{n}\varsigma^{n},\frac{\varsigma^{n}-\varsigma^{n-1}}{\Delta t}\right)\\
&\quad=\left(\psi^{n}\frac{\partial u^{n}}{\partial\tau}-d\frac{u^{n}-\widehat{u}^{n-1}}{\Delta t},\frac{\varsigma^{n}-\varsigma^{n-1}}{\Delta t}\right)\\
&\qquad-\left(d\frac{\eta^{n}-\widehat{\eta}^{n-1}}{\Delta t},\frac{\varsigma^{n}-\varsigma^{n-1}}{\Delta t}\right)-\left(R^{n}\eta^{n},\frac{\varsigma^{n}-\varsigma^{n-1}}{\Delta t}\right),
\end{aligned}\tag{33a}$$

$$\left(\frac{\nabla\varsigma^{n}-\nabla\varsigma^{n-1}}{\Delta t},\xi^{n}\right)-\left(\frac{\theta^{n}-\theta^{n-1}}{\Delta t},\xi^{n}\right)=0,\tag{33b}$$

$$\left(\xi^{n},\frac{\theta^{n}-\theta^{n-1}}{\Delta t}\right)+\left(a^{n}\theta^{n},\frac{\theta^{n}-\theta^{n-1}}{\Delta t}\right)=0.\tag{33c}$$

Adding the three equations and using Cauchy-Schwarz inequality and Young inequality, we obtain

$$\begin{aligned}
&\left(d\frac{\varsigma^{n}-\widetilde{\varsigma}^{n-1}}{\Delta t},\frac{\varsigma^{n}-\varsigma^{n-1}}{\Delta t}\right)+\left(a^{n}\theta^{n},\frac{\theta^{n}-\theta^{n-1}}{\Delta t}\right)\\
&\qquad+\left(R^{n}\varsigma^{n},\frac{\varsigma^{n}-\varsigma^{n-1}}{\Delta t}\right)\\
&\quad=\left(\psi^{n}\frac{\partial u^{n}}{\partial\tau}-d\frac{u^{n}-\widehat{u}^{n-1}}{\Delta t},\frac{\varsigma^{n}-\varsigma^{n-1}}{\Delta t}\right)\\
&\qquad-\left(d\frac{\eta^{n}-\widehat{\eta}^{n-1}}{\Delta t},\frac{\varsigma^{n}-\varsigma^{n-1}}{\Delta t}\right)-\left(R^{n}\eta^{n},\frac{\varsigma^{n}-\varsigma^{n-1}}{\Delta t}\right)\\
&\quad\leq\left(\left\|\psi^{n}\frac{\partial u^{n}}{\partial\tau}-d\frac{u^{n}-\widehat{u}^{n-1}}{\Delta t}\right\|+\left\|d\frac{\eta^{n}-\eta^{n-1}}{\Delta t}\right\|\right.\\
&\qquad\left.+\left\|d\frac{\eta^{n-1}-\widehat{\eta}^{n-1}}{\Delta t}\right\|+\left\|R^{n}\eta^{n}\right\|\right)\left\|\frac{\varsigma^{n}-\varsigma^{n-1}}{\Delta t}\right\|\\
&\quad\leq C\left(d_{1}\right)\Delta t\int_{t_{n-1}}^{t_{n}}\left\|\frac{\partial^{2}u}{\partial\tau^{2}}\right\|^{2}ds+\frac{C\left(d_{1}\right)}{\Delta t}\int_{t_{n-1}}^{t_{n}}\left\|\frac{\partial\eta}{\partial\tau}\right\|^{2}ds\\
&\qquad+C\left(R_{1}\right)\|\eta\|_{L^{\infty}(L^{2})}^{2}+\frac{1}{2}\left\|\frac{\varsigma^{n}-\varsigma^{n-1}}{\Delta t}\right\|^{2}.
\end{aligned}\tag{34}$$

The left-hand side of (34) can be written as

$$\begin{aligned}
&\left(d\frac{\varsigma^n-\widehat{\varsigma}^{n-1}}{\Delta t},\frac{\varsigma^n-\varsigma^{n-1}}{\Delta t}\right)+\left(a^n\theta^n,\frac{\theta^n-\theta^{n-1}}{\Delta t}\right)\\
&\quad+\left(R^n\varsigma^n,\frac{\varsigma^n-\varsigma^{n-1}}{\Delta t}\right)\\
&=\left\|d^{1/2}\frac{\varsigma^n-\varsigma^{n-1}}{\Delta t}\right\|^2+\left(d\frac{\varsigma^{n-1}-\widehat{\varsigma}^{n-1}}{\Delta t},\frac{\varsigma^n-\varsigma^{n-1}}{\Delta t}\right)\\
&\quad+\frac{\left\|\left(a^n\right)^{1/2}\left(\theta^n-\theta^{n-1}\right)\right\|^2}{2\Delta t}\\
&\quad+\frac{\left\|\left(a^n\right)^{1/2}\theta^n\right\|^2-\left\|\left(a^{n-1}\right)^{1/2}\theta^{n-1}\right\|^2}{2\Delta t}\\
&\quad-\left(\frac{a^n-a^{n-1}}{2\Delta t}\theta^{n-1},\theta^{n-1}\right)+\frac{\left\|\left(R^n\right)^{1/2}\left(\varsigma^n-\varsigma^{n-1}\right)\right\|^2}{2\Delta t}\\
&\quad+\frac{\left\|\left(R^n\right)^{1/2}\varsigma^n\right\|^2-\left\|\left(R^{n-1}\right)^{1/2}\varsigma^{n-1}\right\|^2}{2\Delta t}\\
&\quad-\left(\frac{R^n-R^{n-1}}{2\Delta t}\varsigma^{n-1},\varsigma^{n-1}\right).
\end{aligned}\tag{35}$$

