Research Article

The Distance Matrices of Some Graphs Related to Wheel Graphs

Xiaoling Zhang and Chengyuan Song

School of Mathematics and Information Science, Yantai University, Yantai, Shandong 264005, China

Correspondence should be addressed to Xiaoling Zhang; zhangxling04@yahoo.cn

Received 25 November 2012; Revised 29 May 2013; Accepted 30 May 2013

Academic Editor: Maurizio Porfiri

Copyright © 2013 X. Zhang and C. Song. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let $D$ denote the distance matrix of a connected graph $G$. The inertia of $D$ is the triple of integers $(n_+(D), n_0(D), n_-(D))$, where $n_+(D)$, $n_0(D)$, and $n_-(D)$ denote the number of positive, 0, and negative eigenvalues of $D$, respectively. In this paper, we mainly study the inertia of distance matrices of some graphs related to wheel graphs and give a construction for graphs whose distance matrices have exactly one positive eigenvalue.

1. Introduction

A simple graph $G = (V, E)$ consists of $V$, a nonempty set of vertices, and $E$, a set of unordered pairs of distinct elements of $V$ called edges. All graphs considered here are simple and connected. Let $G$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The distance between two vertices $u$, $v \in V(G)$ is denoted by $d_{uv}$ and is defined as the length of the shortest path between $u$ and $v$ in $G$. The distance matrix of $G$ is denoted by $D(G)$ and is defined by $D(G) = (d_{uv})_{u,v\in V(G)}$. Since $D(G)$ is a symmetric matrix, its inertia is the triple of integers $(n_+(D(G)), n_0(D(G)), n_-(D(G)))$, where $n_+(D(G))$, $n_0(D(G))$, and $n_-(D(G))$ denote the number of positive, 0, and negative eigenvalues of $D(G)$, respectively.

The distance matrix of a graph has numerous applications to chemistry [1]. It contains information on various walks and self-avoiding walks of chemical graphs. Moreover, the distance matrix is not only immensely useful in the computation of topological indices such as the Wiener index [1] but also useful in the computation of thermodynamic properties such as pressure and temperature virial coefficients [2]. The distance matrix of a graph contains more structural information compared to a simple adjacency matrix. Consequently, it seems to be a more powerful structure discriminator than the adjacency matrix. In some cases, it can differentiate isospectral graphs although there are nonisomorphic trees with the same distance polynomials [3]. In addition to such applications in chemical sciences, distance matrices find applications in music theory, ornithology [4], molecular biology [5], psychology [4], archeology [6], sociology [7], and so forth. For more information, we can see [1] which is an excellent recent review on the topic and various uses of distance matrices.

Since the distance matrix of a general graph is a complicated matrix, it is very difficult to compute its eigenvalues. People focus on studying the inertia of the distance matrices of some graphs. Unfortunately, up to now, only few graphs are known to have exactly one positive $D$-eigenvalue, such as trees [8], connected unicyclic graphs [9], the polyacenes, honeycomb and square lattices [10], complete bipartite graphs $K_{n,n}$, and iterated line graphs of some regular graphs [12], and cacti [13]. This inspires us to find more graphs whose distance matrices have exactly one positive eigenvalue.

The wheel graph of $n$ vertices $W_n$ is a graph that contains a cycle of length $n-1$ plus a vertex $v$ (sometimes called the hub) not in the cycle such that $v$ is connected to every other vertex. In this paper, we first study the inertia of the distance matrices in wheel graphs if one or more edges are removed from the graph, and then, with the help of the structural characteristics of wheel graphs, we give a construction for graphs whose distance matrices have exactly one positive eigenvalue.

2. Preliminaries

We first give some lemmas that will be used in the main results.
Lemma 1 (see [14]). Let $A$ be a Hermitian matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and $B$ one of its principal submatrices. Let $B$ have eigenvalues $\mu_1 \geq \cdots \geq \mu_m$. Then the inequalities $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$ ($i = 1, \ldots, m$) hold.

