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Research Article

An Iterative Method for the Least-Squares Problems of a General Matrix Equation Subjects to Submatrix Constraints

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An iterative algorithm is proposed for solving the least-squares problem of a general matrix equation $\sum_{i=1}^{t} M_i Z_i N_i = F$, where Z_i $(i = 1, 2, \ldots, t)$ are to be determined centro-symmetric matrices with given central principal submatrices. For any initial iterative matrices, we show that the least-squares solution can be derived by this method within finite iteration steps in the absence of roundoff errors. Meanwhile, the unique optimal approximation solution pair for given matrices $\widetilde{Z}_i$ can also be obtained by the least-norm least-squares solution of matrix equation $\sum_{i=1}^{t} M_i \overline{Z}_i N_i = \overline{F}$, in which $\overline{Z}_i = Z_i - \widetilde{Z}_i$, $\overline{F} = F - \sum_{i=1}^{t} M_i \widetilde{Z}_i N_i$. The given numerical examples illustrate the efficiency of this algorithm.

1. Introduction

Throughout this paper, we denote the set of all $m \times n$ real matrices by $R^{m\times n}$. The symbol A^T represents the transpose of matrix A. J_n and I_n stand for the reverse unit matrix, and identity matrix, respectively. For $A, B \in R^{m\times n}$, the inner product of matrices A and B is defined by $\langle A, B\rangle = \operatorname{trace}(B^T A)$, which leads to the Frobenius norm, that is, $\|A\| = \sqrt{\langle A, A\rangle}$.

A matrix $A = (a_{ij}) \in R^{n\times n}$, is called centro-symmetric (centro-skew symmetric) if and only if

$$a_{ij} = a_{n-i+1,n-j+1} \left(a_{ij} = -a_{n-i+1,n-j+1}\right), \quad i, j = 1, 2, \ldots, n, \tag{1}$$

which can also be characterized equivalently by $J_n A J_n = A$ $(J_n A J_n = -A)$. The set of all centro-symmetric (centro-skew symmetric) matrices is denoted by $\mathrm{CS}R^{n\times n}$ $(\mathrm{CAS}R^{n\times n})$. This kind of matrices plays an important role in many applications (see, e.g., [1–4]), and has been frequently and widely investigated (see, e.g., [5–7]) by using generalized inverse, generalized singular value decomposition (GSVD) [8], and so forth. For more, we refer the readers to [9–16] and therein.

We firstly introduce the concept of the central principal submatrix which is originally put forward by Yin [17].

Definition 1. Let $A \in R^{n\times n}$, if $n - p$ is even, a $p \times p$ central principal submatrix of A, denoted by $A[p]$, is obtained by deleting the first and last $(n - p)/2$ rows and columns of A, namely, $A[p] = (a_{ij})_{(n-p)/2\le i,j\le n-(n-p)/2}$.

Evidently, a matrix with odd (even) order only has central principal submatrices of odd (even) order.

Now, the first problem to be studied here can be stated as follows.

Problem 2. Given $M_i \in R^{m\times p_i}$, $N_i \in R^{p_i\times n}$ and $\mathcal{X}_i \in \mathrm{CS}R^{q_i\times q_i}$ $(i = 1, 2, \ldots, t)$, $F \in R^{m\times n}$. Find the least-squares solution $Z_i \in \Pi_i$ of matrix equation

$$M_1 Z_1 N_1 + M_2 Z_2 N_2 + \cdots + M_t Z_t N_t = F, \tag{2}$$

in which $\Pi_i = \{Z_i \mid Z_i \in \mathrm{CS}R^{p_i\times p_i} \text{ with } Z_i[q_i] = \mathcal{X}_i\}$, $p_i > q_i$, $i \in \Gamma \triangleq \{1, 2, \ldots, t\}$, and $Z_i[q_i]$ represents the $q_i \times q_i$ central principal submatrix of Z_i.

Problem 2 is the submatrix constrained problem of matrix equation (2), which originally arises from a practical subsystem expansion problem, and has been deeply investigated (see, e.g., [7, 18–22]). In these literatures, the generalized inverses or some complicated matrix decompositions such as canonical correlation decomposition (CCD) [23] and GSVD

are employed. However, it is almost impossible to solve (2) by the above methods. The iterative method is an efficient approach. Recently, kinds of iteration methods have been constructed: Zhou and Duan [24] studied the generalized Sylvester matrix equation

$$\sum_{i=0}^{\phi} A_i X F^i + \sum_{k=0}^{\psi} B_k Y F^k = R \tag{3}$$

by so-called generalized Sylvester mapping that has pretty properties. Wu et al. [25] presented an finite iterative method for a class of complex matrix equations including conjugate and transpose of unknown solution. Motivated by the well-known Jacobi and Gauss-Seidel iterations methods, Ding and Chen, in [26], proposed a general family of iterative methods to solve linear matrix equations; meanwhile, these methods were also extended to solve the following coupled Sylvester matrix equations

$$\sum_{j=1}^{p} A_{ij} X_j B_{ij} = C_{ij}, \qquad i = 1, 2, \ldots. \tag{4}$$

Although these iterative algorithms are efficient, there still exist some handicaps when meeting the constrained matrix equation problem (i.e., to find the solution of matrix equation in some matrices sets with specifical structure, for instance, symmetric matrices, centro-symmetric matrices, and bi-symmetric matrices sets) and the submatrix constrained problem, since these methods cannot keep the special properties of the unknown matrix in the iterative process. Based on the classical conjugate gradient (CG) method, Peng et al. [27] gave an iterative method to find the bisymmetric solution of matrix equation (2). Similar method was constructed to solve matrix equations (4) with generalized bisymmetric X_j in [28]. In particular, Li et al. [29] proposed an elegant algorithm for solving the generalized Sylvester (Lyapunov) matrix equation $AXB + CYD = E$ with bisymmetric X and symmetric Y, the two unknown matrices include the given central principal submatrix and leading principal submatrix, respectively. This method shunned the difficulties in numerical instability and computational complexity, and solved the problem, completely. By borrowing the thinking of this iterative algorithm, we will solve Problem 2 by iteration method.

The second problem to be considered is the optimal approximation problem.

Problem 3. Let S_E be the solutions set of Problem 2. For given matrices $\widetilde{Z}_i \in R^{p_i \times p_i}$, find $\widehat{Z}_i$ such that

$$\sum_{i=1}^{t} \left\| \widehat{Z}_i - \widetilde{Z}_i \right\|^2 = \min_{(Z_1, Z_2, \ldots, Z_t) \in S_E} \sum_{i=1}^{t} \left\| Z_i - \widetilde{Z}_i \right\|^2. \tag{5}$$

This problem occurs frequently in experimental design (see for instance [30]). Here, the preliminary estimation $\widetilde{Z}_i$ of the unknown matrix Z_i can be obtained from experiments, but it may not satisfy the structural requirement and/or spectral requirement. The best estimation of Z_i, is the matrix $\widehat{Z}_i$ that satisfies both requirements, which is the optimal approximation of Z_i (see, e.g., [31, 32]). About this problem, we also refer the authors to [9–11, 13, 15, 16, 20–23, 27–29, 33–36] and therein.

The rest of this paper is outlined as follows. In Section 2, an iterative algorithm will be proposed to solve Problem 2, and the properties of which will be investigated. In Section 3, we will consider the optimal approximation Problem 3 by using the iterative algorithm. In Section 4, some numerical examples will be given to verify the efficiency of this algorithm.

2. The Algorithm for Problem 2 and Its Properties

According to the definition of centro-symmetric matrix, when $n - q$ is even, a centro-symmetric matrix $Z \in \mathrm{CSR}^{n \times n}$ can be divided into smaller submatrices, namely,

$$Z = \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & J_q Z_{21} J_{(n-q)/2} \\ J_{(n-q)/2} Z_{13} J_{(n-q)/2} & J_{(n-q)/2} Z_{12} J_q & J_{(n-q)/2} Z_{11} J_{(n-q)/2} \end{pmatrix}, \tag{6}$$

where $Z_{11} \in R^{(n-q)/2 \times (n-q)/2}$, $Z_{12} \in R^{(n-q)/2 \times q}$, $Z_{13} \in R^{(n-q)/2 \times (n-q)/2}$, $Z_{21} \in R^{q \times (n-q)/2}$, and $Z_{22} \in \mathrm{CSR}^{q \times q}$.

