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Research Article

The First Negative Moment of Skew-t and Generalized Student's t-Distributions in the Principal Value Sense

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The (Cauchy) principal value is a method for assigning values to certain improper integrals which would otherwise be undefined. Using the principal value sense, this study derives an explicit expression of the first negative moment of skew-t and generalized Student's t-distributions for practical applications. Some applications obtained from the FNM of skew-t and generalized Student's t-distributions are also discussed.

1. Introduction

The first negative moment (FNM) of a continuous density function defined on the real axis has important applications that can arise in many practical situations. For example, Peng [1] proposed a degradation model based on incorporating random effects into a Wiener process to predict the mean lifetime of highly reliable products as follows. The degradation model $L(t)$ is given by $L(t) = \eta t + \sigma_W W(t)$, where $W(\cdot)$ denotes the standard Wiener process; η and σ_W are the random effects. If the product's lifetime T can be defined as the first-passage time when $L(t)$ crosses a predefined threshold level ω, that is, $T = \inf\{t \mid L(t) \geq \omega\}$, then the product's lifetime on the given η and σ_W is an inverse Gaussian distribution (denoted by $T \mid \eta, \sigma_W \sim \mathcal{IG}(\omega/\eta, \omega^2/\sigma_W^2)$). Hence, the most common measure associated with a lifetime distribution, the first moment, can be expressed as

$$E(T) = E(E(T \mid \eta, \sigma_W)) = \omega E\left(\frac{1}{\eta}\right). \tag{1}$$

The mean lifetime of the products turns out to be the FNM of a prior distribution defined on the real line. An introduction to a Bayesian framework can be found in Berger [2], O'Hagan and Forster [3], and the references given therein. For example, the inverse prediction (or calibration problem) in simple linear regression and the coefficient of variation for normal distributions are commonly used in the fields of econometrics, engineering, physics, and biological sciences (see [4–6]. Thus, existence of the FNM for a distribution defined on the real line and the corresponding evaluation are practical issues in fundamental statistics.

There is a general tendency towards more flexible distributions to reduce unrealistic assumptions, like normality as a prior distribution in the statistical literature. Therefore, the skew-symmetric distribution is an alternative approach which extends the symmetric distributions by allowing a shape parameter to control skewness. A random variable Y is said to have the skew-normal distribution and is denoted by $Y \sim \mathcal{SN}(\mu, \sigma^2, \alpha)$ if its density function is given by

$$f_Y(y) = \frac{2}{\sigma}\phi\left(\frac{y-\mu}{\sigma}\right)\Phi\left(\alpha\frac{y-\mu}{\sigma}\right), \tag{2}$$

where ϕ and Φ are the pdf and the cdf of the standard normal distribution and the parameters μ, σ, and α regulate the location, scale, and skewness, respectively. A systematic treatment of the skew-normal distribution has been developed by Azzalini [7]. Furthermore, Azzalini and Capitanio [8] defined a skewed t variate as the scale mixture

$$X = S^{-1}Y^* + \mu, \tag{3}$$

where $Y^* \sim \mathcal{SN}(0, \sigma^2, \alpha)$, νS^2 is distributed independently of Y^*, according to a chi-squared distribution with degrees of freedom ν, and is denoted by $X \sim \mathcal{S}t(\mu, \sigma, \alpha, \nu)$. Note that the

skew-t distribution approaches the skew-normal distribution as $\nu \to \infty$; that is, $\lim_{\nu\to\infty} f_X(x) = f_Y(x)$, where f_X is the pdf of X and f_Y is defined in (2). Since the skew-normal, skew-Cauchy, and Student's t-distributions were included in the skew-t distribution, they have proved themselves quite adequate for modeling real data sets (refer to [9]).

In the previously-mentioned lifetime example, we can assume the prior distribution of the random effect η to be a skew-t distribution for adequately realistic situations. The same approach for the random effects in regression models has been presented in Azzalini and Capitanio [8]. Then, the product's mean lifetime in (1) turns out to be the FNM of a skew-t distribution. However, by the result in Piegorsch and Casella [10], the FNM of a skew-t distribution does not exist in the usual sense since the value of the density function at origin is larger than zero. Hence, through different viewpoints of the integral of FNM, the concept of the (Cauchy) principal value, widely used in the probability theory such as the weak law of large numbers [11], can be used to avoid the nonexistence of the FNM in the usual sense. From the analytic and practical viewpoint, one can comprehend at least the FNM in the principal value sense (PV-FNM) if that integral does not converge in the usual sense.

We allow for a finite number of discontinuities on the real axis by requiring h to be continuous on the real line except for a finite number of points $x_1 < \cdots < x_n$. Then, we will call the following the principal value for an improper integral that exists if and only if for every $\epsilon_i > 0$, $i = 0, \ldots, n$,

$$\lim_{\epsilon_0,\ldots,\epsilon_n\to 0}\left(\int_{-1/\epsilon_0}^{x_1-\epsilon_1} + \int_{x_1+\epsilon_1}^{x_2-\epsilon_2} + \cdots + \int_{x_{n-1}+\epsilon_{n-1}}^{x_n-\epsilon_n} + \int_{x_n+\epsilon_n}^{1/\epsilon_0}\right) h(x)\,\mathrm{d}x \tag{4}$$

exists and is finite. The improper integral of the FNM in the principal value sense (PV-FNM) can exist even when it does not exist in the usual sense. Based on the principal value sense, Peng [1] gave alternative sufficient conditions for the existence and finiteness of the FNM in which the mild conditions are easy to hold for the most commonly used distributions defined on the real line. The PV-FNM of a skew-normal distribution had also been derived by Peng [1]. Furthermore, an explicit solution is highly desirable and attractive when numerical methods are computationally laborious in evaluating an improper integral. Closed form formula not only avoids time-consuming algorithms but also verifies the numerical evaluation. The aim of this study is to obtain an explicit expression of the PV-FNM corresponding to (3) for practical applications and to provide some interesting results from the PV-FNM of a skew-t distribution.

The remainder of this paper is organized as follows. The exact form of the PV-FNM of skew-t and generalized Student's t-distributions is derived in Section 2. Section 3 provides some applications of the PV-FNM of skew-t and generalized Student's t-distributions. Section 4 contains conclusions.