For a square matrix, let $\text{cof}(A)$ denote the sum of cofactors of $A$. Form the matrix $\tilde{A}$ by subtracting the first row from all other rows then the first column from all other columns and let $\tilde{A}_{11}$ denote the principle submatrix obtained from $A$ by deleting the first row and first column.

Lemma 2 (see [15]). $\text{cof}(A) = \det \tilde{A}_{11}$.

A cut vertex is a vertex the removal of which would disconnect the remaining graph; a block of a graph is defined to be a maximal subgraph having no cut vertices.

Lemma 3 (see [15]). If $G$ is a strongly connected directed graph with blocks $G_1, G_2, \ldots, G_r$, then

$$\text{cof } D(G) = \prod_{i=1}^{r} \text{cof } D(G_i),$$

$$\det D(G) = \sum_{i=1}^{r} \det D(G_i) \prod_{j \neq i} \text{cof } D(G_i).$$

(1)

Lemma 4. Let

$$C = \begin{bmatrix}
1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
-1 & -2 & -1 & 0 & \cdots & 0 & 0 \\
0 & -1 & -2 & -1 & \cdots & 0 & 0 \\
0 & 0 & -1 & -2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -2 & -1 \\
0 & 0 & 0 & 0 & \cdots & -1 & -2
\end{bmatrix}_{n \times n}.\quad (2)$$

Then

$$\det C = \begin{cases}
-\frac{n}{2}, & \text{if } n \text{ is even}, \\
\frac{n+1}{2}, & \text{if } n \text{ is odd}.
\end{cases}\quad (3)$$

Proof. Let

$$C_n = \begin{bmatrix}
1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
-1 & -2 & -1 & 0 & \cdots & 0 & 0 \\
0 & -1 & -2 & -1 & \cdots & 0 & 0 \\
0 & 0 & -1 & -2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -2 & -1 \\
0 & 0 & 0 & 0 & \cdots & -1 & -2
\end{bmatrix}_{n \times n}.\quad (4)$$

$$D_n = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
-2 & -1 & 0 & \cdots & 0 & 0 \\
1 & -1 & -2 & \cdots & 0 & 0 \\
1 & 0 & -1 & -2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & 0 & \cdots & -2 & -1 \\
1 & 0 & 0 & 0 & \cdots & -1 & -2
\end{bmatrix}_{n \times n}.\quad (5)$$

Comparing $C_n$ to $D_n$, we get the following:

$$C_n = D_n + \frac{n}{2} \times (-1)^{n-1}.\quad (6)$$

Expanding the determinant $D_n$ accordingly to the last column and then the last line, we get the following incursion:

$$D_n = -2D_{n-1} - D_{n-2} + 1;\quad (7)$$

that is,

$$D_n + D_{n-1} = -(D_{n-1} + D_{n-2}) + 1.\quad (8)$$

Since $D_1 = 1/2$, $D_2 = 0$, and $D_3 = 1/2$, from the above incursion, we get the following:

$$D_n = \begin{cases}
0, & \text{if } n \text{ is even}, \\
1/2, & \text{if } n \text{ is odd}.
\end{cases}\quad (9)$$

So, we have the following:

$$C_n = \begin{cases}
-\frac{n}{2}, & \text{if } n \text{ is even}, \\
\frac{n+1}{2}, & \text{if } n \text{ is odd}.
\end{cases}\quad (10)$$

This completes the proof.

3. Main Results

In the following, we always assume that $V(W_n) = \{v_0, v_1, \ldots, v_{n-1}\}$, where $v_0$ is the hub of $W_n$.

Theorem 5. Let $e = v_i v_{i+1 \mod (n-1)}$ ($1 \leq i \leq n-1$). Then

$$\det D(W_n - e) = \begin{cases}
-\frac{n^2}{4}, & \text{if } n \text{ is even}, \\
\frac{n^2-1}{4}, & \text{if } n \text{ is odd},
\end{cases}\quad (11)$$

where $n \geq 3$.