Substitute (35) into (34) and multiply by $2\Delta t$ to get

$$\begin{aligned}
&\Delta t\left\|d^{1/2}\frac{\varsigma^n-\varsigma^{n-1}}{\Delta t}\right\|^2+\left\|\left(a^n\right)^{1/2}\left(\theta^n-\theta^{n-1}\right)\right\|^2\\
&\quad+\left\|\left(R^n\right)^{1/2}\left(\varsigma^n-\varsigma^{n-1}\right)\right\|^2\\
&\quad+\left(\left\|\left(a^n\right)^{1/2}\theta^n\right\|^2-\left\|\left(a^{n-1}\right)^{1/2}\theta^{n-1}\right\|^2\right)\\
&\quad+\left(\left\|\left(R^n\right)^{1/2}\varsigma^n\right\|^2-\left\|\left(R^{n-1}\right)^{1/2}\varsigma^{n-1}\right\|^2\right)\\
&\le C\left(d_1\right)(\Delta t)^2\int_{t_{n-1}}^{t_n}\left\|\frac{\partial^2u}{\partial\tau^2}\right\|^2ds\\
&\quad+C\left(d_1\right)\int_{t_{n-1}}^{t_n}\left\|\frac{\partial\eta}{\partial\tau}\right\|^2ds+C\left(R_1\right)\Delta t\|\eta\|^2_{L^\infty(L^2)}\\
&\quad-\Delta t\left(d\frac{\varsigma^{n-1}-\widehat{\varsigma}^{n-1}}{\Delta t},\frac{\varsigma^n-\varsigma^{n-1}}{\Delta t}\right)\\
&\quad+\Delta t\left(\frac{\int_{t_{n-1}}^{t_n}a_tds}{2\Delta t}\theta^{n-1},\theta^{n-1}\right)\\
&\quad-\Delta t\left(\frac{\int_{t_{n-1}}^{t_n}R_tds}{2\Delta t}\varsigma^{n-1},\varsigma^{n-1}\right).
\end{aligned}\tag{36}$$

Summing (36) from $n=1$ to J, the resulting inequality becomes

$$\begin{aligned}
&\Delta t\sum_{n=1}^{J}\left\|d^{1/2}\frac{\varsigma^n-\varsigma^{n-1}}{\Delta t}\right\|^2\\
&\quad+\sum_{n=1}^{J}\left(\left\|\left(a^n\right)^{1/2}\left(\theta^n-\theta^{n-1}\right)\right\|^2+\left\|\left(R^n\right)^{1/2}\left(\varsigma^n-\varsigma^{n-1}\right)\right\|^2\right)\\
&\quad+\left\|\left(a^J\right)^{1/2}\theta^J\right\|^2+\left\|\left(R^J\right)^{1/2}\varsigma^J\right\|^2\\
&\le\left\|\left(a^0\right)^{1/2}\theta^0\right\|^2+C\left(d_1\right)\left((\Delta t)^2\int_{t_0}^{t_J}\left\|\frac{\partial^2u}{\partial\tau^2}\right\|^2ds\right.\\
&\qquad\left.+\int_{t_0}^{t_J}\left\|\frac{\partial\eta}{\partial\tau}\right\|^2ds\right)\\
&\quad+C\left(R_1\right)\|\eta\|^2_{L^\infty(L^2)}+C\left(\overline{a}_1\right)\Delta t\sum_{n=0}^{J-1}\|\theta^n\|^2\\
&\quad+C\left(\overline{R}_1\right)\Delta t\sum_{n=0}^{J-1}\|\varsigma^n\|^2\\
&\quad-\Delta t\sum_{n=1}^{J}\left(d\frac{\varsigma^{n-1}-\widehat{\varsigma}^{n-1}}{\Delta t},\frac{\varsigma^n-\varsigma^{n-1}}{\Delta t}\right).
\end{aligned}\tag{37}$$