2. Preliminaries

The PV-FNM of a skew-t distribution does exist by giving a scale mixture of a skew-normal distribution. In the following, we start with a simple case of skew-t distribution and then increase the complexity of the distributions. Note that the notation PVE$(\cdot)$ can be used for indicating the expectation in the principal value sense. All derivations are given in the Appendices A, B, C, D, and E.

Lemma 1. *If $X \sim t(\mu,\sigma,\nu) \equiv \mathcal{S}t(\mu,\sigma,0,\nu)$, then*

$$\mathrm{PVE}\left(\frac{1}{X}\right) = \begin{cases} \left(\dfrac{\sigma^2\nu}{\mu^2+\sigma^2\nu}\right)^{(\nu+1)/2} \times \displaystyle\sum_{k=0}^{(\nu-1)/2}\binom{\frac{\nu-1}{2}}{k} \times \dfrac{\mu^{2k+1}}{(2k+1)\,\nu^k(\sigma^2)^{k+1}}, & \text{if } \nu \text{ is odd,} \\ \dfrac{\nu!}{\{(\nu/2)!\}^2}\left(\dfrac{\sigma}{2}\right)^{\nu}\left(\dfrac{\nu}{\mu^2+\sigma^2\nu}\right)^{\nu/2+1} \times \left\{\mu\displaystyle\sum_{k=1}^{\nu/2}2^{2k-1}\dfrac{(k-1)!k!}{(2k)!}\left(1+\dfrac{\mu^2}{\sigma^2\nu}\right)^k + \sqrt{\mu^2+\sigma^2\nu} \times \log\left(\dfrac{\sqrt{\mu^2+\sigma^2\nu}+\mu}{\sqrt{\sigma^2\nu}}\right)\right\}, & \text{if } \nu \text{ is even.} \end{cases} \tag{5}$$

In what follows, we deal with the PV-FNM of a skew-t distribution which is more complicated in calculation. Note that to avoid "reverse" sum (product) in the following formula, we define $\sum_{d=n_1}^{n_2} c_d = 0$ and $\prod_{d=n_1}^{n_2} c_d = 1$ for $n_2 < n_1$.

Theorem 2. *If $X \sim \mathcal{S}t(\mu,\sigma,\alpha,\nu)$, then one obtains for odd values of ν*

$$\mathrm{PVE}\left(\frac{1}{X}\right) = \left(\frac{\sigma^2\nu}{\mu^2+\sigma^2\nu}\right)^{(\nu+1)/2} \times \sum_{k=0}^{(\nu-1)/2}\binom{\frac{\nu-1}{2}}{k}\frac{\mu^{2k+1}}{(2k+1)\,\nu^k(\sigma^2)^{k+1}} - \frac{\alpha(\nu+1)\,\Gamma((\nu+1)/2)\left(\sigma^2\nu\right)^{\nu/2}}{\sqrt{\pi}\,\Gamma(\nu/2)}$$

$$\times \left\{ \frac{2\sqrt{1+\alpha^2}}{\nu+2} \sum_{k=0}^{(\nu-1)/2} \frac{\prod_{m=0}^{k}(\nu-2m+2)}{\prod_{m=0}^{k}(\nu-2m+1)} I_{21}(k) + \frac{\prod_{n=0}^{(\nu-1)/2}(\nu-2n)}{\prod_{n=0}^{(\nu-1)/2}(\nu-2n+1)} I_{22} \right\} + \frac{2}{\sqrt{\pi\sigma^2\nu}} \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)} \left(\frac{\sigma^2\nu}{\mu^2+\sigma^2\nu} \right)^{(\nu+1)/2} \times \log\left(\alpha + \sqrt{1+\alpha^2}\right), \tag{6}$$

and for even values of ν

$$\begin{aligned} \mathrm{PVE}\left(\frac{1}{X}\right) &= \frac{\nu}{\nu+1} \frac{\mu}{\mu^2+\sigma^2\nu} \\ &\times \sum_{k=0}^{(\nu/2)-1} \frac{\prod_{m=0}^{k}(\nu-2m+1)}{\prod_{m=0}^{k}(\nu-2m)} \left(\frac{\sigma^2\nu}{\mu^2+\sigma^2\nu} \right)^{k} \\ &+ \frac{\prod_{n=0}^{\nu/2-1}(\nu-2n-1)}{\prod_{n=0}^{\nu/2-1}(\nu-2n)} \\ &\times \sqrt{\frac{\nu}{\sigma^2}} \left(\frac{\sigma^2\nu}{\mu^2+\sigma^2\nu} \right)^{(\nu+1)/2} \\ &\times \log\left(\frac{\sqrt{\mu^2+\sigma^2\nu}+\mu}{\sqrt{\sigma^2\nu}} \right) \\ &+ \frac{2\Gamma((\nu+1)/2)\left(\sigma^2\nu\right)^{\nu/2}}{\sqrt{\pi}\Gamma(\nu/2)} \\ &\times \left\{ \frac{\log\left(\alpha+\sqrt{1+\alpha^2}\right)}{\left(\mu^2+\sigma^2\nu\right)^{(\nu+1)/2}} - \alpha(\nu+1) \right. \\ &\left. \times \sum_{k=0}^{\nu/2} \frac{\binom{\nu/2}{k}\left(1+\alpha^2\right)^{k+1/2}}{2k+1} I_{23}(k) \right\}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} I_{21}(k) &= \frac{\alpha^{\nu-2k-1}}{2\left\{\left(1+\alpha^2\right)\left(\mu^2+\sigma^2\nu\right)\right\}^{(\nu+1)/2}} \\ &\times \left[\sum_{i=1}^{k} \binom{\frac{\nu-1}{2}-i}{\frac{\nu-1}{2}-k} \frac{\left(\mu^2+\sigma^2\nu\right)^{-i} - \left\{\left(1+\alpha^2\right)\mu^2+\sigma^2\nu\right\}^{-i}}{i\left\{\left(1+\alpha^2\right)\left(\mu^2+\sigma^2\nu\right)\right\}^{-i}} \right. \\ &+ \sum_{j=1}^{((\nu-1)/2)-k} \binom{\frac{\nu-1}{2}-j}{k} \frac{\left(\sigma^2\nu\right)^{-j} - \left(\mu^2+\sigma^2\nu\right)^{-j}}{j\alpha^{2j}\left\{\left(1+\alpha^2\right)\left(\mu^2+\sigma^2\nu\right)\right\}^{-j}} \\ &+ \binom{\frac{\nu-1}{2}}{\frac{\nu-1}{2}-k} \\ &\left. \times \log\left\{1+\frac{\left(1+\alpha^2\right)\mu^2}{\sigma^2\nu}\right\} \right], \end{aligned} \tag{8}$$