Now, for some fixed positive integer $i \in \Gamma$, we define two matrix sets.

$$\begin{gathered} \mathrm{CSR}^{p_i \times p_i}_{\star, q_i} = \left\{ Z_i \mid Z_i \in \mathrm{CSR}^{p_i \times p_i} \text{ with } Z_i [q_i] = 0 \right\}, \\ \mathrm{CSR}^{p_i \times p_i}_{\diamond, q_i} = \left\{ Z_i \mid Z_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & X_i & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{CSR}^{p_i \times p_i}, \right. \\ \left. X_i = Z_i [q_i] \in \mathrm{CSR}^{q_i \times q_i} \right\}. \end{gathered} \tag{7}$$

It is clear that both $\mathrm{CSR}^{p_i \times p_i}_{\star, q_i}$ and $\mathrm{CSR}^{p_i \times p_i}_{\diamond, q_i}$ are linear subspaces of $R^{p_i \times p_i}$.

In addition, for any matrix $X \in R^{p_i \times p_i}$, it has uniquely decomposition in direct sum, that is, $X = X_1 \oplus X_2$, here $X_1 = (X + J_{p_i} X J_{p_i})/2$, $X_2 = (X - J_{p_i} X J_{p_i})/2$. Furthermore, $X_1 = X_{11} + X_{12}$ is also the direct sum decomposition of X_1 if $X_{11} \in \mathrm{CSR}^{p_i \times p_i}_{\star, q_i}$, $Z_{12} \in \mathrm{CSR}^{p_i \times p_i}_{\diamond, q_i}$, since $\langle X_{11}, X_{12} \rangle = 0$. Hence, we obtain the following.

Lemma 4. *Consider* $R^{p_i \times p_i} = \mathrm{CSR}^{p_i \times p_i}_{\star, q_i} \oplus \mathrm{CSR}^{p_i \times p_i}_{\diamond, q_i} \oplus \mathrm{CASR}^{p_i \times p_i}$.

Lemma 4 reveals that any matrix $W \in R^{p_i \times p_i}$ can be uniquely written as $W = W_1 + W_2 + W_3$, where $W_1 \in \mathrm{CSR}^{p_i \times p_i}_{\star, q_i}$, $W_2 \in \mathrm{CSR}^{p_i \times p_i}_{\diamond, q_i}$, $W_3 \in \mathrm{CASR}^{p_i \times p_i}$. Then, we can define the following linear projection operators:

$$\begin{gathered} \mathscr{L}_i : R^{p_i \times p_i} \longrightarrow \mathrm{CSR}^{p_i \times p_i}_{\star, q_i} \\ W \longrightarrow W_1 \end{gathered} \tag{8}$$

for $i \in \Gamma$.

According to the definition of $\mathscr{L}_i$, if $W \in R^{p_i\times p_i}$ and $Y \in \mathrm{CSR}^{p_i\times p_i}_{\star,q_i}$, we have

$$\langle W, Y\rangle = \langle W_1, Y\rangle = \langle \mathscr{L}_i(W), Y\rangle. \tag{9}$$

This property will be employed frequently in the residual context.

The following theorem is essential for solving Problem 2, which transforms equivalently Problem 2 into solving the least-square problem of another matrix equation.

Theorem 5. *Any solution group of Problem 2 can be obtained by*

$$(Z_1, Z_2, \ldots, Z_t) = \left(Y_1 + Z_1^{\diamond}, Y_2 + Z_2^{\diamond}, \ldots, Y_t + Z_t^{\diamond}\right), \tag{10}$$

where $Y_i \in CSR^{p_i\times p_i}_{\star,q_i}$ is the least-squares solution of matrix equation

$$\begin{gathered} M_1Y_1N_1 + M_2Y_2N_2 + \cdots + M_tY_tN_t = G, \\ G = F - \left(M_1Z_1^{\diamond}N_1 + M_2Z_2^{\diamond}N_2 + \cdots + M_tZ_t^{\diamond}N_t\right), \\ Z_i^{\diamond} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathscr{X}_i & 0 \\ 0 & 0 & 0 \end{pmatrix} \in CSR^{p_i\times p_i}_{\diamond,q_i}, \end{gathered} \tag{11}$$

$\mathscr{X}_i \in CSR^{q_i\times q_i}$ is the given central principal submatrix of Z_i in Problem 2.

Proof. Noting that the definition of $\mathrm{CSR}^{p_i\times p_i}_{\star,q_i}$, we have

$$\begin{aligned} &\min_{Z_i\in\Pi_i} \left\|\sum_{i=1}^{t} M_iZ_iN_i - F\right\| \\ &\quad\Longleftrightarrow \min_{Y_i\in\mathrm{CSR}^{p_i\times p_i}_{\star,q_i}} \left\|\sum_{i=1}^{t} M_i\left(Y_i + \overline{Z}_i\right)N_i - F\right\| \\ &\quad\Longleftrightarrow \min_{Y_i\in\mathrm{CSR}^{p_i\times p_i}_{\star,q_i}} \left\|\sum_{i=1}^{t} M_iY_iN_i - G\right\|. \end{aligned} \tag{12}$$

The proof is completed. □

Remark 6. It follows, from Theorem 5, that Problem 2 can be solved completely by finding the least-squares solution of matrix equations (11) in subspaces $\mathrm{CSR}^{p_i\times p_i}_{\diamond,q_i}$.

In the next part of this section, we will establish an iterative algorithm for (11) and analysis its properties. For the convenience of expression, we define a matrix function

$$\mathscr{F}(Z_1, Z_2, \ldots, Z_t) = M_1Z_1N_1 + M_2Z_2N_2 + \cdots + M_tZ_tN_t, \tag{13}$$

then matrix equation (11) can be simplified as

$$\mathscr{F}(Z_1, Z_2, \ldots, Z_t) = G. \tag{14}$$

Moreover, we can easily verify that

$$\left\langle X, \mathscr{F}(Z_1, Z_2, \ldots, Z_t)\right\rangle = \sum_{i=1}^{t} \left\langle M_i^TXN_i^T, Z_i\right\rangle \tag{15}$$

holds for arbitrary $X \in R^{m\times n}$.

The iterative algorithm for the least squares problem of matrix equations (11) can be expressed as follows.

Algorithm 7. Consider the following.

Step 1. Let $M_i \in R^{m\times p_i}$, $N_i \in R^{p_i\times n}$, $F \in R^{m\times n}$ and $\mathscr{X}_i \in \mathrm{CSR}^{q_i\times q_i}$ for $i \in \Gamma$.

Input arbitrary matrices $Y_i^{(0)} \in \mathrm{CSR}^{p_i\times p_i}_{\star,q_i}$.