Proof. Without loss of generality, we may assume that $e = v_1 v_{n-1}$. Let

$$A_n = \det D(W_n - e) = \begin{bmatrix}
0 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 0 & 1 & 2 & \cdots & 2 & 2 \\
1 & 1 & 0 & 1 & \cdots & 2 & 2 \\
1 & 0 & 1 & -2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 2 & 2 & 2 & \cdots & 0 & 1 \\
1 & 2 & 2 & 2 & \cdots & 1 & 0
\end{bmatrix}_{n \times n}.\quad (12)$$

(12)
Then

\[
A_n = \begin{vmatrix}
0 & 1 & 1 - \frac{1}{2} & 1 & \cdots & 1 & 1 \\
1 & -2 & 0 & 0 & \cdots & 0 & 0 \\
1 - \frac{1}{2} & 0 & -2 + \frac{1}{2} & -1 & \cdots & 0 & 0 \\
1 & 0 & -1 & -2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & 0 & \cdots & -1 & -2
\end{vmatrix}_{n \times n}.
\] (12)

Expanding the determinant \( A_n \) according to the second line, we get the following incursion:

\[
A_n = (-1)^{n-1} \frac{n}{2} - 2A_{n-1} - C_{n-2} - D_{n-2} - A_{n-2},
\] (13)

where \( C_n \) and \( D_n \) are defined as in Lemma 4.

By Lemma 4, we get the following:

\[
A_n = \begin{cases}
-1 - 2A_{n-1} - A_{n-2}, & \text{if } n \text{ is even,} \\
-2A_{n-1} - A_{n-2}, & \text{if } n \text{ is odd.}
\end{cases}
\] (14)

Since \( A_1 = 2, A_2 = -4, A_3 = 6, \) and \( A_4 = -9, \) according to the above incursion, we get the following:

\[
A_n = \begin{cases}
\frac{-n^2}{4}, & \text{if } n \text{ is even,} \\
\frac{n^2 - 1}{4}, & \text{if } n \text{ is odd},
\end{cases}
\] (15)

where \( n \geq 3 \). This completes the proof.

**Corollary 6.** Let \( e = v_1 v_{i+1} \mod (n-1) \) (\( 1 \leq i \leq n - 1 \)). Then

\[
n_+ (D(W_n - e)) = 1, \quad n_0 (D(W_n - e)) = 0,
\] (16)

\[
n_- (D(W_n - e)) = n - 1.
\]

**Proof.** We will prove the result by induction on \( n \).

If \( n = 3, W_3 = e \equiv P_3 \) is obviously true.

Suppose that the result is true for \( n - 1 \); that is, \( n_+ (D(W_{n-1} - e)) = 1, n_0 (D(W_{n-1} - e)) = 0, n_- (D(W_{n-1} - e)) = n - 2 \).

Since \( D(W_{n-1} - e) \) is a principle submatrix of \( D(W_n - e) \), by Lemma 1, the eigenvalues of \( D(W_{n-1} - e) \) interlace the eigenvalues of \( D(W_n - e) \). By Theorem 5, \( \det D(W_{n-1} - e) \det D(W_n - e) < 0 \). So, \( D(W_n - e) \) has one negative eigenvalue more than \( D(W_{n-1} - e) \). According to the induction hypothesis, we get \( n_+ (D(W_n - e)) = 1, n_0 (D(W_n - e)) = 0, \) and \( n_- (D(W_n - e)) = n - 1 \). This completes the proof.

**Theorem 7.** One has

\[
\det D(W_n) = \begin{cases}
1 - n, & \text{if } n \text{ is even,} \\
0, & \text{if } n \text{ is odd,}
\end{cases}
\] (17)

where \( n \geq 3 \).

**Proof.** Consider the following:

\[
det D(W_n) = \begin{vmatrix}
0 & 1 & 1 & \cdots & 1 & 1 \\
1 & 0 & 1 & \cdots & 2 & 1 \\
1 & 0 & 1 & \cdots & 2 & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 1 & \cdots & 2 & 1 \\
0 & 1 & 1 & \cdots & 1 & 1
\end{vmatrix}_{n \times n}
\]

Expanding the above determinant according to the second line, we get the following:

\[
\det D(W_n) = (-1)^{n-1} (n-1)^3 + 4(n-1) A_{n-1},
\] (19)

where \( A_n \) is defined as in Theorem 5.