$$\begin{aligned} I_{22} &= \frac{1}{\left(\mu^2+\sigma^2\nu\right)^{(\nu+1)/2}} \\ &\times \sum_{i=0}^{(\nu-1)/2} \binom{\frac{\nu-1}{2}}{i} \frac{(-1)^i\alpha^{2i}}{2i+1} \\ &\times \left[\sum_{j=1}^{2i+1} \frac{\left\{(-1)^j+1\right\}\left\{\left(1+\alpha^2\right)\mu^2+\sigma^2\nu\right\}^{j/2-i-1}}{(2i-j+2)\mu^{j-2i-2}\left(1+\alpha^2\right)^{(j-1)/2}} \right. \\ &+ 2\left\{\frac{\mu^2}{\left(1+\alpha^2\right)\mu^2+\sigma^2\nu}\right\}^{i+1/2} \\ &\times \log\left\{\frac{\sqrt{\left(1+\alpha^2\right)\mu^2+\sigma^2\nu}+\mu\sqrt{1+\alpha^2}}{\sqrt{\sigma^2\nu}}\right\} \\ &\left. + \left(\frac{1}{1+\alpha^2}\right)^{i+1/2} \log\left\{\frac{\sigma^2\nu}{\left(1+\alpha^2\right)\mu^2+\sigma^2\nu}\right\} \right], \end{aligned} \tag{9}$$

$$\begin{aligned} I_{23}(k) &= \sum_{i=0}^{k} \binom{k}{i} \left(\mu^2+\sigma^2\nu\right)^{k-i} (-1)^i \\ &\times \left[\sum_{j=1}^{k-i} \frac{\prod_{m=1}^{j}(\nu+2k-2i-2m+3)}{\prod_{m=1}^{j}(2k-2i-2m+1)} \right. \end{aligned}$$

$$\times\left(\left[\left\{\left(1+\alpha^2\right)\mu^2+\sigma^2\nu\right\}^{-\nu/2}\left(\sigma^2\nu\right)^{i+j-k-1/2}\right.\right.$$
$$\left.-\left(\mu^2+\sigma^2\nu\right)^{i+j-k-(\nu+1)/2}\right]$$
$$\times\left(\left(\nu+2k-2i+1\right)\alpha^{2-2j}\right.$$
$$\left.\left.\times\left\{(1+\alpha^2)(\mu^2+\sigma^2\nu)\right\}^{j}\right)^{-1}\right)$$
$$+\frac{\prod_{n=1}^{k-i}(\nu+2n-1)}{\prod_{n=1}^{k-i}(2n-1)}$$
$$\times\left\{\frac{\alpha^{2(k-i)}}{\nu+1}\right.$$
$$\times\sum_{l=1}^{\nu/2}\frac{\prod_{m=1}^{l}(\nu-2m+3)}{\prod_{m=1}^{l}(\nu-2m+2)}$$
$$\times\frac{1}{\left\{(1+\alpha^2)(\mu^2+\sigma^2\nu)\right\}^{l+k-i}}$$
$$\times\left[\frac{1}{\left(\mu^2+\sigma^2\nu\right)^{(\nu+1)/2-l}}\right.$$
$$\left.-\frac{\sqrt{\sigma^2\nu}}{\left\{(1+\alpha^2)\mu^2+\sigma^2\nu\right\}^{\nu/2+1-l}}\right]$$
$$+\frac{\prod_{n=1}^{\nu/2}(\nu-2n+1)}{\prod_{n=1}^{\nu/2}(\nu-2n+2)}$$
$$\times\frac{\alpha^{2k-2i-1}}{\left\{\left(1+\alpha^2\right)\left(\mu^2+\sigma^2\nu\right)\right\}^{(\nu+1)/2+k-i}}$$
$$\times\log\left(\left(\left(\alpha+\sqrt{1+\alpha^2}\right)\right.\right.$$
$$\left.\times\sqrt{\left(1+\alpha^2\right)\mu^2+\sigma^2\nu}\right)$$
$$\times\left(\alpha\sqrt{\sigma^2\nu}\right.$$
$$+\left(\left(1+\alpha^2\right)\right.$$
$$\left.\left.\left.\left.\times\left(\mu^2+\sigma^2\nu\right)\right)^{1/2}\right)^{-1}\right)\right\}\Bigg]. \tag{10}$$

Note that if Y is defined in (2), then the nonexistence of the PV-FNM of PVE(Y^{-r}) pointed out by Peng [1], where $r = 2, 3, \ldots$. Unfortunately, the PV-FNM of a skew-t distribution, PVE(X^{-r}), fails to exist for all $(\mu, \sigma, \alpha, \nu) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ when $r = 2, 3, \ldots$. When $\nu = 1$, the formula of Theorem 2 reduces to that obtained in the skew-Cauchy case as follows.