Step 2. Calculate

$$\begin{gathered} R_0 = G - \mathscr{F}\left(Y_1^{(0)}, Y_2^{(0)}, \ldots, Y_t^{(0)}\right), \\ P_i^{(0)} = \mathscr{L}_i\left(M_i^TR_0N_i^T\right), \quad Q_i^{(0)} = P_i^{(0)}, \\ k = 0. \end{gathered} \tag{16}$$

Step 3. Calculate

$$\begin{gathered} Y_i^{(k+1)} = Y_i^{(k)} + \alpha_kQ_i^{(k)}, \\ \alpha_k = \frac{\sum_{i=1}^{t}\left\|P_i^{(k)}\right\|^2}{\left\|\mathscr{F}\left(Q_1^{(k)}, Q_2^{(k)}, \ldots, Q_t^{(k)}\right)\right\|^2}. \end{gathered} \tag{17}$$

Step 4. Calculate

$$\begin{aligned} R_{k+1} &= G - \mathscr{F}\left(Y_1^{(k+1)}, Y_2^{(k+1)}, \ldots, Y_t^{(k+1)}\right) \\ &= R_k - \alpha_k\mathscr{F}\left(Q_1^{(k)}, Q_2^{(k)}, \ldots, Q_t^{(k)}\right), \\ P_i^{(k+1)} &= \mathscr{L}_i\left(M_i^TR_{k+1}N_i^T\right) \\ &= P_i^{(k)} - \alpha_k\mathscr{L}_i\left(M_i^T\mathscr{F}\left(Q_1^{(k)}, Q_2^{(k)}, \ldots, Q_t^{(k)}\right)N_i^T\right), \\ Q_i^{(k+1)} &= P_i^{(k+1)} + \beta_kQ_i^{(k)}, \\ \beta_k &= \frac{\sum_{i=1}^{t}\left\|P_i^{(k+1)}\right\|^2}{\sum_{i=1}^{t}\left\|P_i^{(k)}\right\|^2}. \end{aligned} \tag{18}$$

Step 5. If $\sum_{i=1}^{t}\left\|P_i^{(k)}\right\|^2 = 0$, stop. Otherwise, $k := k + 1$, go to Step 3.

From Algorithm 7, we can see that $Y_i^{(k)}, P_i^{(k)}, Q_i^{(k)}, \in \mathrm{CSR}^{p_i\times p_i}_{\star,q_i}$. In particular, $(Y_1^{(k)}, Y_2^{(k)}, \ldots, Y_t^{(k)})$ is a least-squares solution group belonging to matrix equation (11) if $P_i^{(k)} = 0$ for all $i \in \Gamma$. The following lemma gives voice to the reason.

Lemma 8. *If $\mathscr{L}_i(M_i^TR_kN_i^T) = 0$ $(i = 1, 2, \ldots, t)$ hold simultaneously for some positive k, then $(Y_1^{(k)}, Y_2^{(k)}, \ldots, Y_t^{(k)})$ generated by Algorithm 7 is a solution group of matrix equation* (11).

Proof. Let $\mathscr{L} = \{L \mid L = \mathscr{F}(Y_i), Y_i \in \mathrm{CSR}^{p_i\times p_i}_{\star,q_i}\}$ and $\widehat{G} = \mathscr{F}(Y_i^{(k)})$. Obviously, $\widehat{G} \in \mathscr{L}$. Then, from the Project Theorem, $(Y_1^{(k)}, Y_2^{(k)}, \ldots, Y_t^{(k)})$ is a least-square solution group of matrix equation (11) if and only if $G - \widehat{G} \perp \mathscr{L}$. That is to say, for

any matrices $Y_i \in CSR^{p_i\times p_i}_{\star,q_i}$, noting that Lemma 4, we have $\langle G-\widehat{G}, \mathscr{F}(Y_1,Y_2,\ldots,Y_t)\rangle = 0$, that is,

$$\begin{aligned}\left\langle G-\widehat{G}, \mathscr{F}\left(Y_1,Y_2,\ldots,Y_t\right)\right\rangle &= \left\langle R_{k+1}, \mathscr{F}\left(Y_1,Y_2,\ldots,Y_t\right)\right\rangle \\ &= \sum_{i=1}^{t}\left\langle M_i^T R_{k+1} N_i^T, Y_i\right\rangle \\ &= \sum_{i=1}^{t}\left\langle \mathscr{L}_i\left(M_i^T R_{k+1} N_i^T\right), Y_i\right\rangle = 0,\end{aligned} \tag{19}$$

which completes the proof. □

In addition, the sequences $\{Y_i^{(l)}\}$, $\{P_i^{(l)}\}$, $\{Q_i^{(l)}\}$ generated by Algorithm 7 are self-orthogonal, that is, as follows.

Lemma 9. *Suppose that the sequences $\{Y_i^{(l)}\}$, $\{P_i^{(l)}\}$, $\{Q_i^{(l)}\}$ generated by Algorithm 7 not equal null for $l \le k$, then*

$$(1)\ \sum_{i=1}^{t}\left\langle P_i^{(j)}, P_i^{(l)}\right\rangle = 0, \tag{20}$$

$$(2)\ \sum_{i=1}^{t}\left\langle Q_i^{(j)}, P_i^{(l)}\right\rangle = 0, \tag{21}$$

$$(3)\ \left\langle \mathscr{F}\left(Q_1^{(j)},Q_2^{(j)},\ldots,Q_t^{(j)}\right), \mathscr{F}\left(Q_1^{(l)},Q_2^{(l)},\ldots,Q_t^{(l)}\right)\right\rangle = 0, \tag{22}$$

where $j,l = 1,2,\ldots,k$, $j \ne l \le k$.

Proof. In view of the symmetry of the inner product, we only prove (20)–(22) when $j < l$. According to Algorithm 7, when $k = 1$, we have

$$\begin{aligned}&\sum_{i=1}^{t}\left\langle P_i^{(0)}, P_i^{(1)}\right\rangle \\ &= \sum_{i=1}^{t}\left\langle P_i^{(0)}, P_i^{(0)} - \alpha_0 \mathscr{L}_i M_i^T\left(\mathscr{F}\left(Q_1^{(0)},Q_2^{(0)},\ldots,Q_t^{(0)}\right)N_i^T\right)\right\rangle \\ &= \sum_{i=1}^{t}\left[\left\langle P_i^{(0)}, P_i^{(0)}\right\rangle \right. \\ &\qquad \left. -\alpha_0\left\langle P_i^{(0)}, \mathscr{L}_i M_i^T\left(\mathscr{F}\left(Q_1^{(0)},Q_2^{(0)},\ldots,Q_t^{(0)}\right)N_i^T\right)\right\rangle\right] \\ &= \sum_{i=1}^{t}\left\|P_i^{(0)}\right\|^2 - \alpha_0 \\ &\quad \times \sum_{i=1}^{t}\left\langle P_i^{(0)}, M_i^T \mathscr{F}\left(Q_1^{(0)},Q_2^{(0)},\ldots,Q_t^{(0)}\right)N_i^T\right\rangle \\ &= \sum_{i=1}^{t}\left\|P_i^{(0)}\right\|^2 - \alpha_0 \sum_{i=1}^{t}\left\langle M_i P_i^{(0)} N_i, \mathscr{F}\left(Q_1^{(0)},Q_2^{(0)},\ldots,Q_t^{(0)}\right)\right\rangle \\ &= \sum_{i=1}^{t}\left\|P_i^{(0)}\right\|^2 - \alpha_0\left\|\mathscr{F}\left(Q_1^{(0)},Q_2^{(0)},\ldots,Q_t^{(0)}\right)\right\|^2 \\ &= 0,\end{aligned} \tag{23}$$

which also deduces that

$$\begin{aligned}&\sum_{i=1}^{t}\left\langle Q_i^{(0)}, P_i^{(1)}\right\rangle \\ &= \sum_{i=1}^{t}\left\langle Q_i^{(0)}, P_i^{(0)} - \alpha_0 \mathscr{L}_i\left(M_i^T \mathscr{F}\left(Q_1^{(0)},Q_2^{(0)},\ldots,Q_t^{(0)}\right)N_i^T\right)\right\rangle \\ &= \sum_{i=1}^{t}\left\langle Q_i^{(0)}, P_i^{(0)}\right\rangle - \alpha_0 \\ &\quad \times \sum_{i=1}^{t}\left\langle Q_i^{(0)}, M_i^T \mathscr{F}\left(Q_1^{(0)},Q_2^{(0)},\ldots,Q_t^{(0)}\right)N_i^T\right\rangle \\ &= \sum_{i=1}^{t}\left\langle P_i^{(0)}, P_i^{(0)}\right\rangle - \alpha_0 \\ &\quad \times \sum_{i=1}^{t}\left\langle M_i Q_i^{(0)} N_i, \mathscr{F}\left(Q_1^{(0)},Q_2^{(0)},\ldots,Q_t^{(0)}\right)\right\rangle \\ &= 0,\end{aligned}$$

