**Research Article**

**Strong Convergence for Hybrid S-Iteration Scheme**

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We establish a strong convergence for the hybrid S-iterative scheme associated with nonexpansive and Lipschitz strongly pseudocontractive mappings in real Banach spaces.

1. Introduction and Preliminaries

Let $E$ be a real Banach space and let $K$ be a nonempty convex subset of $E$. Let $J$ denote the normalized duality mapping from $E$ to $2^{E^*}$ defined by

$$ J(x) = \{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2, \|f^*\| = \|x\| \}, \quad \forall x, y \in E, $$

where $E^*$ denotes the dual space of $E$ and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We will denote the single-valued duality map by $j$.

Let $T: K \rightarrow K$ be a mapping.

**Definition 1.** The mapping $T$ is said to be Lipschitzian if there exists a constant $L > 1$ such that

$$ \|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in K. $$

**Definition 2.** The mapping $T$ is said to be nonexpansive if

$$ \|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in K. $$

**Definition 3.** The mapping $T$ is said to be pseudocontractive if for all $x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$ \langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2. $$

**Definition 4.** The mapping $T$ is said to be strongly pseudocontractive if for all $x, y \in K$, there exists $k \in (0, 1)$ such that

$$ \langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2. $$

Let $K$ be a nonempty convex subset $C$ of a normed space $E$.

(a) The sequence $\{x_n\}$ defined by, for arbitrary $x_1 \in K$,

$$ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n Ty_n, $$

$$ y_n = (1 - \beta_n) x_n + \beta_n Tx_n, \quad n \geq 1, $$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, is known as the Ishikawa iteration process [1]. If $\beta_n = 0$ for $n \geq 1$, then the Ishikawa iteration process becomes the Mann iteration process [2].

(b) The sequence $\{x_n\}$ defined by, for arbitrary $x_1 \in K$,

$$ x_{n+1} = Ty_n, $$

$$ y_n = (1 - \beta_n) x_n + \beta_n Tx_n, \quad n \geq 1, $$

where $\{\beta_n\}$ is a sequence in $[0, 1]$, is known as the $S$-iteration process [3, 4].

In the last few years or so, numerous papers have been published on the iterative approximation of fixed points of Lipschitz strongly pseudocontractive mappings using the Ishikawa iteration scheme (see, e.g., [1]). Results which had
been known only in Hilbert spaces and only for Lipschitz mappings have been extended to more general Banach spaces (see, e.g., [5–10] and the references cited therein).

In 1974, Ishikawa [1] proved the following result.

**Theorem 5.** Let $K$ be a compact convex subset of a Hilbert space $H$ and let $T : K \to K$ be a Lipschitzian pseudocontractive mapping. For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence defined iteratively by

$$
x_{n+1} = (1 - \alpha_n) x_n + \alpha_n Ty_n,
$$

$$
y_n = (1 - \beta_n) x_n + \beta_n Tx_n , \quad n \geq 1,
$$

(8)

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying

(i) $0 \leq \alpha_n \leq \beta_n \leq 1$,

(ii) $\lim_{n \to \infty} \beta_n = 0$,

(iii) $\sum_{n=1}^\infty \alpha_n \beta_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly at a fixed point of $T$.

In [6], Chidume extended the results of Schu [9] from Hilbert spaces to the much more general class of real Banach spaces and approximated the fixed points of (strongly) pseudocontractive mappings.

In [11], Zhou and Jia gave the more general answer of the question raised by Chidume [5] and proved the following.

If $X$ is a real Banach space with a uniformly convex dual $X^*$, $K$ is a nonempty bounded closed convex subset of $X$, and $T : K \to K$ is a continuous strongly pseudocontractive mapping, then the Ishikawa iteration scheme converges strongly at the unique fixed point of $T$.

In this paper, we establish the strong convergence for the hybrid $S$-iterative scheme associated with nonexpansive and Lipschitz strongly pseudocontractive mappings in real Banach spaces. We also improve the result of Zhou and Jia [11].

### 2. Main Results

We will need the following lemmas.

