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Research Article

Convergence Theorems for Total Asymptotically Nonexpansive Mappings in Hyperbolic Spaces

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The purpose of this paper is to introduce the concept of *total asymptotically nonexpansive mappings* and to prove some Δ-convergence theorems of the iteration process for this kind of mappings in the setting of *hyperbolic spaces*. The results presented in the paper extend and improve some recent results announced in the current literature.

1. Introduction and Preliminaries

Most of the problems in various disciplines of science are nonlinear in nature whereas fixed point theory proposed in the setting of normed linear spaces or Banach spaces majorly depends on the linear structure of the underlying spaces. A nonlinear framework for fixed point theory is a metric space embedded with a "convex structure." The class of hyperbolic spaces, nonlinear in nature, is a general abstract theoretic setting with rich geometrical structure for metric fixed point theory. The study of hyperbolic spaces has been largely motivated and dominated by questions about hyperbolic groups, one of the main objects of study in geometric group theory.

Throughout this paper, we work in the setting of hyperbolic spaces introduced by Kohlenbach [1], defined below, which is more restrictive than the hyperbolic type introduced in [2] and more general than the concept of hyperbolic space in [3].

A hyperbolic space is a metric space (X, d) together with a mapping $W : X^2 \times [0,1] \to X$ satisfying

(i) $d(u, W(x, y, \alpha)) \le (1-\alpha)d(u, x) + \alpha d(u, y)$;

(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$;

(iii) $W(x, y, \alpha) = W(y, x, (1-\alpha))$;

(iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \le (1-\alpha)d(x, y) + \alpha d(z, w)$,

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0,1]$. A nonempty subset K of a hyperbolic space X is convex if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0,1]$. The class of hyperbolic spaces contains normed spaces and convex subsets thereof, the Hilbert ball equipped with the hyperbolic metric [4], Hadamard manifolds, and CAT(0) spaces in the sense of Gromov (see [5]).

A hyperbolic space is uniformly convex [6] if for any $r > 0$ and $\epsilon \in (0,2]$ there exists a $\delta \in (0,1]$, such that, for all $u, x, y \in X$, we have

$$d\left(W\left(x, y, \frac{1}{2}\right), u\right) \le (1-\delta) r, \tag{1}$$

provided $d(x, u) \le r$, $d(y, u) \le r$ and $d(x, y) \ge \epsilon r$.

A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$, which provides such a $\delta = \eta(r, \epsilon)$ for given $r > 0$ and $\epsilon \in (0, 2]$, is known as a modulus of uniform convexity of X. We call η monotone if it decreases with r (for a fixed ϵ), that is, for all $\epsilon > 0$, for all $r_2 \ge r_1 > 0$ $(\eta(r_2, \epsilon) \le \eta(r_1, \epsilon))$.

In the sequel, let (X, d) be a metric space and let K be a nonempty subset of X. We will denote the fixed point set of a mapping T by $F(T) = \{x \in K : Tx = x\}$.

A mapping $T : K \to K$ is said to be *nonexpansive*, if

$$d(Tx, Ty) \le d(x, y), \quad \forall x, y \in K. \tag{2}$$

A mapping $T : K \to K$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that

$$d(T^n x, T^n y) \le k_n d(x,y), \quad \forall n \ge 1,\ x,y \in K. \tag{3}$$

A mapping $T : K \to K$ is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \le L d(x,y), \quad \forall n \ge 1,\ x,y \in K. \tag{4}$$

Definition 1. A mapping $T : K \to K$ is said to be $(\{\mu_n\},\{\xi_n\},\phi)$*-total asymptotically nonexpansive*, if there exist nonnegative sequences $\{\mu_n\},\{\xi_n\}$ with $\mu_n \to 0$, $\xi_n \to 0$ and a strictly increasing continuous function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$d(T^n x, T^n y) \le d(x,y) + \mu_n \phi(d(x,y)) + \xi_n, \quad \forall n \ge 1,\ x,y \in K. \tag{5}$$

Remark 2. From the definitions, it is clear that each nonexpansive mapping is an asymptotically nonexpansive mapping with a sequence $\{k_n = 1\}$ and each asymptotically nonexpansive mapping is a $(\{\mu_n\},\{\xi_n\},\phi)$-total asymptotically nonexpansive mapping with $\xi_n = 0$, $\mu_n = k_n - 1$, $n \ge 1$ and $\phi(t) = t$, $t \ge 0$.