Using the same analysis as that in [29], we obtain

$$\begin{aligned}
&\Delta t\sum_{n=1}^{J}\left(d\frac{\varsigma^{n-1}-\widehat{\varsigma}^{n-1}}{\Delta t},\frac{\varsigma^n-\varsigma^{n-1}}{\Delta t}\right)\\
&=\Delta t\sum_{n=1}^{J}\left(d\frac{\varsigma^{n-1}-\widehat{\varsigma}^{n-1}}{\Delta t},\frac{\varsigma^n}{\Delta t}\right)\\
&\quad-\Delta t\sum_{n=1}^{J}\left(d\frac{\varsigma^{n-1}-\widehat{\varsigma}^{n-1}}{\Delta t},\frac{\varsigma^{n-1}}{\Delta t}\right)\\
&=\Delta t\sum_{n=1}^{J}\left(d\frac{\varsigma^{n-1}-\varsigma^n-\widehat{\varsigma}^{n-1}+\widehat{\varsigma}^n}{\Delta t},\frac{\varsigma^n}{\Delta t}\right)\\
&\quad+\Delta t\sum_{n=1}^{J}\left(d\frac{\varsigma^n-\widehat{\varsigma}^n}{\Delta t},\frac{\varsigma^n}{\Delta t}\right)\\
&\quad-\Delta t\sum_{n=1}^{J}\left(d\frac{\varsigma^{n-1}-\widehat{\varsigma}^{n-1}}{\Delta t},\frac{\varsigma^{n-1}}{\Delta t}\right)\\
&=\Delta t\sum_{n=1}^{J}\left(d\frac{\varsigma^{n-1}-\varsigma^n-\widehat{\varsigma}^{n-1}+\widehat{\varsigma}^n}{\Delta t},\frac{\varsigma^n}{\Delta t}\right)+\left(d\frac{\varsigma^J-\widehat{\varsigma}^J}{\Delta t},\varsigma^J\right)
\end{aligned}$$

$$\leq \Delta t \sum_{n=1}^{J} \left\| \frac{\varsigma^{n-1} - \varsigma^{n} - \widehat{\varsigma}^{n-1} + \widehat{\varsigma}^{n}}{\Delta t} \right\| \left\| \frac{d\varsigma^{n}}{\Delta t} \right\| + \left(d \frac{\varsigma^{J} - \widehat{\varsigma}^{J}}{\Delta t}, \varsigma^{J} \right)$$
$$\leq \varepsilon \Delta t \sum_{n=1}^{J} \left\| \frac{\varsigma^{n-1} - \varsigma^{n}}{\Delta t} \right\|^{2} + C(\varepsilon) \Delta t \sum_{n=1}^{J} \left\| \varsigma^{n} \right\|^{2} + C(d_1) \left\| \varsigma^{J} \right\|^{2}. \tag{38}$$

Substitute (31) and (38) into (37) to obtain

$$\begin{aligned}
&(d_0 - \varepsilon) \Delta t \sum_{n=1}^{J} \left\| \frac{\varsigma^{n} - \varsigma^{n-1}}{\Delta t} \right\|^{2} \\
&\quad + \sum_{n=1}^{J} \left(\left\| (a^{n})^{1/2} \left(\theta^{n} - \theta^{n-1} \right) \right\|^{2} + \left\| (R^{n})^{1/2} \left(\varsigma^{n} - \varsigma^{n-1} \right) \right\|^{2} \right) \\
&\quad + \left\| \left(a^{J} \right)^{1/2} \theta^{J} \right\|^{2} + \left\| \left(R^{J} \right)^{1/2} \varsigma^{J} \right\|^{2} \\
&\leq C(a_0, d_1, R_1) \left((\Delta t)^{2} \int_{t_0}^{t_J} \left\| \frac{\partial^{2} u}{\partial \tau^{2}} \right\|^{2} ds + \int_{t_0}^{t_J} \left\| \frac{\partial \eta}{\partial \tau} \right\|^{2} ds \right) \\
&\quad + C\left(a_0, d_1, R_1, \overline{R}_1 \right) \left\| \eta \right\|_{L^{\infty}(L^{2})}^{2} + C(\overline{a}_1) \Delta t \sum_{n=0}^{J-1} \left\| \theta^{n} \right\|^{2}.
\end{aligned} \tag{39}$$

Using Gronwall lemma, we have

$$\begin{aligned}
&\Delta t \sum_{n=1}^{J} \left\| \frac{\varsigma^{n} - \varsigma^{n-1}}{\Delta t} \right\|^{2} + \left\| \theta^{J} \right\|^{2} + \left\| \left(R^{J} \right)^{1/2} \varsigma^{J} \right\|^{2} \\
&\leq C(a_0, d_1, R_1) \left((\Delta t)^{2} \int_{t_0}^{t_J} \left\| \frac{\partial^{2} u}{\partial \tau^{2}} \right\|^{2} ds + \int_{t_0}^{t_J} \left\| \frac{\partial \eta}{\partial \tau} \right\|^{2} ds \right) \\
&\quad + C\left(a_0, d_1, R_1, \overline{R}_1 \right) \left\| \eta \right\|_{L^{\infty}(L^{2})}^{2}.
\end{aligned} \tag{40}$$