Corollary 3. *If in Theorem 2, one lets $\nu = 1$, then $X \sim \mathcal{SC}(\mu, \sigma, \alpha) \equiv \mathcal{S}t(\mu, \sigma, \alpha, 1)$ and*

$$\begin{aligned}\mathrm{PVE}\left(\frac{1}{X}\right) &= \frac{\mu}{\mu^2+\sigma^2}+\frac{2}{\pi}\frac{\sigma}{\left(\mu^2+\sigma^2\right)}\log\left(\alpha+\sqrt{1+\alpha^2}\right)\\ &\quad-\frac{2\alpha}{\pi}\frac{\mu}{\left(\mu^2+\sigma^2\right)}\\ &\quad\times\frac{\log\left(\sqrt{1+\left(1+\alpha^2\right)\mu^2/\sigma^2}+\sqrt{1+\alpha^2}\mu/\sigma\right)}{\sqrt{1+\left(1+\alpha^2\right)\mu^2/\sigma^2}}.\end{aligned} \tag{11}$$

Corollary 4. *If in Theorem 2, one lets $\nu = 2$, then*

$$\begin{aligned}\mathrm{PVE}\left(\frac{1}{X}\right) &= \frac{\mu}{\mu^2+2\sigma^2}-\frac{\alpha\mu^2\sqrt{\left(2\sigma^2\right)\left(1+\alpha^2\right)}}{\left(\mu^2+2\sigma^2\right)\left\{\left(1+\alpha^2\right)\mu^2+2\sigma^2\right\}}\\ &\quad+\frac{2\sigma^2}{\left(\mu^2+2\sigma^2\right)^{3/2}}\\ &\quad\times\log\left[\left(\sqrt{\mu^2+2\sigma^2}+\mu\right)\right.\\ &\quad\times\left\{\alpha+\sqrt{\left(1+\alpha^2\right)\left(1+\mu^2/\left(2\sigma^2\right)\right)}\right\}\\ &\quad\left.\times\left(\sqrt{\left(1+\alpha^2\right)\mu^2+2\sigma^2}\right)^{-1}\right].\end{aligned} \tag{12}$$

As shown before, the formula of the PV-FNM of a skew-t distribution becomes complicated and difficult to calculate due to the nonzero location parameter in the integral. However, if $\mu = 0$ in Theorem 2, the complicated formula can be simplified significantly as the following results show.

Corollary 5. *If in Theorem 2, one lets $\mu = 0$, then*

$$\mathrm{PVE}\left(\frac{1}{X}\right)=\frac{2}{\sqrt{\pi\sigma^2}}\frac{\Gamma\left(\left(\nu+1\right)/2\right)}{\sqrt{\nu}\Gamma\left(\nu/2\right)}\log\left(\alpha+\sqrt{1+\alpha^2}\right), \tag{13}$$

which is an increasing function in ν.

Moreover, by using the property of the Gamma function, that is, $\Gamma((\nu+1)/2)/(\sqrt{\nu}\Gamma(\nu/2)) \to 1/\sqrt{2}$ as $\nu \to \infty$, we have the following result immediately for $\mu = 0$:

$$\mathrm{PVE}\left(\frac{1}{X}\right)=\sqrt{\frac{2}{\pi\sigma^2}}\log\left(\alpha+\sqrt{1+\alpha^2}\right)\quad\text{as } \nu\to\infty, \tag{14}$$

which coincides with the fact given by Peng [1].

The PV-FNM of a generalized Student's t-distribution can be obtained as follows. The generalized Student's t-distribution ($Y \sim \mathcal{GT}(\mu, \sigma^2, N, r, \alpha, \beta)$) is defined as a scale mixture of the Kotz distribution and an inverse gamma distribution. That is,

$$Y = \mu + \sqrt{\alpha\sigma^2 V} Z, \tag{15}$$

where $Z \sim \mathcal{K}(0, 1, N, r)$ and $V \sim \mathcal{I}nv\mathcal{G}(\alpha, \beta)$, and Z is independent of V. The Kotz distribution ($X \sim \mathcal{K}(\mu, \sigma^2, N, r)$) introduced in Kotz [12] is given by

$$f(x) = \frac{r^N ((x-\mu)/\sigma)^{2(N-1)}}{\pi\Gamma(N)\sigma} \exp\left\{-\frac{r(x-\mu)^2}{\sigma^2}\right\}, \tag{16}$$

where $r > 0$ and $N > 0$. The inverse gamma distribution ($V \sim \mathcal{I}nv\mathcal{G}(\alpha, \beta)$) is given by

$$f(v) = \frac{1}{\Gamma(\alpha)\beta}\left(\frac{\beta}{v}\right)^{\alpha+1} \exp\left(\frac{-\beta}{v}\right), \tag{17}$$

where $\alpha > 0$ and $\beta > 0$. For $N = 1$, $r = 1/2$, $\alpha = \nu/2$, and $\beta = 1$, we obtain the t distribution. Setting $N = r = 1$, $\alpha\beta = \kappa$, and $\alpha = \nu/2$ yields the generalized version of the t distributions family defined by Arellano-Valle and Bolfarine [13]. The case $\beta = 1$ and $\alpha \to \infty$ gives the Kotz distribution. Setting $N = r = 1$, $\beta = 1$, and $\alpha \to \infty$ yields the normal distribution.

Theorem 6. *If $Y \sim \mathcal{GT}(\mu, \sigma^2, N, r, \alpha, \beta)$, then*

$$\begin{aligned}
\mathrm{PVE}\left(\frac{1}{Y}\right) &= \frac{\sqrt{\pi} r^{N-1}}{\Gamma(\alpha)\Gamma(N-1/2)} \\
&\times \sum_{i=1}^{2N-2}\sum_{j=0}^{i-1}\binom{2N-2}{i}\binom{i-1}{j} \\
&\times (-1)^{2N-2-i}\mu^{2N-j-3}E\left(Z^{*j}\right) \\
&\times \frac{\Gamma(N+\alpha-j/2-1)}{(\alpha\beta\sigma^2)^{N-j/2-1}} \\
&+ \frac{\Gamma(N+\alpha)\,2\sqrt{\pi}\beta^{\alpha}\mu^{2N-2}}{\Gamma(N-1/2)\Gamma(\alpha)} \\
&\times \left(\frac{r}{\alpha\sigma^2}\right)^{N-1/2} \\
&\times \begin{cases}
\left(\dfrac{\alpha\sigma^2}{\alpha\beta\sigma^2 + r\mu^2}\right)^{N+\alpha-1/2} \\
\quad \cdot \displaystyle\sum_{k=0}^{N+\alpha-3/2} \frac{\binom{N+\alpha-3/2}{k}}{2k+1} \\
\qquad \times \left(\dfrac{r\mu^2}{\alpha\beta\sigma^2}\right)^{k+1/2}, \\
\qquad\qquad \text{if } N+\alpha-\dfrac{1}{2} \in \mathbb{N}, \\
\dfrac{\{2(N+\alpha-1)\}!}{\{(N+\alpha-1)!\}^2}\left(\dfrac{\alpha\sigma^2}{4(\alpha\beta\sigma^2+r\mu^2)}\right)^{N+\alpha-1} \\
\quad \times \left\{\dfrac{\sqrt{r\alpha\sigma^2\mu^2}}{\alpha\beta\sigma^2+r\mu^2}\right. \\
\qquad \times \displaystyle\sum_{k=1}^{N+\alpha-1} 2^{2k-1}\frac{(k-1)!k!}{(2k)!}\left(1+\frac{r\mu^2}{\alpha\beta\sigma^2}\right)^k \\
\qquad + \sqrt{\dfrac{\alpha\sigma^2}{\alpha\beta\sigma^2+r\mu^2}} \\
\qquad \times \log\left(\sqrt{1+\dfrac{r\mu^2}{\alpha\beta\sigma^2}}\right. \\
\qquad\qquad \left.\left. + \sqrt{\dfrac{r\mu^2}{\alpha\beta\sigma^2}}\right)\right\}, \\
\qquad\qquad \text{if } N+\alpha-1 \in \mathbb{N},
\end{cases}
\end{aligned} \tag{18}$$