The existence of fixed points of various nonlinear mappings has relevant applications in many branches of nonlinear analysis and topology. On the other hand, there are certain situations where it is difficult to derive conditions for the existence of fixed points for certain types of nonlinear mappings. It is worth mentioning that fixed point theory for nonexpansive mappings, a limit case of a contraction mapping when the Lipschitz constant is allowed to be 1, requires tools far beyond from metric fixed point theory. Iteration schemas are the only main tool for analysis of generalized nonexpansive mappings. Fixed point theory has a computational flavor as one can define effective iteration schemas for the computation of fixed points of various nonlinear mappings. The problem of finding a common fixed point of some nonlinear mappings acting on a nonempty convex domain often arises in applied mathematics.

The purpose of this paper is to introduce the concept of total asymptotically nonexpansive mappings and to prove some Δ-convergence theorems of the iteration process for the approximation of total asymptotically nonexpansive mappings in hyperbolic spaces. The results presented in the paper extend and improve some recent results given in [6–18].

In order to define the concept Δ-convergence in the general setup of hyperbolic spaces, we first collect some basic concepts.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, we define a continuous functional $r(\cdot,\{x_n\}) : X \to [0,\infty)$ by

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n). \tag{6}$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\}. \tag{7}$$

The *asymptotic center* $A_k(\{x_n\})$ of a bounded sequence $\{x_n\}$ with respect to $K \subset X$ is the set

$$A_K(\{x_n\}) = \{x \in X : r(x,\{x_n\}) \le r(y,\{x_n\}),\ \forall y \in K\}. \tag{8}$$

This is the set of minimizers of the functional $r(\cdot,\{x_n\})$. If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_n\})$. It is known that uniformly convex Banach spaces and CAT(0) spaces enjoy the property that "bounded sequences have unique asymptotic centers with respect to closed convex subsets." The following lemma is due to Leuştean [19] and ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 3 (see [19]). *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then, every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X.*

Recall that a sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

A mapping $T : K \to K$ is semicompact if every bounded sequence $\{x_n\} \subset K$, satisfying $d(x_n, Tx_n) \to 0$, has a convergent subsequence.

Lemma 4 (see [8]). *Let $\{a_n\}$, $\{b_n\}$, and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \le (1+\delta_n) a_n + b_n, \quad \forall n \ge 1. \tag{9}$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the limit $\lim_{n\to\infty} a_n$ exists. If there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \to 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 5 (see [11]). *Let (X,d,W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that*

$$\limsup_{n\to\infty} d(x_n,x) \le c, \qquad \limsup_{n\to\infty} d(y_n,x) \le c, \qquad \lim_{n\to\infty} d(W(x_n,y_n,\alpha_n),x) = c, \tag{10}$$

for some $c \ge 0$, then $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Lemma 6 (see [11]). *Let K be a nonempty closed convex subset of uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m,\{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