Taking $\mathbf{z}_h = \xi^n$ in (21a), (21b), and (21c), we get

$$\left\| \xi^{n} \right\|^{2} \leq C \left\| \theta^{n} \right\|^{2}. \tag{41}$$

Substitute (40) into (41) to get

$$\begin{aligned}
\left\| \xi^{J} \right\|^{2} &\leq C(a_0, d_1, R_1) \left((\Delta t)^{2} \int_{t_0}^{t_J} \left\| \frac{\partial^{2} u}{\partial \tau^{2}} \right\|^{2} ds + \int_{t_0}^{t_J} \left\| \frac{\partial \eta}{\partial \tau} \right\|^{2} ds \right) \\
&\quad + C\left(a_0, d_1, R_1, \overline{R}_1 \right) \left\| \eta \right\|_{L^{\infty}(L^{2})}^{2}.
\end{aligned} \tag{42}$$

Taking $\mathbf{w}_h = \nabla \varsigma^n$ in (21b), we have

$$\begin{aligned}
\left\| \nabla \varsigma^{n} \right\|^{2} &\leq \left\| \theta^{n} \right\|^{2} \\
&\leq C(a_0, d_1, R_1) \left((\Delta t)^{2} \int_{t_0}^{t_J} \left\| \frac{\partial^{2} u}{\partial \tau^{2}} \right\|^{2} ds + \int_{t_0}^{t_J} \left\| \frac{\partial \eta}{\partial \tau} \right\|^{2} ds \right) \\
&\quad + C\left(a_0, d_1, R_1, \overline{R}_1 \right) \left\| \eta \right\|_{L^{\infty}(L^{2})}^{2}.
\end{aligned} \tag{43}$$

Combining (14)–(16), (31), (40), (43), (42), and the triangle inequality, we complete the proof. □

Remark 9. Compared to the results in [29], we can obtain the optimal a priori error estimates in H^1-norm for the scalar unknown u in Theorem 8.

4. Numerical Experiment

In this section, in order to confirm our theoretical results for the novel characteristic expanded mixed finite element method, we consider the following test problem:

$$\begin{aligned}
&d(\mathbf{x}) \frac{\partial u}{\partial t} + \mathbf{c}(\mathbf{x}, t) \cdot \nabla u - \nabla \cdot (a(\mathbf{x}, t) \nabla u) + R(\mathbf{x}, t) u \\
&\quad = f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times J,
\end{aligned} \tag{44}$$

with boundary condition

$$u(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times \overline{J}, \tag{45}$$

and initial condition

$$u(\mathbf{x}, 0) = x_1 x_2 (x_1 - 1)(x_2 - 1)(2x_2 - 1), \quad \mathbf{x} \in \Omega, \tag{46}$$

where $\Omega = [0,1] \times [0,1]$, $\mathbf{x} = (x_1, x_2)$, $J = (0,1]$, $a(\mathbf{x}, t) = 1 + 2x_1^2 + x_2^2$, $\mathbf{c}(\mathbf{x}, t) = (1 + x_1^2 + x_2^2 + t^2, 1 + x_1^2 + x_2^2 + t^2)$, $R(\mathbf{x}, t) = 1 + x_1^2 + 2x_2^2$, and $d(\mathbf{x}) = 1$ and $f(\mathbf{x}, t)$ is chosen so that the exact solution for the scalar unknown function is

$$u(\mathbf{x}, t) = e^{-t} x_1 x_2 (x_1 - 1)(x_2 - 1)(2x_2 - 1). \tag{47}$$

The corresponding exact gradient function is

$$\lambda = \nabla u = e^{-t} \begin{bmatrix} x_2 (2x_1 - 1)(x_2 - 1)(2x_2 - 1) \\ x_1 (x_1 - 1)\left(6x_2^2 - 6x_2 + 1\right) \end{bmatrix}, \tag{48}$$

and its exact flux function is

$$\begin{aligned}
\sigma &= -a \nabla u \\
&= -e^{-t} \left(1 + 2x_1^2 + x_2^2\right) \begin{bmatrix} x_2 (2x_1 - 1)(x_2 - 1)(2x_2 - 1) \\ x_1 (x_1 - 1)\left(6x_2^2 - 6x_2 + 1\right) \end{bmatrix}.
\end{aligned} \tag{49}$$

We divide the domain $\Omega = [0,1] \times [0,1]$ into the triangulations of spatial mesh parameter h uniformly and use the backward Euler procedure with uniform time discretization parameter Δt. We consider the piecewise linear space V_h with index $k = 1$ and the corresponding piecewise constant space $\mathbf{W}_h$ with index $k = 0$.

In Table 1, we get the optimal a priori error estimate in L^2-norm for the scalar unknown u with $h = 2\sqrt{2}\Delta t = \sqrt{2}/16, \sqrt{2}/32, \sqrt{2}/64$. At the same time, we also obtain the optimal a priori error estimate in H^1-norm for the scalar unknown u.