where $Z^* \sim \mathcal{N}(0, (2r)^{-1})$.

To show the applicability and effectiveness of the main results, some applications are presented in the following section.

3. Applications

One special property of the Student's t-distribution is now obtained from Lemma 1.

Corollary 7. *If in Lemma 1, one lets $\nu = 1$, then $X \sim \mathcal{C}(\mu, \sigma) \equiv t(\mu, \sigma, 1)$ and*

$$\mathrm{PVE}\left(\frac{1}{X}\right) = \frac{\mu}{\mu^2 + \sigma^2}. \tag{19}$$

It is of interest to note that if $X \sim \mathcal{C}(\mu, \sigma)$, then $\mathrm{PVE}(X^r)$ only exists for all $(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+$ when $r = \pm 1$. That is,

$$\mathrm{PVE}(X) = \mu. \tag{20}$$

Moreover, it is well known that the random variable $1/X$ still follows the Cauchy distribution with location parameter

$\mu/(\mu^2+\sigma^2)$ and scale parameter $\sigma/(\mu^2+\sigma^2)$. Therefore, assume that $X_1,\ldots,X_n \overset{\text{i.i.d.}}{\sim} \mathscr{C}(\mu,\sigma)$,

$$M_{-1} = \frac{1}{n}\sum_{i=1}^{n} X_i^{-1}, \qquad M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i. \tag{21}$$

The moment estimators in the principal value sense can be obtained by (19) and (20) as

$$\tilde{\mu} = M_1, \qquad \tilde{\sigma}^2 = \frac{M_1}{M_{-1}}\left(1 - M_1 M_{-1}\right). \tag{22}$$

This result provides one kind of initial value to compute the maximum likelihood estimators for the Cauchy distribution.

Another application of Lemma 1 is a representation of Dawson's integral, which is defined as

$$\mathscr{D}(z) = \exp\left(-z^2\right)\int_0^z \exp\left(t^2\right)\mathrm{d}t \tag{23}$$

for all real z. Dawson's integral comes up in the theory of propagation of electromagnetic waves along the earth's surface. A precise computation is usually necessary for Dawson's integral in mathematical physics. Thus, how to represent this special function $\mathscr{D}(z)$ becomes a more practical issue. The representation of Dawson's integral and a related result are investigated in the following corollaries.

Corollary 8. *Dawson's integral can be represented as the limit of a Gaussian hypergeometric function. That is,*

$$\mathscr{D}(z) = z\exp\left(-z^2\right)\lim_{n\to\infty} {}_2F_1\left(\frac{1}{2}, -n; \frac{3}{2}; \frac{-2z^2}{2n+1}\right), \tag{24}$$

where a (Gaussian) hypergeometric function is given by

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(a)}\frac{\Gamma(b+k)}{\Gamma(b)}\frac{\Gamma(c)}{\Gamma(c+k)}\frac{x^k}{k!}. \tag{25}$$

The accuracy of this representation is omitted here as it is not essential for this work. Moreover, if a (Kummerian) confluent hypergeometric function is defined as

$${}_1F_1(a;b;x) = \sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(a)}\frac{\Gamma(b)}{\Gamma(b+k)}\frac{x^k}{k!}, \tag{26}$$

then a confluent hypergeometric function can be shown to be a limiting case of a hypergeometric function as follows.

Corollary 9. *Consider the following:*

$$\lim_{n\to\infty} {}_2F_1\left(\frac{1}{2}, -n; \frac{3}{2}; \frac{-2z^2}{2n+1}\right) = {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; z^2\right). \tag{27}$$

Note that we also double-checked the correctness of our results in this study via numerical evaluations with MathCAD mathematical software.

4. Conclusion

The primary objective considered in this study is to derive the exact form of the PV-FNM of a skew-t distribution. In addition to our attempt, the exact form of the PV-FNM of other distributions will be of great interest. For this objective, the principal value sense is used for the improper integral with singular points, so the PV-FNM can exist and be finite if that integral cannot converge in the usual sense. Hence, an explicit expression for the PV-FNM of a univariate skew-t distribution is obtained. The complex formula also has direct bearing on computations in theoretical studies. Some applications discovered from the PV-FNM of a skew-t distribution are discussed.

Many other interesting distributions and FNM-related issues are worthy of further investigation. For example, it is well known that the skew-normal distribution presents singular Fisher information matrix (see Gómez et al. [14]). This brings difficulty to inference close to the singularity point. Using the principal value on inferring this issue is a challenge worthy of research.