2. Main Results

Theorem 7. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T_i : K \to K$, $i = 1,2$, be uniformly L_i-Lipschitzian and $(\{\mu_{in}\},\{\xi_{in}\},\phi_i)$-total*

asymptotically nonexpansive mappings with sequences $\{\mu_{in}\}$ *and* $\{\xi_{in}\}$ *satisfying* $\lim_{n\to\infty}\mu_{in} = 0$, $\lim_{n\to\infty}\xi_{in} = 0$, *and strictly increasing function* $\phi_i : [0,\infty) \to [0,\infty)$ *with* $\phi_i(0) = 0$, $i = 1,2$. *Assume that* $F := \bigcap_{i=1}^{2} F(T_i) \neq \emptyset$, *and for arbitrarily chosen* $x_1 \in K$, $\{x_n\}$ *is defined as follows:*

$$\begin{aligned} x_{n+1} &= W\left(x_n, T_1^n y_n, \alpha_n\right), \\ y_n &= W\left(x_n, T_2^n x_n, \beta_n\right), \end{aligned} \tag{11}$$

where $\{\mu_{in}\}$, $\{\xi_{in}\}$, ϕ_i, $i = 1,2$, $\{\alpha_n\}$, *and* $\{\beta_n\}$ *satisfy the following conditions:*

(1) $\sum_{n=1}^{\infty} \mu_{in} < \infty$, $\sum_{n=1}^{\infty} \xi_{in} < \infty$, $i = 1,2$;

(2) *there exist constants* $a, b \in (0,1)$ *with* $0 < b(1-a) \leq 1/2$ *such that* $\{\alpha_n\} \subset [a,b]$ *and* $\{\beta_n\} \subset [a,b]$;

(3) *there exist a constant* $M^* > 0$ *such that* $\phi_i(r) \leq M^* r$, $r > 0$, $i = 1,2$.

Then, the sequence $\{x_n\}$ *defined by* (11) Δ*-converges to a common fixed point of* $F := \bigcap_{i=1}^{2} F(T_i)$.

Proof. The proof of Theorem 7 is divided into four steps.

Step 1. First, we prove that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.

Set $\mu_n = \max\{\mu_{1n}, \mu_{2n}\}$, $\xi_n = \max\{\xi_{1n}, \xi_{2n}\}$, and $L = \max\{L_1, L_2\}$. Since $\sum_{n=1}^{\infty} \mu_{in} < \infty$, $\sum_{n=1}^{\infty} \xi_{in} < \infty$, $i = 1,2$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \xi_n < \infty$. For any $p \in F(T_1) \cap F(T_2)$, by (11) we have

$$\begin{aligned} &d\left(x_{n+1}, p\right) \\ &= d\left(W\left(x_n, T_1^n y_n, \alpha_n\right), p\right) \\ &\leq \left(1-\alpha_n\right) d\left(x_n, p\right) + \alpha_n d\left(T_1^n y_n, p\right) \\ &\leq \left(1-\alpha_n\right) d\left(x_n, p\right) + \alpha_n \left[d\left(y_n, p\right) + \mu_n \phi_1\left(d\left(y_n, p\right)\right) + \xi_n\right] \\ &\leq \left(1-\alpha_n\right) d\left(x_n, p\right) + \alpha_n \left[\left(1+\mu_n M^*\right) d\left(y_n, p\right) + \xi_n\right], \end{aligned} \tag{12}$$

where

$$\begin{aligned} &d\left(y_n, p\right) \\ &= d\left(W\left(x_n, T_2^n x_n, \beta_n\right), p\right) \\ &\leq \left(1-\beta_n\right) d\left(x_n, p\right) + \beta_n d\left(T_2^n x_n, p\right) \\ &\leq \left(1-\beta_n\right) d\left(x_n, p\right) + \beta_n \left[d\left(x_n, p\right) + \mu_n \phi_2\left(d\left(x_n, p\right)\right) + \xi_n\right] \\ &\leq \left(1-\beta_n\right) d\left(x_n, p\right) + \beta_n \left[\left(1+\mu_n M^*\right) d\left(x_n, p\right) + \xi_n\right] \\ &\leq \left(1+\beta_n \mu_n M^*\right) d\left(x_n, p\right) + \beta_n \xi_n. \end{aligned} \tag{13}$$