$$\begin{aligned}&\left\langle \mathscr{F}\left(Q_1^{(0)},Q_2^{(0)},\ldots,Q_t^{(0)}\right), \mathscr{F}\left(Q_1^{(1)},Q_2^{(1)},\ldots,Q_t^{(1)}\right)\right\rangle \\ &= \left\langle \mathscr{F}\left(Q_1^{(0)},Q_2^{(0)},\ldots,Q_t^{(0)}\right), \sum_{i=1}^{t}\left[M_i\left(P_i^{(1)} + \beta_0 Q_i^{(0)}\right)N_i\right]\right\rangle \\ &= \left\langle \mathscr{F}\left(Q_1^{(0)},Q_2^{(0)},\ldots,Q_t^{(0)}\right), \mathscr{F}\left(P_1^{(1)},P_2^{(1)},\ldots,P_t^{(1)}\right)\right\rangle \\ &\quad + \beta_0\left\|\mathscr{F}\left(Q_1^{(0)},\ldots,Q_t^{(0)}\right)\right\|^2 \\ &= \frac{1}{\alpha_0}\left\langle R_0 - R_1, \mathscr{F}\left(P_1^{(1)},P_2^{(1)},\ldots,P_t^{(1)}\right)\right\rangle \\ &\quad + \beta_0\left\|\mathscr{F}\left(Q_1^{(0)},Q_2^{(0)},\ldots,Q_t^{(0)}\right)\right\|^2 \\ &= \frac{1}{\alpha_0}\left[\sum_{i=1}^{t}\left\langle M_i^T R_0 N_i^T, P_i^{(1)}\right\rangle - \sum_{i=1}^{t}\left\langle M_i^T R_1 N_i^T, P_i^{(1)}\right\rangle\right] \\ &\quad + \beta_0\left\|\mathscr{F}\left(Q_1^{(0)},Q_2^{(0)},\ldots,Q_t^{(0)}\right)\right\|^2 \\ &= -\frac{1}{\alpha_0}\sum_{i=1}^{t}\left\|P_i^{(1)}\right\|^2 + \beta_0\left\|\mathscr{F}\left(Q_1^{(0)},Q_2^{(0)},\ldots,Q_t^{(0)}\right)\right\|^2 \\ &= 0.\end{aligned} \tag{24}$$

Assume that (20), (21), and (22) hold for positive integer $s\ (< k)$, that is, for $j = 1, 2, \ldots, s-1$,

$$\sum_{i=1}^{t} \left\langle P_i^{(j)}, P_i^{(s)} \right\rangle = 0,$$
$$\sum_{i=1}^{t} \left\langle Q_i^{(j)}, P_i^{(s)} \right\rangle = 0,$$
$$\left\langle \mathscr{F}\left(Q_1^{(j)}, Q_2^{(j)}, \ldots, Q_t^{(j)}\right), \mathscr{F}\left(Q_1^{(s)}, Q_2^{(s)}, \ldots, Q_t^{(s)}\right) \right\rangle = 0. \tag{25}$$

Then, similar to the above proof, noting that the assumptions, we have

$$\begin{aligned}
&\sum_{i=1}^{t} \left\langle P_i^{(j)}, P_i^{(s+1)} \right\rangle \\
&= \sum_{i=1}^{t} \left\langle P_i^{(j)}, P_i^{(s)} - \alpha_s \mathscr{L}_i \left(M_i^T \mathscr{F}\left(Q_1^{(s)}, Q_2^{(s)}, \ldots, Q_t^{(s)}\right) N_i^T\right) \right\rangle \\
&= \sum_{i=1}^{t} \left\langle P_i^{(j)}, P_i^{(s)} \right\rangle - \alpha_s \\
&\quad \times \sum_{i=1}^{t} \left\langle P_i^{(j)}, \mathscr{L}_i \left(M_i^T \mathscr{F}\left(Q_1^{(s)}, Q_2^{(s)}, \ldots, Q_t^{(s)}\right) N_i^T\right) \right\rangle \\
&= -\alpha_s \sum_{i=1}^{t} \left\langle M_i P_i^{(j)} N_i, \mathscr{F}\left(Q_1^{(s)}, Q_2^{(s)}, \ldots, Q_t^{(s)}\right) \right\rangle \\
&= -\alpha_s \sum_{i=1}^{t} \left\langle M_i \left(Q_i^{(j)} - \beta_{j-1} Q_{j-1,i}\right) N_i, \right. \\
&\qquad \left. \mathscr{F}\left(Q_1^{(s)}, Q_2^{(s)}, \ldots, Q_t^{(s)}\right) \right\rangle \\
&= -\alpha_s \left\langle \mathscr{F}\left(Q_1^{(j)}, Q_2^{(j)}, \ldots, Q_t^{(j)}\right), \mathscr{F}\left(Q_1^{(s)}, Q_2^{(s)}, \ldots, Q_t^{(s)}\right) \right\rangle \\
&\quad + \alpha_s \beta_{j-1} \left\langle \mathscr{F}\left(Q_1^{(j-1)}, Q_2^{(j-1)}, \ldots, Q_t^{(j-1)}\right), \right. \\
&\qquad \left. \mathscr{F}\left(Q_1^{(s)}, Q_2^{(s)}, \ldots, Q_t^{(s)}\right) \right\rangle \\
&= 0.
\end{aligned} \tag{26}$$

Furthermore,

$$\begin{aligned}
&\sum_{i=1}^{t} \left\langle Q_i^{(j)}, P_i^{(s+1)} \right\rangle \\
&= \sum_{i=1}^{t} \left\langle Q_i^{(j)}, P_i^{(s)} - \alpha_s \mathscr{L}_i \left(M_i^T \mathscr{F}\left(Q_1^{(s)}, Q_2^{(s)}, \ldots, Q_t^{(s)}\right) N_i^T\right) \right\rangle \\
&= \sum_{i=1}^{t} \left\langle P_i^{(j)}, P_i^{(s)} \right\rangle - \alpha_s \\
&\quad \times \sum_{i=1}^{t} \left\langle M_i Q_i^{(j)} N_i, \mathscr{F}\left(Q_1^{(s)}, Q_2^{(s)}, \ldots, Q_t^{(s)}\right) \right\rangle \\
&= 0, \\
&\left\langle \mathscr{F}\left(Q_1^{(j)}, Q_2^{(j)}, \ldots, Q_t^{(j)}\right), \mathscr{F}\left(Q_1^{(s+1)}, Q_2^{(s+1)}, \ldots, Q_t^{(s+1)}\right) \right\rangle \\
&= \left\langle \mathscr{F}\left(Q_1^{(j)}, Q_2^{(j)}, \ldots, Q_t^{(j)}\right), \mathscr{F}\left(P_1^{(s+1)}, P_2^{(s+1)}, \ldots, P_t^{(s+1)}\right) \right\rangle \\
&\quad + \beta_s \left\langle \mathscr{F}\left(Q_1^{(j)}, Q_2^{(j)}, \ldots, Q_t^{(j)}\right), \mathscr{F}\left(Q_1^{(s)}, Q_2^{(s)}, \ldots, Q_t^{(s)}\right) \right\rangle \\
&= \frac{1}{\alpha_j} \left\langle R_j - R_{j+1}, \mathscr{F}\left(P_1^{(s+1)}, P_2^{(s+1)}, \ldots, P_t^{(s+1)}\right) \right\rangle \\
&= \frac{1}{\alpha_j} \left[\sum_{i=1}^{t} \left\langle M_i^T R_j N_i^T, P_i^{(s+1)} \right\rangle - \sum_{i=1}^{t} \left\langle M_i^T R_{j+1} N_i^T, P_i^{(s+1)} \right\rangle \right] \\
&= \frac{1}{\alpha_j} \left[\sum_{i=1}^{t} \left\langle P_i^{(j)}, P_i^{(s+1)} \right\rangle - \sum_{i=1}^{t} \left\langle P_{j+1,i}, P_i^{(s+1)} \right\rangle \right] \\
&= 0.
\end{aligned} \tag{27}$$