Appendices

A. Proof of Lemma 1

Proof. It is well known that $X \mid S^2 \sim \mathscr{N}(\mu, \sigma^2/S^2)$. Hence, by Theorem 2 in Peng [1] and using the PV-FNM of the normal distribution showed in the appendix of Quenouille [15], we obtain

$$\begin{aligned}\mathrm{PVE}\left(\frac{1}{X}\right) &= \mathrm{PVE}\left(\mathrm{PVE}\left(\frac{1}{X} \mid S^2\right)\right) \\ &= \nu^{\nu/2+1}\sigma^{\nu}\int_0^{\mu}\frac{\mathrm{d}y}{\left(\mu^2+\sigma^2\nu-y^2\right)^{\nu/2+1}}.\end{aligned} \tag{A.1}$$

For odd degrees of freedom, making use of formula 2.271 (6) in Gradshteyn and Ryzhik [16], we have

$$\begin{aligned}&\int_0^{\mu}\frac{\mathrm{d}y}{\left(\mu^2+\sigma^2\nu-y^2\right)^{\nu/2+1}} \\ &\quad = \frac{1}{\left(\mu^2+\sigma^2\nu\right)^{(\nu+1)/2}} \\ &\qquad \times \sum_{k=0}^{(\nu-1)/2}\binom{\frac{\nu-1}{2}}{k}\frac{\mu^{2k+1}}{(2k+1)\left(\sigma^2\nu\right)^{k+1/2}}.\end{aligned} \tag{A.2}$$

For even degrees of freedom, by using formulas 2.171 (4) and 2.124 (1) in Gradshteyn and Ryzhik [16], we get

$$\begin{aligned}
&\int_0^{\mu} \frac{\mathrm{d}y}{\left(\mu^2+\sigma^2\nu-y^2\right)^{\nu/2+1}} \\
&\quad = \frac{\mu}{\nu+1} \\
&\qquad \times \sum_{k=0}^{(\nu/2)-1} \left(\prod_{m=0}^{k} (\nu-2m+1) \right. \\
&\qquad\qquad \times \left(\prod_{m=0}^{k} (\nu-2m) \right. \\
&\qquad\qquad\qquad \left. \left. \times \left(\mu^2+\sigma^2\nu\right)^{k+1} \left(\sigma^2\nu\right)^{\nu/2-k} \right)^{-1} \right) \\
&\qquad + \frac{\prod_{n=0}^{\nu/2-1} (\nu-2n-1)}{\prod_{n=0}^{\nu/2-1} (\nu-2n) \left(\mu^2+\sigma^2\nu\right)^{\nu/2}} \int_0^{\mu} \frac{\mathrm{d}y}{\mu^2+\sigma^2\nu-y^2},
\end{aligned} \tag{A.3}$$

$$\int_0^{\mu} \frac{\mathrm{d}y}{\mu^2+\sigma^2\nu-y^2} = \frac{1}{\sqrt{\mu^2+\sigma^2\nu}} \log\left(\frac{\sqrt{\mu^2+\sigma^2\nu}+\mu}{\sqrt{\sigma^2\nu}} \right), \tag{A.4}$$

respectively. By combining (A.2), (A.3), and (A.4) and some algebraic manipulations, the result then follows. □

B. Proof of Theorem 2

To prove Theorem 2, we need the following auxiliary results.

Lemma 10. *For $p > \mu^2$, α, $\mu \in \mathbb{R}$ and $r = 0, 1, 2, \ldots$, one gets*

$$\begin{aligned}
&\int_0^{\mu} \frac{1}{\left(p+\alpha^2 y^2\right)^{r+3/2}} \\
&\quad \times \log\left\{ \frac{p+\alpha^2 y^2 + y\sqrt{\left(1+\alpha^2\right)\left(p+\alpha^2 y^2\right)}}{p+\alpha^2 y^2 - y\sqrt{\left(1+\alpha^2\right)\left(p+\alpha^2 y^2\right)}} \right\} dy \\
&\quad = \frac{1}{p^{r+1}} \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^i \alpha^{2i}}{2i+1} \\
&\qquad \times \left[\sum_{j=1}^{2i+1} \frac{\left\{(-1)^j+1\right\}\left(\alpha^2\mu^2+p\right)^{j/2-i-1}}{(2i-j+2)\,\mu^{j-2i-2}\left(1+\alpha^2\right)^{(j-1)/2}} \right. \\
&\qquad\qquad + 2\left(\frac{\mu^2}{\alpha^2\mu^2+p} \right)^{i+1/2} \\
&\qquad\qquad \times \log\left(\frac{\sqrt{\alpha^2\mu^2+p}+\mu\sqrt{1+\alpha^2}}{\sqrt{p-\mu^2}} \right) \\
&\qquad\qquad \left. + \left(\frac{1}{1+\alpha^2} \right)^{i+1/2} \log\left(\frac{p-\mu^2}{\alpha^2\mu^2+p} \right) \right].
\end{aligned} \tag{B.1}$$

Proof. By using the change of variable $t = y/\sqrt{p+\alpha^2 y^2}$, the binomial theorem, and term-by-term integration, the original integral can be expressed as

$$\frac{1}{p^{r+1}} \sum_{i=0}^{r} \binom{r}{i} (-1)^i \alpha^{2i} \int_0^{\mu/\sqrt{p+\alpha^2\mu^2}} t^{2i} \log\left(\frac{1+\sqrt{1+\alpha^2}t}{1-\sqrt{1+\alpha^2}t} \right) \mathrm{d}t. \tag{B.2}$$

Now, applying formula 2.729 (1) in Gradshteyn and Ryzhik [16], the result thus follows. □

Lemma 11. *For $p > \mu^2$, α, $\mu \in \mathbb{R}$, and s, $k = 0, 1, 2, \ldots$, one gets*

$$\begin{aligned}
&\int_0^{\mu} \frac{y^{2k+1}}{\left(p+\alpha^2 y^2\right)^{s+1}\left(p-y^2\right)^{k+1/2}} dy \\
&\quad = \sum_{i=0}^{k} \binom{k}{i} p^{k-i} (-1)^i \\
&\qquad \times \left[\sum_{j=1}^{k-i} \frac{\prod_{m=1}^{j} (2s+2k-2i-2m+3)}{\prod_{m=1}^{j} (2k-2i-2m+1)} \right. \\
&\qquad\qquad \times \left(\left\{ \left(\alpha^2\mu^2+p\right)^{-s} \left(p-\mu^2\right)^{i+j-k-1/2} \right. \right. \\
&\qquad\qquad\qquad \left. - p^{i+j-k-s-1/2} \right\} \\
&\qquad\qquad \times \left((2s+2k-2i+1) \right. \\
&\qquad\qquad\qquad \left. \left. \times \alpha^{2-2j} \left\{ \left(1+\alpha^2\right) p \right\}^j \right)^{-1} \right) \\
&\qquad + \frac{\prod_{n=1}^{k-i} (2s+2n-1)}{\prod_{n=1}^{k-i} (2n-1)} \\
&\qquad \times \left\{ \frac{\alpha^{2(k-i)}}{2s+1} \sum_{l=1}^{s} \frac{\prod_{m=1}^{l} (2s-2m+3)}{\prod_{m=1}^{l} (2s-2m+2)} \right. \\
&\qquad\qquad \times \frac{1}{\left\{ \left(1+\alpha^2\right) p \right\}^{l+k-i}} \\
&\qquad\qquad \times \left\{ \frac{1}{p^{s+1/2-l}} - \frac{\sqrt{p-\mu^2}}{\left(\alpha^2\mu^2+p\right)^{s+1-l}} \right\}
\end{aligned}$$