By Theorem 5, when \( n \geq 3 \), we get the following:

\[
\det D(W_n) = \begin{cases}
1 - n, & \text{if } n \text{ is even,} \\
0, & \text{if } n \text{ is odd.}
\end{cases}
\] (20)

This completes the proof.

Similar to Corollary 6, we can get the following corollary.

**Corollary 8.** (i) If \( n \) is even, \( n_+ (D(W_n)) = 1, n_0 (D(W_n)) = 0, n_- (D(W_n)) = n - 1 \).

(ii) If \( n \) is odd, \( n_+ (D(W_n)) = 1, n_0 (D(W_n)) = 1, n_- (D(W_n)) = n - 2 \).

Denote by \( W_n - v_0 \) the graph obtained from \( W_n \) by deleting the vertex \( v_0 \) and all the edges adjacent to \( v_0 \); that is, \( W_n - v_0 \equiv C_{n-1} \). Let \( E_k \) (\( 1 \leq k \leq n \)) be any subset of \( E(W_n - v_0) \) with \( |E_k| = k \). In the following, we always denote by \( W_n - E_k \) the graph obtained from \( W_n \) by deleting all the edges in \( E_k \).
Theorem 9. One has \( n_+(D(W_n - E_k)) = 1, n_0(D(W_n - E_k)) = 0, n_-(D(W_n - E_k)) = n - 1 \).

Proof. Denote the components of \( W_n - E_k - v_0 \) by \( C_1, \ldots, C_s \). Let \( G_i \) denote the graph that contains \( C_i \) plus the vertex \( v_0 \) such that \( v_0 \) is connected to every other vertex, \( 1 \leq i \leq s \). Then each \( G_i \) is an edge of \( |V(W_i)| - v_0 \). By Lemma 2 and some direct calculations, we get the following:

\[
\text{cof}(G_i) = \det D\left( \overrightarrow{G_i} - e_i \right)_{11} = (-1)^{|V(G_i)| - 1} |V(G_i)|.
\]

(21)

It is easy to check that \( \text{cof}(G_i) = (-1)^{|V(G_i)| - 1} |V(G_i)| \) is also true when \( G_i \) is isomorphic to \( K_2 \).

In the following, we will prove the theorem by induction on \( s \). For \( s = 1 \), \( G \equiv W_n - e \), where \( e \) is an edge of \( W_n - v_0 \), by Corollary 6, we get the result.

Suppose the result is true for \( s - 1 \). For \( s \), let \( G'_2 = G_1 \cup G_2 \cup \cdots \cup G_{s-1} \). Then by the induction hypothesis, \( n_+(D(G'_2)) = 1, n_0(D(G'_2)) = 0 \), and \( n_-(D(G'_2)) = |V(G'_2)| - 1 \), which implies that

\[
\det D(G'_2) = (-1)^{|V(G'_2)| - 1} a,
\]

(22)

where \( a \) is a positive integer.

Since

\[
\text{cof}(G_i) = (-1)^{|V(G_i)| - 1} |V(G_i)|, \quad 1 \leq i \leq s,
\]

by Lemma 3,

\[
\text{cof}D(G'_2) = \prod_{j=1}^{s-1} \text{cof}D(G_j) = (-1)^{|V(G'_2)| - 1} \prod_{j=1}^{s-1} |V(G_j)|.
\]

(23)

Then

\[
\det D(W_n - E_k)
\]

\[
= \det D(G'_2) \text{cof}D(G_i) + \det D(G_i) \text{cof}D(G'_2)
\]

\[
= (-1)^{|V(G'_2)| - 1} a \times (-1)^{|V(G_i)| - 1} |V(G_i)|
\]

\[
+ (-1)^{|V(G'_2)| - 1} b \times (-1)^{|V(G'_2)| - 1} \prod_{j=1}^{s-1} |V(G_j)|
\]

\[
= (-1)^{n-1} \left( a |V(G_i)| + b \prod_{j=1}^{s-1} |V(G_j)| \right),
\]

(24)

where \( b = n^2/4 \), if \( n \) is even and \( b = (n^2 - 1)/4 \), if \( n \) is odd.