In Table 2, we obtain some convergence results in $(L^2(\Omega))^2$-norm for the gradient λ with $h = 2\sqrt{2}\Delta t = \sqrt{2}/16$, $\sqrt{2}/32$, $\sqrt{2}/64$ in Table 2. The similar results are obtained for the flux σ in Table 2. From the data obtained in Table 2, we can

Table 1: The errors and convergence rate for u.

$(h, \Delta t)$	$\|u - u_h\|_{L^\infty(L^2(\Omega))}$	Rate	$\|u - u_h\|_{L^\infty(H^1(\Omega))}$	Rate
$(\sqrt{2}/8, 1/16)$	$1.3622e-003$		$2.6286e-002$	
$(\sqrt{2}/16, 1/32)$	$4.1958e-004$	1.6989	$1.3583e-002$	0.9525
$(\sqrt{2}/32, 1/64)$	$1.3172e-004$	1.6715	$6.8843e-003$	0.9804

Table 2: The errors and convergence rate for λ and σ.

$(h, \Delta t)$	$\|\lambda - \lambda_h\|_{L^\infty((L^2(\Omega))^2)}$	Rate	$\|\sigma - \sigma_h\|_{L^\infty((L^2(\Omega))^2)}$	Rate
$(\sqrt{2}/8, 1/16)$	$2.6251e-002$		$5.4313e-002$	
$(\sqrt{2}/16, 1/32)$	$1.3576e-002$	0.9513	$2.9081e-002$	0.9013
$(\sqrt{2}/32, 1/64)$	$6.8830e-003$	0.9800	$1.5018e-002$	0.9534

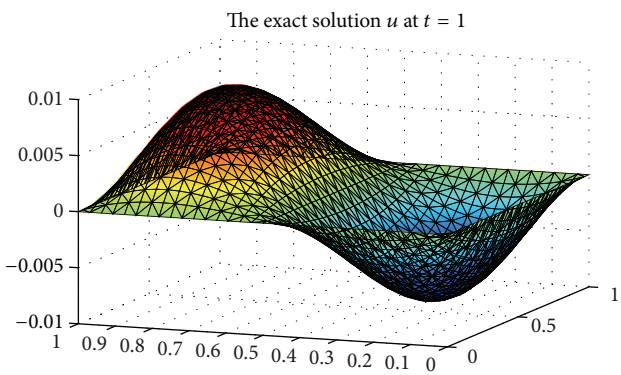

Figure 1: The surface for the exact solution u.

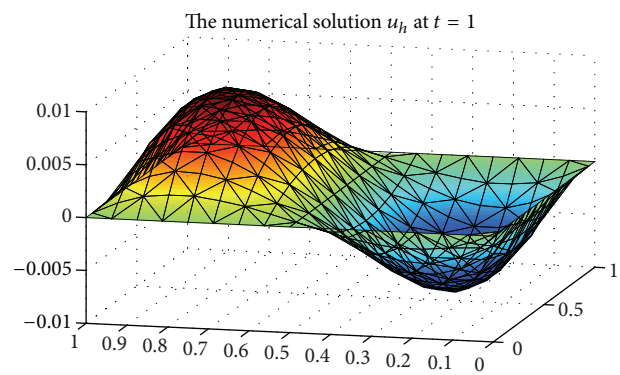

Figure 2: The surface for the numerical solution u_h.

find that the numerical results confirm the theoretical results of Theorem 8.

Figure 1 shows the surface for the exact solution u at $t = 1$, and Figure 2 describes the corresponding surface for the numerical solution u_h with $h = 2\sqrt{2}\Delta t = \sqrt{2}/16$ at $t = 1$. In Figures 1 and 2, we can find easily that the exact solution u is approximated very well by the numerical solution u_h.

Figures 3 and 4 show the surface of the exact gradient function λ and the surface of the numerical gradient function λ_h with $h = 2\sqrt{2}\Delta t = \sqrt{2}/16$ at t, respectively. A similar comparison between the surface of the exact flux function σ and the surface of the numerical flux function σ_h in Figures 5 and 6 also is made.

From the convergence results for $\|u - u_h\|_{L^\infty(L^2(\Omega))}$, $\|u - u_h\|_{L^\infty(H^1(\Omega))}$, $\|\lambda - \lambda_h\|_{L^\infty((L^2(\Omega))^2)}$, and $\|\sigma - \sigma_h\|_{L^\infty((L^2(\Omega))^2)}$ in Tables 1 and 2 and Figures 1 and 2, we find that the numerical results confirm our theoretical analysis.