$$+\frac{\prod_{n=1}^{s}(2s-2n+1)}{\prod_{n=1}^{s}(2s-2n+2)}\times\frac{\alpha^{2k-2i-1}}{\{(1+\alpha^2)p\}^{s+k+1/2-i}}\times\log\left(\frac{\left(\alpha+\sqrt{1+\alpha^2}\right)\sqrt{\alpha^2\mu^2+p}}{\alpha\sqrt{p-\mu^2}+\sqrt{(1+\alpha^2)p}}\right)\Bigg\}\Bigg]. \tag{B.3}$$

Proof. Using the change of variable $t=\sqrt{p-y^2}$, the binomial theorem, and term-by-term integration, we get

$$\int_0^{\mu}\frac{y^{2k+1}}{(p+\alpha^2y^2)^{s+1}(p-y^2)^{k+1/2}}\mathrm{d}y=\frac{(-1)^{s+1}}{\alpha^{2s+2}}\sum_{i=0}^{k}\binom{k}{i}p^{k-i}(-1)^i\times\int_{\mathscr{A}}\frac{\mathrm{d}t}{t^{2(k-i)}(t^2-q)^{s+1}}, \tag{B.4}$$

where $q=p(1+\alpha^2)/\alpha^2$ and $\mathscr{A}=\{t\mid\sqrt{p-\mu^2}\le t\le\sqrt{p}\}$. By using formulas 1.2.11 (2) and 1.2.10 (13) in Prudnikov et al. [17], one can obtain

$$\begin{aligned}\int_{\mathscr{A}}\frac{\mathrm{d}t}{t^{2(k-i)}(t^2-q)^{s+1}}&=\frac{(-1)^{s+1}\alpha^{2s+2}}{2s+2k-2i+1}\times\sum_{j=1}^{k-i}\frac{\prod_{m=1}^{j}(2s+2k-2i-2m+3)}{\prod_{m=1}^{j}(2k-2i-2m+1)}\\&\times\frac{\left\{\left(\alpha^2\mu^2+p\right)^{-s}\left(p-\mu^2\right)^{i+j-k-1/2}-p^{i+j-k-s-1/2}\right\}}{\alpha^{2-2j}\{(1+\alpha^2)p\}^j}\\&+\frac{\prod_{n=1}^{k-i}(2s+2n-1)}{\prod_{n=1}^{k-i}(2n-1)}\frac{\alpha^{2(k-i)}}{\{(1+\alpha^2)p\}^{k-i}}\int_{\mathscr{A}}\frac{\mathrm{d}t}{(t^2-q)^{s+1}},\end{aligned} \tag{B.5}$$

$$\begin{aligned}\int_{\mathscr{A}}\frac{\mathrm{d}t}{(t^2-q)^{s+1}}&=\frac{(-1)^{s+1}}{2s+1}\times\sum_{l=1}^{s}\frac{\prod_{m=1}^{l}(2s-2m+3)}{\prod_{m=1}^{l}(2s-2m+2)}\times\frac{\alpha^{2s+2}}{\{(1+\alpha^2)p\}^l}\times\left\{\frac{1}{p^{s+1/2-l}}-\frac{\sqrt{p-\mu^2}}{(\alpha^2\mu^2+p)^{s+1-l}}\right\}\\&+\frac{\prod_{n=1}^{s}(2s-2n+1)}{\prod_{n=1}^{s}(2s-2n+2)}\times\frac{(-1)^{s+1}\alpha^{2s+1}}{\{(1+\alpha^2)p\}^{s+1/2}}\times\log\left\{\frac{\left(\alpha+\sqrt{1+\alpha^2}\right)\sqrt{\alpha^2\mu^2+p}}{\alpha\sqrt{p-\mu^2}+\sqrt{(1+\alpha^2)p}}\right\},\end{aligned} \tag{B.6}$$

respectively. Substituting (B.5) and (B.6) into (B.4) yields the desired result. □

Now, we return to the proof of Theorem 2.

Proof. First, we represent the expression (8) of the PV-FNM of a skew-normal distribution in Peng [1] as follows. If $Y\sim\mathscr{SN}(\mu,\sigma^2,\alpha)$, then

$$\begin{aligned}\mathrm{PVE}\left(\frac{1}{Y}\right)&=\frac{1}{\sigma^2}\int_0^{\mu}\exp\left(\frac{x^2-\mu^2}{2\sigma^2}\right)\mathrm{d}x-\frac{\alpha\sqrt{2(1+\alpha^2)}}{\sigma^3\sqrt{\pi}}\exp\left(\frac{-\mu^2}{2\sigma^2}\right)\times\int_0^{\mu}\exp\left(\frac{-\alpha^2x^2}{2\sigma^2}\right)\int_0^{x}\exp\left\{\frac{(1+\alpha^2)y^2}{2\sigma^2}\right\}\mathrm{d}y\mathrm{d}x\\&+\sqrt{\frac{2}{\pi\sigma^2}}\exp\left(\frac{-\mu^2}{2\sigma^2}\right)\log\left(\alpha+\sqrt{1+\alpha^2}\right).\end{aligned} \tag{B.7}$$