Substituting (13) into (12), we have

$$\begin{aligned} &d\left(x_{n+1}, p\right) \\ &\leq \left(1-\alpha_n\right) d\left(x_n, p\right) \\ &\quad + \alpha_n \left[\left(1+\mu_n M^*\right)\left(\left(1+\beta_n \mu_n M^*\right) d\left(x_n, p\right) + \beta_n \xi_n\right) + \xi_n\right] \\ &\leq \left[1 + \left(1+\beta_n + \beta_n \mu_n M^*\right) \alpha_n \mu_n M^*\right] d\left(x_n, p\right) \\ &\quad + \left(1+\beta_n + \beta_n \mu_n M^*\right) \alpha_n \xi_n. \end{aligned} \tag{14}$$

Applying Lemma 4 to the inequality, we get that $\lim_{n\to\infty} d(x_n, p)$ exist for $p \in F(T_1) \cap F(T_2)$.

Step 2. We show that $\lim_{n\to\infty} d(x_n, T_1 x_n) = \lim_{n\to\infty} d(x_n, T_2 x_n) = 0$.

For each $p \in F(T_1) \cap F(T_2)$, from the proof of Step 1, we know that $\lim_{n\to\infty} d(x_n, p)$ exists. We may assume that $\lim_{n\to\infty} d(x_n, p) = c \geq 0$. The case $c = 0$ is trivial. Next, we deal with the case $c > 0$. From (13), we have

$$d\left(y_n, p\right) \leq \left(1+\beta_n \mu_n M^*\right) d\left(x_n, p\right) + \beta_n \xi_n. \tag{15}$$

Taking limsup on both sides in (15), we have

$$\limsup_{n\to\infty} d\left(y_n, p\right) \leq c. \tag{16}$$

In addition, since

$$\begin{aligned} d\left(T_1^n y_n, p\right) &\leq d\left(y_n, p\right) + \mu_n \phi_1\left(d\left(y_n, p\right)\right) + \xi_n \\ &\leq \left(1+\mu_n M^*\right) d\left(y_n, p\right) + \xi_n, \end{aligned} \tag{17}$$

we have

$$\limsup_{n\to\infty} d\left(T_1^n y_n, p\right) \leq c. \tag{18}$$

Since $\lim_{n\to\infty} d(x_{n+1}, p) = c$, it is easy to prove that

$$\lim_{n\to\infty} d\left(W\left(x_n, T_1^n y_n, \alpha_n\right), p\right) = c. \tag{19}$$

It follows from Lemma 5 that

$$\lim_{n\to\infty} d\left(x_n, T_1^n y_n\right) = 0. \tag{20}$$

On the other hand, since

$$\begin{aligned} d\left(x_n, p\right) &\leq d\left(x_n, T_1^n y_n\right) + d\left(T_1^n y_n, p\right) \\ &\leq d\left(x_n, T_1^n y_n\right) + d\left(y_n, p\right) + \mu_n \phi_1\left(d\left(y_n, p\right)\right) + \xi_n \\ &\leq d\left(x_n, T_1^n y_n\right) + \left(1+\mu_n M^*\right) d\left(y_n, p\right) + \xi_n, \end{aligned} \tag{21}$$

we have $\liminf_{n\to\infty} d(y_n, p) \geq c$. Combined with (16), it yields that

$$\lim_{n\to\infty} d\left(y_n, p\right) = c. \tag{22}$$

This implies that

$$\lim_{n\to\infty} d\left(W\left(x_n, T_2^n x_n, \beta_n\right), p\right) = c. \tag{23}$$