**Lemma 6.** (see [12]). Let $J : E \to 2^E$ be the normalized duality mapping. Then for any $x, y \in E$, one has

$$
\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, j(x + y) \rangle ,
$$

$$
\forall j(x + y) \in J(x + y).
$$

(9)

**Lemma 7.** (see [10]). Let $\{\beta_n\}$ be nonnegative sequence satisfying

$$
\rho_{n+1} \leq (1 - \theta_n) \rho_n + \omega_n,
$$

(10)

where $\theta_n \in [0, 1], \sum_{n=1}^\infty \theta_n = \infty$, and $\omega_n = o(\theta_n)$. Then

$$
\lim_{n \to \infty} \rho_n = 0.
$$

(11)

The following is our main result.

**Theorem 8.** Let $K$ be a nonempty closed convex subset of a real Banach space $E$, let $S : K \to K$ be nonexpansive, and let $T : K \to K$ be Lipschitzian strongly pseudocontractive mappings such that $p \in F(S) \cap F(T) = \{x \in K : Sx = Tx = x\}$ and

$$
\|x - Sy\| \leq \|x - Ty\| , \quad \forall x, y \in K,
$$

$$
\|x - Ty\| \leq \|Tx - Ty\| , \quad \forall x, y \in K.
$$

(12)

Let $\{\beta_n\}$ be a sequence in $[0, 1]$ satisfying

(iv) $\sum_{n=1}^\infty \beta_n = \infty$,

(v) $\lim_{n \to \infty} \beta_n = 0$.

For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence iteratively defined by

$$
x_{n+1} = Sy_n ,
$$

(12)

$$
y_n = (1 - \beta_n) x_n + \beta_n Tx_n , \quad n \geq 1.
$$

Then the sequence $\{x_n\}$ converges strongly at the common fixed point $p$ of $S$ and $T$.

**Proof.** For strongly pseudocontractive mappings, the existence of a fixed point follows from Delmling [13]. It is shown in [11] that the set of fixed points for strongly pseudocontractions is a singleton.

By (v), since $\lim_{n \to \infty} \beta_n = 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$
\beta_n \leq \min \left\{ \frac{1}{2k}, \frac{1 - k}{(1 + L)(1 + 3L)} \right\},
$$

(13)

where $k < 1/2$. Consider

$$
\|x_{n+1} - p\|^2 = (x_{n+1} - p, j(x_{n+1} - p))
$$

$$
= (Sy_n - p, j(x_{n+1} - p))
$$

$$
= (Tx_{n+1} - p, j(x_{n+1} - p))
$$

$$
+ (Sy_n - Tx_{n+1}, j(x_{n+1} - p))
$$

$$
\leq k\|x_{n+1} - p\|^2 + \|Sy_n - T_{n+1}\| \|x_{n+1} - p\| ,
$$

(14)

which implies that

$$
\|x_{n+1} - p\| \leq \frac{1}{1 - k} \|Sy_n - T_{n+1}\| ,
$$

(15)

where

$$
\|Sy_n - T_{n+1}\| \leq \|Sy_n - Ty_n\| + \|Ty_n - T_{n+1}\|
$$

$$
\leq \|x_n - Sy_n\| + \|x_n - Ty_n\| + \|Ty_n - T_{n+1}\|
$$

$$
\leq \|SX_n - Sy_n\| + \|Tx_n - Ty_n\| + \|Ty_n - T_{n+1}\|
$$

$$
\leq \|SX_n - Sy_n\| + L(\|x_n - y_n\| + \|y_n - x_{n+1}\|) .
$$

(16)