The last equal sign "=" holds due to $\sum_{i=1}^{t} \langle P_i^{(s)}, P_i^{(s+1)} \rangle = 0$. In fact, from the hypothesis, we deduce

$$\begin{aligned}
&\sum_{i=1}^{t} \left\langle P_i^{(s)}, P_i^{(s+1)} \right\rangle \\
&= \sum_{i=1}^{t} \left\langle P_i^{(s)}, P_i^{(s)} - \alpha_s \mathscr{L}_i \left(M_i^T \mathscr{F}\left(Q_1^{(s)}, Q_2^{(s)}, \ldots, Q_t^{(s)}\right) N_i^T\right) \right\rangle \\
&= \sum_{i=1}^{t} \left\| P_i^{(s)} \right\|^2 \\
&\quad - \alpha_s \sum_{i=1}^{t} \left\langle M_i \left(Q_i^{(s)} - \beta_{s-1} Q_i^{(j-1)}\right) N_i, \mathscr{F}\left(Q_1^{(s)}, Q_2^{(s)}, \ldots, Q_t^{(s)}\right) \right\rangle \\
&= \sum_{i=1}^{t} \left\| P_i^{(s)} \right\|^2 - \alpha_s \left\| \mathscr{F}\left(Q_1^{(s)}, Q_2^{(s)}, \ldots, Q_t^{(s)}\right) \right\|^2 \\
&\quad + \alpha_s \beta_{s-1} \left\langle \mathscr{F}\left(Q_1^{(s-1)}, Q_2^{(s-1)}, \ldots, Q_t^{(s-1)}\right), \right. \\
&\qquad \left. \mathscr{F}\left(Q_1^{(s)}, Q_2^{(s)}, \ldots, Q_t^{(s)}\right) \right\rangle \\
&= 0.
\end{aligned} \tag{28}$$

Moreover,

$$\sum_{i=1}^{t} \left\langle Q_i^{(s)}, P_i^{(s)} \right\rangle = \sum_{i=1}^{t} \left\langle P_i^{(s)} + \beta_{s-1} Q_i^{(s-1)}, P_i^{(s)} \right\rangle$$

$$= \sum_{i=1}^{t} \left\| P_i^{(s)} \right\|^2 + \beta_{s-1} \sum_{i=1}^{t} \left\langle Q_i^{(s-1)}, P_i^{(s)} \right\rangle$$
$$= \sum_{i=1}^{t} \left\| P_i^{(s)} \right\|^2. \tag{29}$$

It follows from (29) that

$$\begin{aligned}
&\sum_{i=1}^{t} \left\langle Q_i^{(s)}, P_i^{(s+1)} \right\rangle \\
&= \sum_{i=1}^{t} \left\langle Q_i^{(s)}, P_i^{(s)} - \alpha_s \mathscr{L}_i \left(M_i^T \mathscr{F} \left(Q_1^{(s)}, Q_2^{(s)}, \ldots, Q_t^{(s)} \right) N_i^T \right) \right\rangle \\
&= \sum_{i=1}^{t} \left[\left\langle Q_i^{(s)}, P_i^{(s)} \right\rangle \right. \\
&\qquad \left. - \alpha_s \left\langle Q_i^{(s)}, M_i^T \mathscr{F} \left(Q_1^{(s)}, Q_2^{(s)}, \ldots, Q_t^{(s)} \right) N_i^T \right\rangle \right] \\
&= \sum_{i=1}^{t} \left\| P_i^{(s)} \right\|^2 - \alpha_s \left\| \mathscr{F} \left(Q_1^{(s+1)}, Q_2^{(s+1)}, \ldots, Q_t^{(s+1)} \right) \right\|^2 \\
&= 0, \\
&\left\langle \mathscr{F} \left(Q_1^{(s)}, Q_2^{(s)}, \ldots, Q_t^{(s)} \right), \mathscr{F} \left(Q_1^{(s+1)}, Q_2^{(s+1)}, \ldots, Q_t^{(s+1)} \right) \right\rangle \\
&= \left\langle \mathscr{F} \left(Q_1^{(s)}, \ldots, Q_t^{(s)} \right), \mathscr{F} \left(P_1^{(s+1)}, P_2^{(s+1)}, \ldots, P_t^{(s+1)} \right) \right\rangle \\
&\quad + \beta_s \left\| \mathscr{F} \left(Q_1^{(s)}, \ldots, Q_t^{(s)} \right) \right\|^2 \\
&= \frac{1}{\alpha_s} \left\langle R_s - R_{s+1}, \mathscr{F} \left(P_1^{(s+1)}, P_2^{(s+1)}, \ldots, P_t^{(s+1)} \right) \right\rangle \\
&\quad + \beta_s \sum_{i=1}^{t} \left\| M_i Q_i^{(s)} N_i \right\|^2 \\
&= -\frac{1}{\alpha_s} \sum_{i=1}^{t} \left\langle M_i^T R_{s+1} N_i^T, P_i^{(s+1)} \right\rangle \\
&\quad + \beta_s \sum_{i=1}^{t} \left\| M_i Q_i^{(s)} N_i \right\|^2 \\
&= 0.
\end{aligned} \tag{30}$$

Therefore, for arbitrary integers number l, the conclusions hold. The proof is completed. □

Remark 10. We know from Lemma 4 that the matrices sequences

$$\begin{pmatrix} P_{0,1} & & \\ & \ddots & \\ & & P_{0,t} \end{pmatrix}, \begin{pmatrix} P_{1,1} & & \\ & \ddots & \\ & & P_{1,t} \end{pmatrix}, \ldots, \begin{pmatrix} P_{k,1} & & \\ & \cdots & \\ & & P_{k,t} \end{pmatrix}, \ldots \tag{31}$$

are orthogonal to each other. Hence, it can be regarded as an orthogonal basis of matrix space $R^{\sum_{i=1}^{t} p_i \times \sum_{i=1}^{t} p_i}$. Hence, the iteration will be terminated at most $\sum_{i=1}^{t}(p_i + q_i)(p_i - q_i)/2$ steps in the absence of roundoff errors. Therefore, there exists a positive integer $k \leq \sum_{i=1}^{t}(p_i + q_i)(p_i - q_i)/2$ such that $\sum_{i=1}^{t} \|P_i^{(k)}\|^2 = 0$, in this case, $(Y_1^{(k)}, Y_2^{(k)}, \ldots, Y_t^{(k)})$ can be regarded as a least-squares solution group of matrix equation (11).

In addition, we should point out that if $\alpha_k = 0$ or ∞, the conclusions may not be true, and the iteration will break down before $P_i^{(k)} = 0$ for $k < \sum_{i=1}^{t}(p_i + q_i)(p_i - q_i)/2$. Actually, $\alpha_k = 0$ implies that $\sum_{i=1}^{t} \|P_i^{(k)}\|^2 = 0$, so $P_i^{(k)} = 0$ for $i \in \Gamma$. While $\alpha_k = \infty$ leads to $\mathscr{F}(Q_1^{(k)}, Q_2^{(k)}, \ldots, Q_t^{(k)}) = 0$, making inner product with R_i by both sides, it follows from Algorithm 7 that

$$\begin{aligned}
&\left\langle R_i, \mathscr{F} \left(Q_1^{(k)}, Q_2^{(k)}, \ldots, Q_t^{(k)} \right) \right\rangle \\
&= \sum_{i=1}^{t} \left\langle P_i^{(k)}, P_i^{(k)} + \beta_{k-1} Q_i^{(k-1)} \right\rangle \\
&= \sum_{i=1}^{t} \left\| P_i^{(k)} \right\|^2 + \beta_{k-1} \sum_{i=1}^{t} \left\langle P_i^{(k)}, Q_i^{(k-1)} \right\rangle \\
&= \sum_{i=1}^{t} \left\| P_i^{(k)} \right\|^2 = 0,
\end{aligned} \tag{32}$$

which also implies the same situation as $\alpha_k = 0$. Hence, if there exists a positive integer s such that the coefficient $\alpha_s = 0$ or $\alpha_s = \infty$, then the corresponding matrix group $Y_1^{(s)}, Y_2^{(s)}, \ldots, Y_t^{(s)}$ is just the solution of matrix equation (11).