Since

$$\begin{aligned} d\left(T_2^n x_n, p\right) &\le d\left(x_n, p\right) + \mu_n \phi_2\left(d\left(x_n, p\right)\right) + \xi_n \\ &\le \left(1 + \mu_n M^*\right) d\left(x_n, p\right) + \xi_n, \end{aligned} \tag{24}$$

we have

$$\limsup_{n\to\infty} d\left(T_2^n x_n, p\right) \le c. \tag{25}$$

So, it follows from (25) and Lemma 5 that

$$\lim_{n\to\infty} d\left(x_n, T_2^n x_n\right) = 0. \tag{26}$$

Observe that

$$\begin{aligned} &d\left(x_n, T_1^n x_n\right) \\ &\quad \le d\left(x_n, T_1^n y_n\right) + d\left(T_1^n y_n, T_1^n x_n\right) \\ &\quad \le d\left(x_n, T_1^n y_n\right) + d\left(y_n, x_n\right) + \mu_n \phi_1\left(d\left(y_n, x_n\right)\right) + \xi_n \\ &\quad = d\left(x_n, T_1^n y_n\right) + \left(1 + \mu_n M^*\right) d\left(y_n, x_n\right) + \xi_n, \end{aligned} \tag{27}$$

where

$$d\left(y_n, x_n\right) = d\left(W\left(x_n, T_2^n x_n, \beta_n\right), x_n\right) \le \beta_n d\left(x_n, T_2^n x_n\right). \tag{28}$$

It follows from (26) that

$$\lim_{n\to\infty} d\left(y_n, x_n\right) = 0. \tag{29}$$

Thus, from (20), (27), and (29), we have

$$\lim_{n\to\infty} d\left(x_n, T_1^n x_n\right) = 0. \tag{30}$$

In addition, since

$$\begin{aligned} d\left(x_{n+1}, x_n\right) &= d\left(W\left(x_n, T_1^n y_n, \alpha_n\right), x_n\right) \\ &\le \alpha_n d\left(x_n, T_1^n y_n\right), \end{aligned} \tag{31}$$

from (20), we have

$$\lim_{n\to\infty} d\left(x_{n+1}, x_n\right) = 0. \tag{32}$$

Finally, since

$$\begin{aligned} d\left(x_n, T_1 x_n\right) &\le d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, T_1^{n+1} x_{n+1}\right) \\ &\quad + d\left(T_1^{n+1} x_{n+1}, T_1^{n+1} x_n\right) + d\left(T_1^{n+1} x_n, T_1 x_n\right) \\ &\le (1 + L) d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, T_1^{n+1} x_{n+1}\right) \\ &\quad + L d\left(T_1^n x_n, x_n\right), \end{aligned} \tag{33}$$

it follows from (30) and (32) that

$$\lim_{n\to\infty} d\left(x_n, T_1 x_n\right) = 0. \tag{34}$$

Similarly, we also can show that

$$\lim_{n\to\infty} d\left(x_n, T_2 x_n\right) = 0. \tag{35}$$

Step 3. Now we prove that the sequence $\{x_n\}$ Δ-converges to a common fixed point of $F(T_1) \cap F(T_2)$.

In fact, since, for each $p \in F$, $\lim_{n\to\infty} d(x_n, p)$ exists, this implies that the sequence $\{d(x_n, p)\}$ is bounded, so is the sequence $\{x_n\}$. Hence, by virtue of Lemma 3, $\{x_n\}$ has a unique asymptotic center $A_K(\{x_n\}) = \{x\}$.

Let $\{u_n\}$ be any subsequence of $\{x_n\}$ with $A_K(\{u_n\}) = \{u\}$. It follows from (34) that

$$\lim_{n\to\infty} d\left(u_n, T_1 u_n\right) = 0. \tag{36}$$

Now, we show that $u \in F(T_1)$. For this, we define a sequence $\{z_n\}$ in K by $z_k = T_1^k u$. So, we calculate

$$\begin{aligned} &d\left(z_k, u_n\right) \\ &\quad \le d\left(T_1^k u, T_1^k u_n\right) + d\left(T_1^k u_n, T_1^{k-1} u_n\right) + \cdots + d\left(T_1 u_n, u_n\right) \\ &\quad \le d\left(u, u_n\right) + \mu_n \phi_1\left(d\left(u, u_n\right)\right) + \xi_n + \sum_{i=1}^{k} d\left(T_1^i u_n, T_1^{i-1} u_n\right) \\ &\quad \le \left(1 + \mu_n M^*\right) d\left(u, u_n\right) + \xi_n + \sum_{i=1}^{k} d\left(T_1^i u_n, T_1^{i-1} u_n\right). \end{aligned} \tag{37}$$