5. Concluding Remarks

In this paper, we propose and study a novel characteristic expanded mixed finite element method, which combines the novel expanded mixed method [24] applied to approximating the diffusion term and the characteristic method that handled the hyperbolic part, for reaction-convection-diffusion equations. Compared to Chen's expanded mixed method, the gradient for our method belongs to the square integrable space instead of the classical $\mathbf{H}(\mathrm{div}; \Omega)$ space. We derive a priori error estimates based on backward Euler method. Moreover, we prove the optimal a priori error estimates in L^2- and H^1-norms for the scalar unknown u and a priori error estimates in $(L^2)^2$-norm for its gradient λ and its flux σ. Finally, we choose a test problem to confirm our theoretical results. In the near future, the proposed characteristic expanded mixed scheme will be applied to other linear/nonlinear evolution equations, such as nonlinear reaction-diffusion equations, linear/nonlinear convection-dominated Sobolev equations, and time-dependent convection-diffusion optimal control problems.

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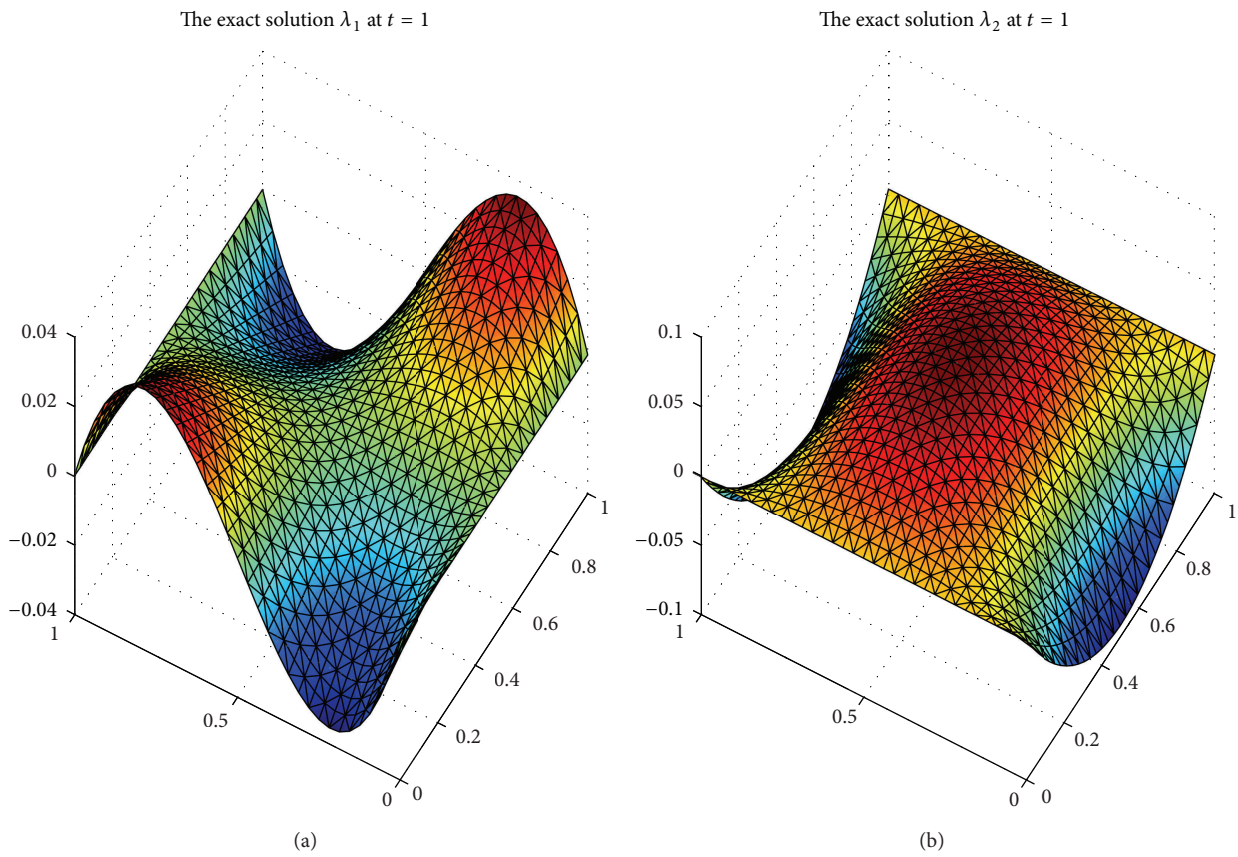

FIGURE 3: The surface for the exact solution $\lambda = (\lambda_1, \lambda_2)$.