Together with the above analysis and Lemma 9, we can conclude the following theorem.

Theorem 11. *For any initial iteration matrices $Y_i^{(0)} \in CSR_{\star,q_i}^{n\times n}$, $i = 1, 2, \ldots, t$, the least-squares solution of matrix equation* (11) *can be obtained within finite iteration steps. Moreover, Suppose that $(\check{Y}_1, \check{Y}_2, \ldots, \check{Y}_t)$ is a least-squares solution group of* (11), *then the general solution to Problem 2 can be expressed as $Z_i = \check{Y}_i + X_i + Z_i^{\diamond} (i = 1, 2, \ldots, t)$, where $X_i \in CSR_{\star,q_i}^{n\times n}$ satisfy homogeneous equation*

$$\mathscr{F} \left(X_1, X_2, \ldots, X_t \right) = 0, \tag{33}$$

$Z_i^{\diamond}$ as in Theorem 5.

In order to show the validity of Theorem 11, it is adequate to prove the following conclusion.

Proposition 12. *The least-squares solution group* $(\check{Y}_1, \check{Y}_2, \ldots, \check{Y}_t)$ *of matrix equation* (11) *can be expressed as* $\check{Y}_i + X_i$, *where* X_i *satisfy equality* (33).

Proof. According to the assumptions, we obtain

$$\min_{Y_i \in \mathrm{CSR}^{m\times n}_{*,di}} \left\| \mathscr{F}(Y_1, Y_2, \ldots, Y_t) - G \right\| = \left\| \mathscr{F}\left(\check{Y}_1, \check{Y}_2, \ldots, \check{Y}_t\right) - G \right\|. \tag{34}$$

On the other hand, noting that $\mathscr{F}(X_1, X_2, \ldots, X_t) = 0$, then

$$\begin{aligned} &\left\| \mathscr{F}\left(\check{Y}_1 + X_1, \check{Y}_2 + X_2, \ldots, \check{Y}_t + X_t\right) - G \right\|^2 \\ &\quad = \left\| \mathscr{F}\left(\check{Y}_1, \check{Y}_2, \ldots, \check{Y}_t\right) - G + \mathscr{F}(X_1, X_2, \ldots, X_t) \right\|^2 \\ &\quad = \left\| \mathscr{F}\left(\check{Y}_1, \check{Y}_2, \ldots, \check{Y}_t\right) - G \right\|^2. \end{aligned} \tag{35}$$

The proof is completed. □

Next, we will show that the unique least norm solution of matrix equation (11) can be derived by choosing a special kind of initial iteration matrices.

Theorem 13. *Let the initial iteration matrices* $Y_i^{(0)} = \mathscr{L}_i(M_i^T H N_i^T)$ *with arbitrary* $H \in R^{m\times n}$, $i = 1, 2, \ldots, t$, *and then* $(Y_1^*, Y_2^*, \ldots, Y_t^*)$ *generated by Algorithm 7 is the least-norm least-squares solution group of matrix equation* (11). *Furthermore, the least-norm solution group to Problem 2 can be expressed by*

$$(Z_1^*, Z_2^*, \ldots, Z_t^*) = \left(Y_1^* + Z_i^\diamond, Y_2^* + Z_2^\diamond, \ldots, Y_t^* + Z_t^\diamond\right). \tag{36}$$

Proof. From Algorithm 7 and Theorem 11, for initial iteration matrices $Y_i^{(0)} = \mathscr{L}_i(M_i^T H N_i^T)$, we can obtain a least-squares solution Y_i^* of matrix equation (11) and there exists a matrix H^* such that $Y_i^* = \mathscr{L}_i(M_i^T H^* N_i^T)$. Hence, it is enough to prove that the Y_i^* is the least-norm solution. In fact, noting that (33) and Proposition 12, we have

$$\begin{aligned} &\sum_{i=1}^t \left\| Y_i^* + X_i \right\|^2 \\ &\quad = \sum_{i=1}^t \left(\left\| Y_i^* \right\|^2 + \left\| X_i \right\|^2 + 2\left\langle Y_i^*, X_i \right\rangle \right) \\ &\quad = \sum_{i=1}^t \left\| Y_i^* \right\|^2 + \sum_{i=1}^t \left\| X_i \right\|^2 + 2\sum_{i=1}^t \left\langle \mathscr{L}_i\left(M_i^T H^* N_i^T\right), X_i \right\rangle \\ &\quad \geq \sum_{i=1}^t \left\| Y_i^* \right\|^2, \end{aligned} \tag{37}$$

as required. □

Theorems 11 and 13 display the efficiency of Algorithm 7. Actually, the iteration sequence $\{Y_i^{(k)}\}$ converges smoothly to the solution Y_i, that is the minimization property of Algorithm 7.

Theorem 14. *For any initial iteration matrices* $Y_i^{(0)}$, *the* $Y_i^{(k)}$ *generated by Algorithm 7 satisfy the minimization problem*

$$\left\| \mathscr{F}\left(Y_1^{(k)}, Y_2^{(k)}, \ldots, Y_t^{(k)}\right) - G \right\| = \min_{Y_i \in \mathbb{L}_i} \left\| \mathscr{F}(Y_1, Y_2, \ldots, Y_t) - G \right\|, \tag{38}$$

where $\mathbb{L}_i = Y_i^{(0)} + \mathrm{span}\{Q_i^{(0)}, Q_i^{(1)}, \ldots, Q_i^{(k-1)}\}$, $i = 1, 2, \ldots, t$.

Proof. From the definition of $\mathbb{L}_i$, there exist a series of real numbers $a_0, a_1, \ldots, a_{k-1}$ such that $Y_i = Y_i^{(0)} + \sum_{l=0}^{k-1} a_l Q_i^{(l)}$.

Define a function of k variables $f(a_0, a_1, \ldots, a_{k-1})$, that is,

$$f(a_0, a_1, \ldots, a_{k-1}) = \left\| \sum_{i=1}^t \left[M_i \left(Y_i^{(0)} + \sum_{l=0}^{k-1} a_l Q_i^{(l)} \right) N_i - G \right] \right\|^2. \tag{39}$$

In addition, from Algorithm 7, we know that

$$\begin{aligned} R_0 = R_l &+ \alpha_{l-1} \mathscr{F}\left(Q_1^{(l-1)}, Q_2^{(l-1)}, \ldots, Q_t^{(l-1)}\right) \\ &+ \cdots + \alpha_0 \mathscr{F}\left(Q_1^{(0)}, Q_2^{(0)}, \ldots, Q_t^{(0)}\right). \end{aligned} \tag{40}$$

Noting that (22) and making the inner product with $\mathscr{F}(Q_1^{(l)}, Q_2^{(l)}, \ldots, Q_t^{(l)})$ on both sides of (40) yield

$$\left\langle \mathscr{F}\left(Q_1^{(l)}, Q_2^{(l)}, \ldots, Q_t^{(l)}\right), R_0 \right\rangle = \left\langle \mathscr{F}\left(Q_1^{(l)}, Q_2^{(l)}, \ldots, Q_t^{(l)}\right), R_l \right\rangle. \tag{41}$$