In this case, similar to Corollary 6, we can easily get \( n_+(D(W_n - E_k)) = 1, n_0(D(W_n - E_k)) = 0 \), and \( n_-(D(W_n - E_k)) = n - 1 \).

Up to now, we have proved the result.

\[
\text{Lemma 10 (see [13]). Let } G \times H \text{ denote the Cartesian product of connected graphs } G \text{ and } H, \text{ where } V(G) = \{ u_1, \ldots, u_m \} \text{ and } V(H) = \{ v_1, \ldots, v_n \}. \text{ Then we have}
\]

(i) \( n_+(G \times H) = n_+(G_u \times v_H) \);

(ii) \( n_0(G \times H) = (m - 1)(n - 1) + n_0(G_u \times v_H) \);

(iii) \( n_+(G \times H) = n_+(G_u \times v_H) \).

Theorem 11. Let \( u_0 \) and \( v_0 \) be the hubs of \( W_n \) and \( W_m \), respectively. Suppose \( E_p \) \((0 \leq p \leq n - 1)\) and \( E_q \) \((0 \leq q \leq m - 1)\) are any subsets of \( E(W_n - u_0) \) and \( E(W_m - v_0) \) with \(|E_p| = p\), \(|E_q| = q\), respectively. Then, the distance matrix of the graph \((W_n - E_p) \times (W_m - E_q)\) has exactly one positive eigenvalue.

Proof. Since \( u_0 \) and \( v_0 \) are the hubs of \( W_n \) and \( W_m \), respectively, \((W_n - E_p)u_0, v_0, (W_m - E_q)\) must be isomorphic to some connected graphs \((W_n - u_0) \times (W_m - v_0)\) where \( u_0 \) is the hub of \( W_n - u_0 \) and \( v_0 \) is any subset of \( E(W_n - u_0) \) with \(|E_p| = p + q \). By Theorem 9 and Lemma 10, we get the result.

Given an arbitrary integer \( m \), for \( 1 \leq i \leq m \), let \( v_i \) be the hub of \( W_i \) and \( E_p \) any subset of \( E(W_i - v_0) \). Suppose \( V(W_i) = \{ v_i0, v_i1, \ldots, v_i(n-1) \} \).

Theorem 12. For an arbitrary integer \( m \), the distance matrix of the graph \( G = (W_n - E_p)u_0, v_0, (W_m - E_q)\) has exactly one positive eigenvalue.

Proof. We will prove the conclusion by induction on \( m \).

If \( m = 1 \), by Theorem 9, the conclusion is true.

Suppose the conclusion is true for \( m - 1 \). For convenience, let \( H = (W_n - E_p)u_0, v_0, (W_m - E_q) \). Then \( G = H v_0, v_0, (W_m - E_p) \). By Lemma 10, we have the following:

\[
n_+(G) = n_+(H (W_{n-1} - E_{p-1}), v_0, v_0, (W_m - E_p))
\]

\[
= n_+(H (W_{n-1} - E_{p-1}), v_0, v_0, (W_m - E_p))
\]

(25)

Since \((W_m - E_p) \times (W_m - E_q)\), \( v_0, v_0, (W_m - E_p) \) has exactly one positive eigenvalue.

Remark 13. Let \( G_1 \) and \( G_2 \) be any two graphs with the same form as \( G \) in Theorem 12. Making Cartesian product of graphs \( G_1 \) and \( G_2 \), by Lemma 10 and Theorem 12, we get a series of graphs whose distance matrices have exactly one positive eigenvalue.

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and suggestions. This work was supported by NSFC (11126256) and NSF of Shandong Province of China (ZR2012AQ022).
References