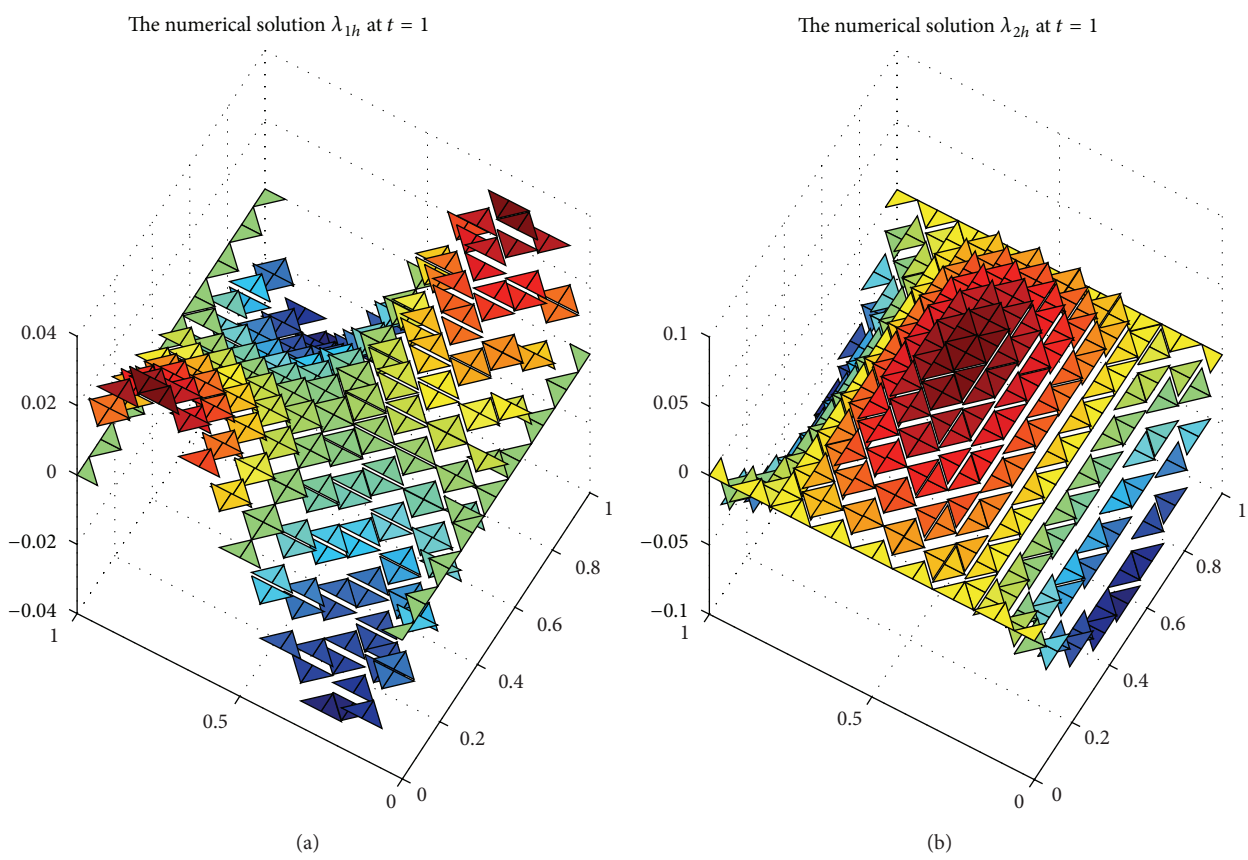

FIGURE 4: The surface for the numerical solution $\lambda_h = (\lambda_{1h}, \lambda_{2h})$.