Hence, by simple calculation, (40) and (41), the function can be rewritten as

$$\begin{aligned} f(a_0, a_1, \ldots, a_{k-1}) &= \left\| R_0 \right\|^2 + \sum_{l=0}^{k-1} a_l^2 \left\| \mathscr{F}\left(Q_1^{(l)}, Q_2^{(l)}, \ldots, Q_t^{(l)}\right) \right\|^2 \\ &\quad - 2\sum_{l=0}^{k-1} a_l \left\langle \mathscr{F}\left(Q_1^{(l)}, Q_2^{(l)}, \ldots, Q_t^{(l)}\right), R_l \right\rangle. \end{aligned} \tag{42}$$

Then,

$$\min_{a_l} f(a_0, a_1, \ldots, a_{k-1}) \Longleftrightarrow \min_{Y_i \in \mathbb{L}_i} \left\| \mathscr{F}(Y_1, Y_2, \ldots, Y_t) - G \right\|. \tag{43}$$

Since $f(a_0, a_1, \ldots, a_{k-1}) = \min$ only if $\partial f/\partial a_l = 0$, it follows from (29) that

$$\begin{aligned} a_l &= \frac{\left\langle \mathscr{F}\left(Q_1^{(l)}, Q_2^{(l)}, \ldots, Q_t^{(l)}\right), R_l \right\rangle}{\left\| \mathscr{F}\left(Q_1^{(l)}, Q_2^{(l)}, \ldots, Q_t^{(l)}\right) \right\|^2} \\ &= \frac{\sum_{i=1}^{t} \left\langle Q_i^{(l)}, P_{l,i} \right\rangle}{\left\| \mathscr{F}\left(Q_1^{(l)}, Q_2^{(l)}, \ldots, Q_t^{(l)}\right) \right\|^2} \\ &= \frac{\sum_{i=1}^{t} \left\| P_{l,i} \right\|^2}{\left\| \mathscr{F}\left(Q_1^{(l)}, Q_2^{(l)}, \ldots, Q_t^{(l)}\right) \right\|^2} = \alpha_l. \end{aligned} \tag{44}$$

Combined with (43), we complete the proof. □

Theorem 14 reveals that the sequence

$$\left\{ \left\| \mathscr{F}\left(Y_1^{(k)}, Y_2^{(k)}, \ldots, Y_t^{(k)}\right) - G \right\| \right\} \tag{45}$$

monotonically decreases with respect to increasing integer k. The descent property of the residual norm of matrix equation (11) leads to the smoothly convergence of Algorithm 7.

3. The Solution of Problem 3

In this section, we discuss the optimal approximation Problem 3. Since the least squares problem is always consistent, it is easy to verify that the solution set S_E of Problem 2 is a nonempty convex cone, so the optimal approximation solution is unique.

Without loss of generality, we can assume that the given matrices $\widetilde{Z}_i \in \mathrm{CSR}_{\star,q_i}^{p_i \times p_i}$. In fact, from Lemma 4, arbitrary $\widetilde{Z}_i \in R^{p_i \times p_i}$ can be divided into

$$\begin{gathered} \widetilde{Z}_i = \widetilde{Z}_{i1} + \widetilde{Z}_{i2} + \widetilde{X}_{i3}, \quad \text{with } \widetilde{Z}_{i1} \in \mathrm{CSR}_{\star,q_i}^{p_i \times p_i}, \\ \widetilde{Z}_{i2} \in \mathrm{CSR}_{\diamond,q_i}^{p_i \times p_i}, \ \widetilde{Z}_{i3} \in \mathrm{CASR}^{p_i \times p_i}. \end{gathered} \tag{46}$$

Furthermore, if $Z_i \in \mathrm{CSR}_{\star,q_i}^{p_i \times p_i}$, then

$$\left\| Z_i - \widetilde{Z}_i \right\|^2 = \left\| Z_i - \widetilde{Z}_{i1} \right\|^2 + \left\| \widetilde{Z}_{i2} \right\|^2 + \left\| \widetilde{Z}_{i3} \right\|^2, \tag{47}$$

which meets the claim.

Denote $\overline{Z}_i = Z_i - \widetilde{Z}_i$, $\overline{F} = F - \mathscr{F}(\widetilde{Z}_1, \widetilde{Z}_2, \ldots, \widetilde{Z}_t)$, then to solve Problem 3 is equivalent to find the least-norm solution of the new matrix equation

$$\mathscr{F}\left(\overline{Z}_1, \overline{Z}_2, \ldots, \overline{Z}_t\right) = \overline{F}. \tag{48}$$

Furthermore, similar to the construction of (11), Problem 2 is transformed equivalently into finding the least-norm least-squares solution of matrix equation

$$\mathscr{F}\left(\overline{Y}_1, \overline{Y}_2, \ldots, \overline{Y}_t\right) = \overline{G}, \quad \text{with } \overline{Y}_i \in \mathrm{CSR}_{\star,q_i}^{p_i \times p_i}, \tag{49}$$

in which $\overline{G} = \overline{F} - \mathscr{F}(Z_1^{\diamond}, Z_2^{\diamond}, \ldots, Z_t^{\diamond})$.

Therefore, we can apply Algorithm 7 to derive the required solution of matrix equation (49). Virtually, it follows from Theorem 13 that if let the initial iteration matrices $\overline{Y}_i^{(0)} = \mathscr{L}_i(M_i^T \widetilde{H} N_i^T)$ with arbitrary $\widetilde{H} \in R^{m \times n}$, or especially $\overline{Y}_i^{(0)} = 0 \in R^{p_i \times p_i}$, then the iteration solutions $\overline{Y}_i^*$ consist of the least-norm least-squares solution of which. In this case, the unique optimal approximation solution to Problem 3 can be obtained by

$$\begin{aligned} &\left(\widehat{Z}_1, \widehat{Z}_2, \ldots, \widehat{Z}_t\right) \\ &\quad = \left(\overline{Y}_1^* + \widetilde{Z}_1 + Z_1^{\diamond}, \overline{Y}_2^* + \widetilde{Z}_2 + Z_2^{\diamond}, \ldots, \overline{Y}_t^* + \widetilde{Z}_t + Z_t^{\diamond}\right). \end{aligned} \tag{50}$$

4. Numerical Example

In this section, we illustrate the efficiency and reliability of Algorithm 7 by some numerical experiments.

All the numerical experiments are performed by using Matlab 6.5. In addition, because of the influence of the round-off errors, $P_i^{(k)}$ may not equal zero within finite iteration steps, so the iteration will be terminated if $\sum_{i=1}^{t} \left\| P_i^{(k)} \right\|^2 < \epsilon$, for example, let $\epsilon = 1.0e-008$. At this time, $(Y_1^{(k)}, Y_1^{(k)}, \ldots, Y_t^{(k)})$ can be regarded as a solution of matrix equation (11), and $Y_i^{(k)} + Z_i^{\diamond}$ $(i = 1, 2, \ldots, t)$ consist of the solution group to Problem 2. In particular, let the initial iteration matrices $Y_i^{(0)} = 0$, then we will obtain the least-norm solution by (36).