$$
\|y_n - x_{n+1}\| \leq \|y_n - x_n\| + \|x_n - x_{n+1}\|
$$

$$
= \|y_n - x_n\| + \|x_n - Sy_n\|
$$

$$
\leq \|y_n - x_n\| + \|SX_n - Sy_n\| ,
$$

(17)
and consequently from (16), we obtain
\[ \|S_{y_n} - Tx_{n+1}\| \leq (1 + L) \|Sx_n - S_{y_n}\| + 2L \|x_n - y_n\| 
\leq (1 + 3L) \|x_n - y_n\| 
= (1 + 3L) \beta_n \|x_n - Tx_n\| 
\leq (1 + L) (1 + 3L) \beta_n \|x_n - p\|. \]
Substituting (18) in (15) and using (13), we get
\[
\|x_{n+1} - p\| \leq \frac{(1 + L)(1 + 3L)}{1 - k} \beta_n \|x_n - p\| 
\leq \|x_n - p\|.
\]
So, from the above discussion, we can conclude that the sequence \(\{x_n - p\}\) is bounded. Since \(T\) is Lipschitzian, so \(\{Tx_n - p\}\) is also bounded. Let \(M_1 = \sup_{n \geq 1} \|x_n - p\| + \sup_{n \geq 1} \|Tx_n - p\|\). Also by (ii), we have
\[
\|x_n - y_n\| = \beta_n \|x_n - Tx_n\| 
\leq M_1 \beta_n 
\rightarrow 0
\]
as \(n \rightarrow \infty\), implying that \(\{x_n - y_n\}\) is bounded, so let \(M_2 = \sup_{n \geq 1} \|x_n - y_n\| + M_1\). Further,
\[
\|y_n - p\| \leq \|x_n - y_n\| + \|x_n - p\| 
\leq M_2,
\]
which implies that \(\{y_n - p\}\) is bounded. Therefore, \(\{Ty_n - p\}\) is also bounded.
Set
\[
M_3 = \sup_{n \geq 1} \|y_n - p\| + \sup_{n \geq 1} \|Ty_n - p\|.
\]
Denote \(M = M_1 + M_2 + M_3\). Obviously, \(M < \infty\).
Now from (12) for all \(n \geq 1\), we obtain
\[
\|x_{n+1} - p\|^2 = \|S_n - p\|^2 \leq \|y_n - p\|^2,
\]
and by Lemma 6, we get
\[
\|y_n - p\|^2 = \|(1 - \beta_n)(x_n - p) + \beta_n(Tx_n - p)\|^2 
= \|(1 - \beta_n)(x_n - p) + \beta_n(Tx_n - p)\|^2 
\leq (1 - \beta_n)\|x_n - p\|^2 + 2\beta_n\|(Tx_n - p, y_n - p)\| 
= (1 - \beta_n)\|x_n - p\|^2 + 2\beta_n\|(Ty_n - p, y_n - p)\| 
+ 2\beta_n\|(Tx_n - Ty_n, y_n - p)\| 
\leq (1 - \beta_n)\|x_n - p\|^2 + 2k\|\beta_n\|\|y_n - p\|^2 
+ 2\|\beta_n\|\|Tx_n - Ty_n\|\|y_n - p\| 
\leq (1 - \beta_n)\|x_n - p\|^2 + 2k\|\beta_n\|\|y_n - p\|^2 
+ 2M\|\beta_n\|\|x_n - y_n\|,
\]
which implies that
\[
\|y_n - p\|^2 \leq (1 - \beta_n)\|x_n - p\|^2 + \frac{2M\|\beta_n\|\|x_n - y_n\|}{1 - 2k\|\beta_n\|} 
\leq (1 - \beta_n)\|x_n - p\|^2 + 4M\|\beta_n\|\|x_n - y_n\|.
\]
Corollary 9. Let \(K\) be a nonempty closed convex subset of a real Hilbert space \(H\), let \(S : K \rightarrow K\) be nonexpansive, and let \(T : K \rightarrow K\) be Lipschitz strongly pseudocontractive mappings such that \(p \in F(S) \cap F(T)\) and the condition (C). Let \(\{\beta_n\}\) be a sequence in \([0, 1]\) satisfying the conditions (iv) and (v).
For arbitrary \(x_1 \in K\), let \(\{x_n\}\) be a sequence iteratively defined by (12). Then the sequence \(\{x_n\}\) converges strongly at the common fixed point \(p\) of \(S\) and \(T\).

Example 10. As a particular case, we may choose, for instance, \(\beta_n = 1/n\).

Remark 11. (1) The condition (C) is not new and it is due to Liu et al. [14].
(2) We prove our results for a hybrid iteration scheme, which is simple in comparison to the previously known iteration schemes.

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References