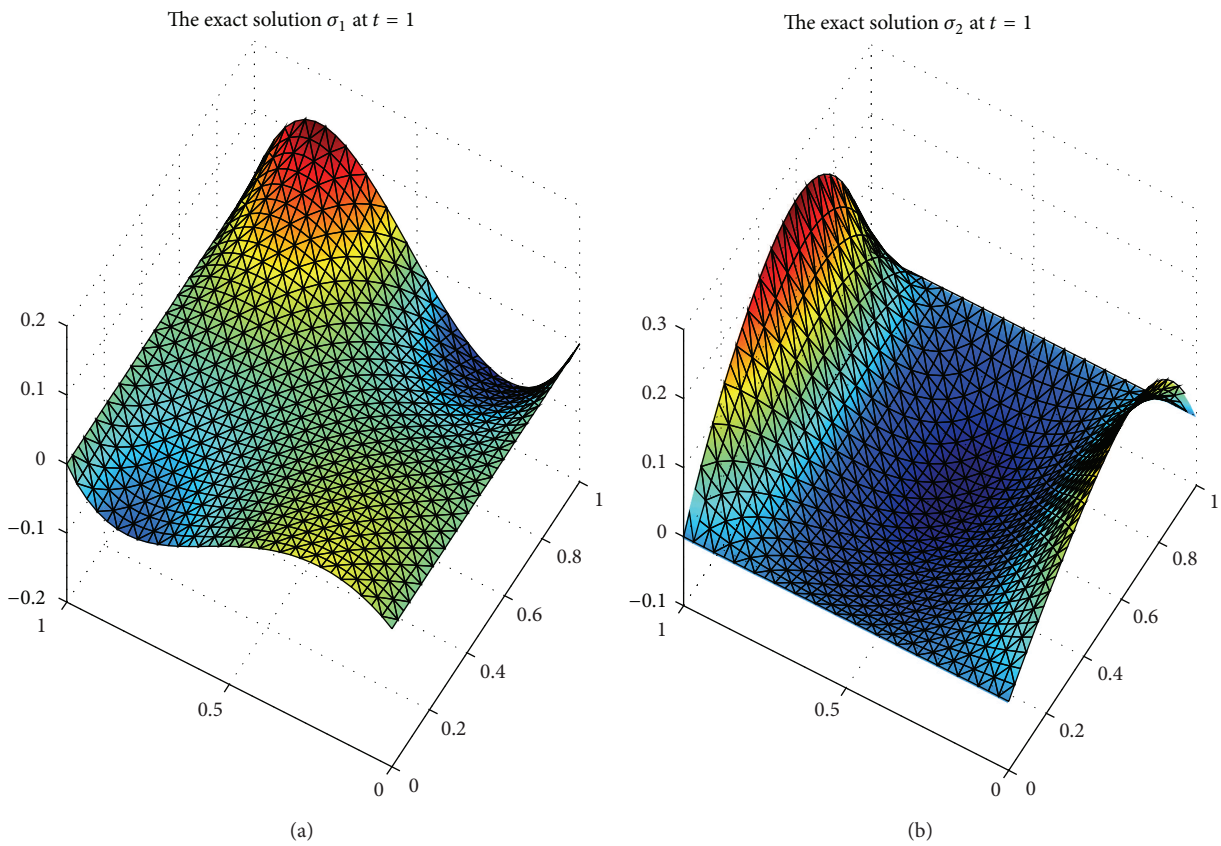

FIGURE 5: The surface for the exact solution $\sigma = (\sigma_1, \sigma_2)$.

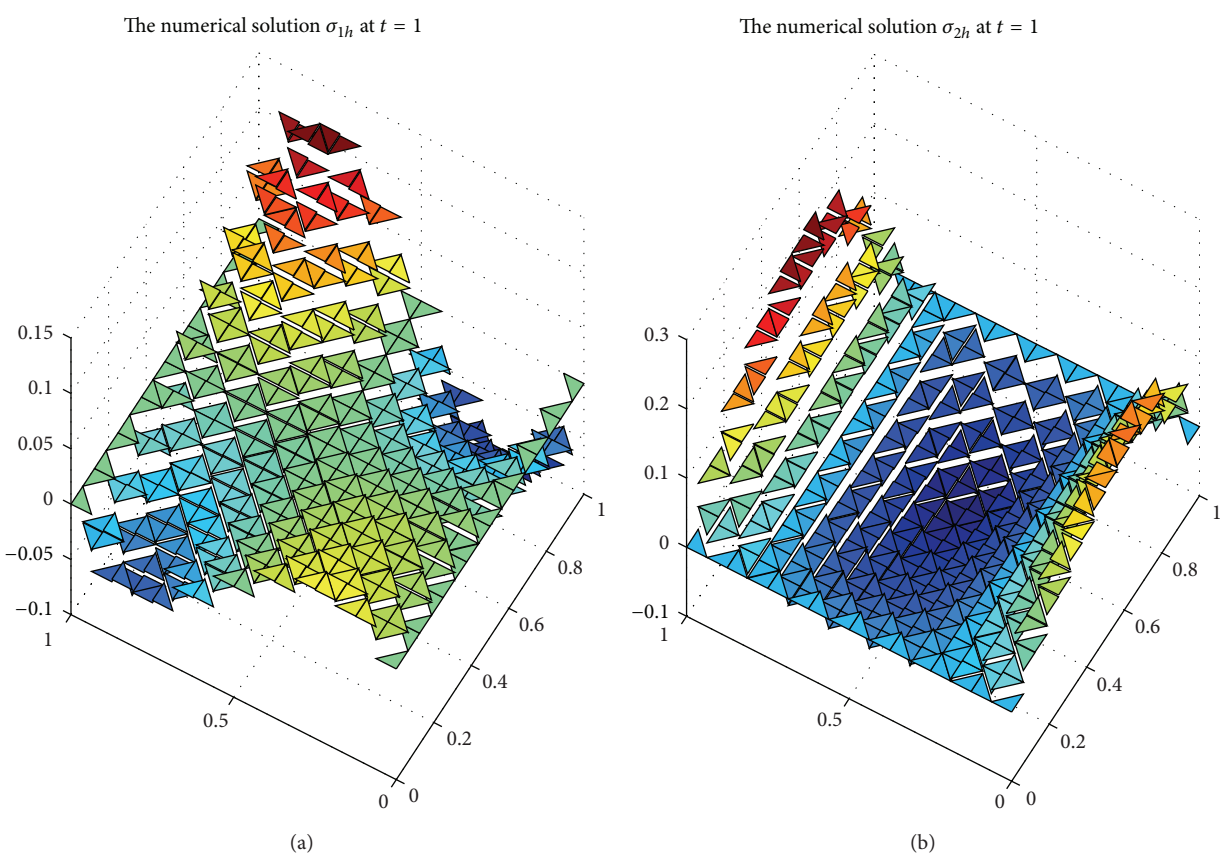

FIGURE 6: The surface for the numerical solution $\sigma_h = (\sigma_{1h}, \sigma_{2h})$.

Talents of Inner Mongolia University (125119, Z200901004, and 30105-125132).

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