Example 1. Input matrices M_1, M_2, M_3, N_1, N_2, N_3, and F as follows:

$$\begin{gathered} M_1 = \begin{bmatrix} \mathrm{hilb}\left(\frac{r}{2}\right) & \mathrm{ones}\left(\frac{r}{2}\right) \\ \mathrm{hankel}\left(1 : \frac{r}{2}\right) & \mathrm{zeros}\left(\frac{r}{2}\right) \end{bmatrix}, \\ M_2 = \begin{bmatrix} \mathrm{toeplitz}\left(1 : \frac{r}{2}\right) & \mathrm{hilb}\left(\frac{r}{2}\right) \\ \mathrm{ones}\left(\frac{r}{2}\right) & \mathrm{hankel}\left(1 : \frac{r}{2}\right) \end{bmatrix}, \\ M_3 = \begin{bmatrix} \mathrm{zeros}\left(\frac{r}{2}\right) & \mathrm{hankel}\left(1 : \frac{r}{2}\right) \\ \mathrm{hilb}\left(\frac{r}{2}\right) & \mathrm{ones}\left(\frac{r}{2}\right) \end{bmatrix}, \\ N_1 = \mathrm{eye}\,(r), \qquad N_2 = \mathrm{ones}\,(r), \\ N_3 = \mathrm{tridiag}\,([7, 1, -1], r), \\ F = \begin{bmatrix} 3 & -2 & -1 & & \\ -2 & 3 & -2 & \ddots & \\ -1 & -2 & \ddots & -2 & -1 \\ & \ddots & -2 & 3 & -2 \\ & & -1 & -2 & 3 \end{bmatrix}, \end{gathered} \tag{51}$$

where toeplitz(k), hilb(k), hankel(k), zeros(k), and eye(k)

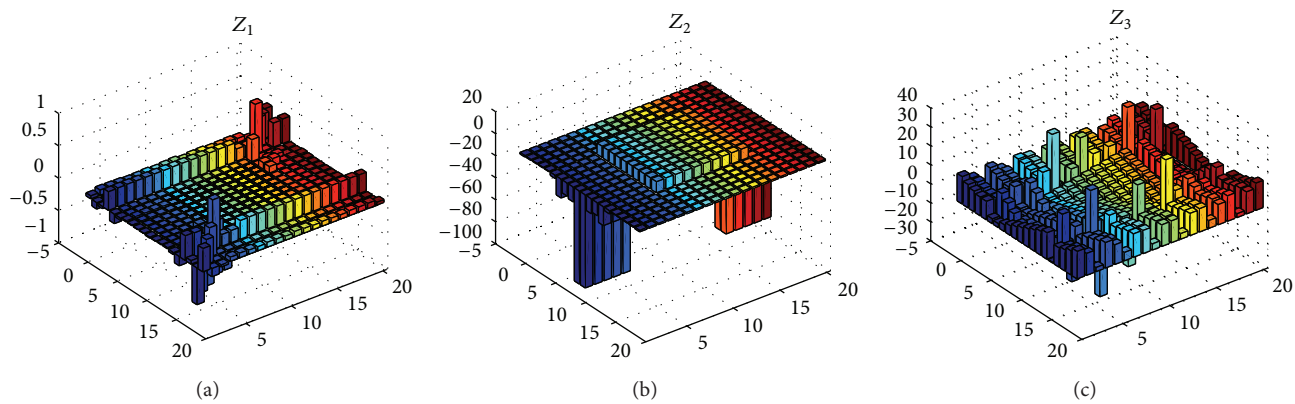

FIGURE 1: The bars graphs of Z_i when $r = 20$.

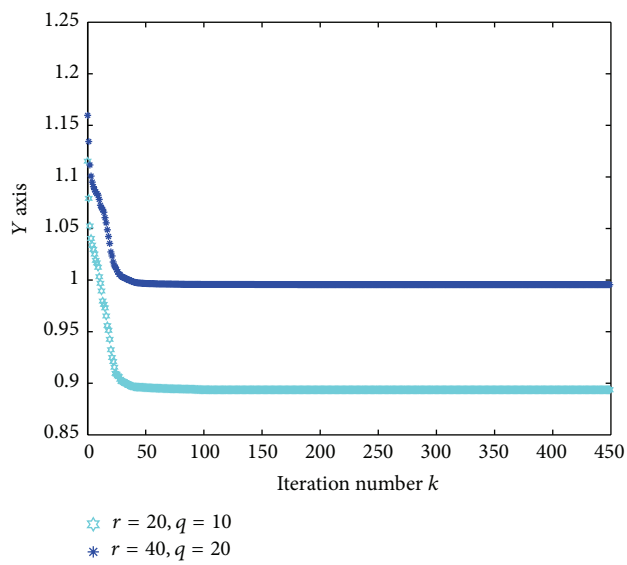

FIGURE 2: The convergence curve of the Frobenius norm of the residual.

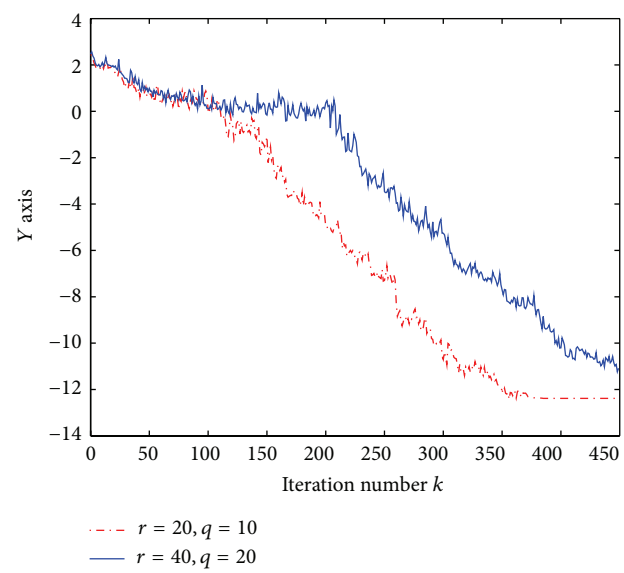

FIGURE 3: The convergence curve of the Frobenius norm of the terminated condition.

denote the Toeplitz matrix, Hilbert matrix, Hankel matrix, null matrix, identity matrix with orders k, and the elements of matrix ones$(\cdot)$ are one, tridiag$([7, 1, -1], r)$ represents $r \times r$ tri-diagonal matrix produced by vector $[7, 1, -1]$.

Let the given central principal matrices

$$\mathcal{X}_1 = \text{zeros}\left(\frac{r}{2}\right),$$
$$\mathcal{X}_2 = \text{ones}\left(\frac{r}{2}\right) * 10, \qquad \mathcal{X}_3 = \text{toeplitz}\left(1 : \frac{r}{2}\right). \tag{52}$$

By using the Algorithm 7, we obtain the solution to Problem 2. To save space, we shall not report the explicit datum of the solution, but the bars graphs of the components for the solution matrices will be given. Let $r = 20$, Figure 1 shows the bars graphs of Z_1, Z_2, Z_3 when we choose the initial iterative matrices

$$Y_i = \begin{pmatrix} \text{ones}\left(\frac{r-q}{2}\right) & \text{ones}\left(\frac{r-q}{2}, q\right) & \text{ones}\left(\frac{r-q}{2}\right) \\ \text{ones}\left(q, \frac{r-q}{2}\right) & \text{zeros}\left(\frac{r}{2}\right) & \text{ones}\left(q, \frac{r-q}{2}\right) \\ \text{ones}\left(\frac{r-q}{2}\right) & \text{ones}\left(\frac{r-q}{2}, q\right) & \text{ones}\left(\frac{r-q}{2}\right) \end{pmatrix}, \quad i = 1, 2, 3, \tag{53}$$

and the terminal condition $\|P_1^{(k)}\| + \|P_2^{(k)}\| + \|P_3^{(k)}\| < \epsilon = 1.0e - 012$.

Moreover, when $r = 20$ and $r = 40$, the convergence curves for the Frobenius norm of the residual denoted by RES $= \|R_k\|$ and the termination condition denoted by TC $= \|P_1^{(k)}\| + \|P_2^{(k)}\| + \|P_3^{(k)}\|$ are plotted in Figures 2 and 3, respectively.

From Figure 2, we can see that the residual norm of Algorithm 7 is monotonically decreasing, which is in accordance with the theory established in Theorem 14, namely, this algorithm is numerical stable. While Figure 3 shows that the terminated condition $\|P_1^{(k)}\|+\|P_2^{(k)}\|+\|P_3^{(k)}\|$ is oscillating back and forth and approaches to zero as iterative process. Hence, the iterative Algorithm 7 is efficient, but it lacks of smooth convergence. Of course, for a problem with large and sparse matrices, Algorithm 7 may not terminate in a finite number of steps because of roundoff errors. How to establish an efficient and smooth algorithm is an important problem which we should study in a future work.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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