Since T_1 is uniformly L-Lipschitzian, from (37) we have

$$d\left(z_k, u_n\right) \le \left(1 + \mu_n M^*\right) d\left(u, u_n\right) + \xi_n + kL d\left(T_1 u_n, u_n\right). \tag{38}$$

Taking limsup on both sides of the previous estimate and using (36), we have

$$\begin{aligned} r\left(z_k, \{u_n\}\right) &= \limsup_{n\to\infty} d\left(z_k, u_n\right) \le \limsup_{n\to\infty} d\left(u, u_n\right) \\ &= r\left(u, \{u_n\}\right). \end{aligned} \tag{39}$$

Since $A_K(\{u_n\}) = \{u\}$, by the definition of asymptotic center $A_K(\{u_n\})$ of a bounded sequence $\{u_n\}$ with respect to $K \subset X$ and (8), this implies that $r(z_k, \{u_n\}) = r(u, \{u_n\})$, for all $k \ge 1$. Therefore, $|r(z_k, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $k \to \infty$. It follows from Lemma 6 that $\lim_{k\to\infty} T_1^k u = u$. As T_1 is uniformly continuous, $T_1 u = T_1(\lim_{k\to\infty} T_1^k u) = \lim_{k\to\infty} T_1^{k+1} u = u$. That is, $u \in F(T_1)$. Similarly, we also can show that $u \in F(T_2)$. Hence, u is the common fixed point of T_1 and T_2. Reasoning as previously mentioned by utilizing the uniqueness of asymptotic centers, we get that $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, $A(\{u_n\}) = \{u\}$ for all subsequence $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ Δ-converges to a common fixed point of T_1 and T_2. This completes the proof. □

The following theorem can be obtained from Theorem 7 immediately.

Theorem 8. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T_i : K \to K$, $i = 1, 2$, be asymptotically nonexpansive mappings with sequence $\{t_{in}\} \subset [1, \infty)$ satisfying $\lim_{n\to\infty} t_{in} = 1$. Assume that $F := \bigcap_{i=1}^{2} F(T_i) \neq \emptyset$; for arbitrarily chosen $x_1 \in K$, $\{x_n\}$ is defined as follows:*

$$\begin{aligned} x_{n+1} &= W\left(x_n, T_1^n y_n, \alpha_n\right), \\ y_n &= W\left(x_n, T_2^n x_n, \beta_n\right), \end{aligned} \tag{40}$$

where $\{t_{in}\}$, $i = 1, 2$, $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(1) $\sum_{n=1}^{\infty}(t_{in} - 1) < \infty$, $i = 1, 2$;

(2) there exist constants $a, b \in (0, 1)$ with $0 < b(1-a) \leq 1/2$ such that $\{\alpha_n\} \subset [a, b]$ and $\{\beta_n\} \subset [a, b]$.

Then, the sequence $\{x_n\}$ defined in (40) Δ-converges to a common fixed point of $F := \bigcap_{i=1}^{2} F(T_i)$.

Proof. Take $\phi_i(t) = t$, $t \geq 0, \xi_{in} = 0$, $\mu_{in} = t_{in} - 1$ in Theorem 7. Since all conditions in Theorem 7 are satisfied, it follows from Theorem 7 that the sequence $\{x_n\}$ Δ-converges to a common fixed point of $F := \bigcap_{i=1}^{2} F(T_i)$.

This completes the proof of Theorem 8. □

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