By using (B.7), the definition in (3), and the conditional expectation, we obtain

$$\mathrm{PVE}\left(\frac{1}{X}\right)=I_1-I_2+\sqrt{\frac{2}{\pi\sigma^2}}\log\left(\alpha+\sqrt{1+\alpha^2}\right)I_3, \tag{B.8}$$

where

$$I_1 = \mathrm{PVE}\left\{\int_0^{\mu} \frac{S^2}{\sigma^2} \exp\left(\frac{x^2-\mu^2}{2\sigma^2}S^2\right)\mathrm{d}x\right\},$$

$$I_2 = \frac{\alpha\sqrt{2\left(1+\alpha^2\right)}}{\sigma^3\sqrt{\pi}}\mathrm{PVE} \times\left[\int_0^{\mu}\int_0^{y} S^3 \times\exp\left\{-\frac{\mu^2+\alpha^2y^2-\left(1+\alpha^2\right)x^2}{2\sigma^2}S^2\right\}\mathrm{d}x\mathrm{d}y\right],$$

$$I_3 = \mathrm{PVE}\left\{S\exp\left(\frac{-\mu^2}{2\sigma^2}S^2\right)\right\}. \tag{B.9}$$

Now, I_1 can be calculated as (18) from Lemma 1. By Fubini's theorem, I_2 turns out to be

$$I_2 = \frac{2\alpha\sqrt{1+\alpha^2}\,(\nu+1)\,\Gamma\left((\nu+1)/2\right)\left(\sigma^2\nu\right)^{\nu/2}}{\sqrt{\pi}\Gamma(\nu/2)}I_2^*, \tag{B.10}$$

where

$$I_2^* = \int_0^{\mu}\int_0^{y}\frac{\mathrm{d}x\mathrm{d}y}{\left\{\mu^2+\sigma^2\nu+\alpha^2y^2-(1+\alpha^2)x^2\right\}^{(\nu+3)/2}}. \tag{B.11}$$

The evaluation of this integral is divided into two parts.

(i) For odd degrees of freedom, by using formulas 2.171 (4) and 2.124 (1) in Gradshteyn and Ryzhik [16], we get

$$I_2^* = \frac{1}{\nu+2} \times\sum_{k=0}^{(\nu-1)/2}\frac{\prod_{m=0}^{k}(\nu-2m+2)}{\prod_{m=0}^{k}(\nu-2m+1)}I_{21}(k) + \frac{\prod_{n=0}^{(\nu-1)/2}(\nu-2n)}{\prod_{n=0}^{(\nu-1)/2}(\nu-2n+1)}\frac{I_{22}}{2\sqrt{1+\alpha^2}}, \tag{B.12}$$

where

$$I_{21}(k) = \int_0^{\mu}\frac{y}{\left(p+\alpha^2y^2\right)^{k+1}\left(p-y^2\right)^{(\nu+1)/2-k}}\mathrm{d}y,$$

$$I_{22} = \int_0^{\mu}\frac{1}{\left(p+\alpha^2y^2\right)^{\nu/2+1}} \times\log\left\{\frac{p+\alpha^2y^2+y\sqrt{(1+\alpha^2)\left(p+\alpha^2y^2\right)}}{p+\alpha^2y^2-y\sqrt{(1+\alpha^2)\left(p+\alpha^2y^2\right)}}\right\}\mathrm{d}y, \tag{B.13}$$

in which $p=\mu^2+\sigma^2\nu$. Therefore, the change of variable $t=y^2$ implies

$$I_{21}(k) = \frac{(-1)^{((\nu+1)/2)-k}}{2\alpha^{2k+2}}\int_0^{\mu^2}\left(t+\frac{p}{\alpha^2}\right)^{-k-1}(t-p)^{k-(\nu+1)/2}\,\mathrm{d}t. \tag{B.14}$$

Then, using formula 1.2.7 (6) in Prudnikov et al. [17], one can obtain $I_{21}(k)$ as (8). From Lemma 10, let $p=\mu^2+\sigma^2\nu$ and $r=(\nu-1)/2$; then I_{22} can be expressed as (9).

(ii) For even degrees of freedom, making use of formula 2.271 (6) in Gradshteyn and Ryzhik [16], I_2^* can be expanded as

$$I_2^* = \sum_{k=0}^{\nu/2}\frac{\binom{\nu/2}{k}\left(1+\alpha^2\right)^k}{2k+1}I_{23}(k), \tag{B.15}$$

where

$$I_{23}(k) = \int_0^{\mu}\frac{y^{2k+1}}{\left(\mu^2+\sigma^2\nu+\alpha^2y^2\right)^{\nu/2+1}\left(\mu^2+\sigma^2\nu-y^2\right)^{k+1/2}}\mathrm{d}y. \tag{B.16}$$

Let $p=\mu^2+\sigma^2\nu$ and $s=\nu/2$ in Lemma 11; then $I_{23}(k)$ can be worked out as (10).

Finally, I_3 is easy to calculate as

$$I_3 = \sqrt{\frac{2}{\nu}}\frac{\Gamma\left((\nu+1)/2\right)}{\Gamma(\nu/2)}\left(\frac{\sigma^2\nu}{\mu^2+\sigma^2\nu}\right)^{(\nu+1)/2}. \tag{B.17}$$

Consequently, combining (18), (8), (9), (10), and (B.17) yields the desired result. □

C. Proof of Theorem 6

We need the following lemma.

Lemma 12. *Letting $Y \sim \mathscr{K}(\mu,\sigma^2,N,r)$, where N is integer, then the PV-FNM of a Kotz distribution is given by*

$$\mathrm{PVE}\left(\frac{1}{Y}\right) = \left(\frac{\mu}{\sigma}\right)^{2N-2}\cdot\frac{2\sqrt{\pi}r^N}{\sigma^2\Gamma(N-1/2)} \times\int_0^{\mu}\exp\left(\frac{r\left(x^2-\mu^2\right)}{\sigma^2}\right)dx + \frac{\sqrt{\pi}r^{N-1}}{\sigma\Gamma(N-1/2)} \times\sum_{i=1}^{2N-2}\sum_{j=0}^{i-1}\binom{2N-2}{i}\binom{i-1}{j} \times(-1)^{2N-2-i}\cdot\left(\frac{\mu}{\sigma}\right)^{2N-j-3}E\left(Z^{*j}\right), \tag{C.1}$$

where $Z^* \sim \mathscr{N}(0,(2r)^{-1})$.

Proof. Let $x=y/\sigma$ and $a=\mu/\sigma$; then

$$\mathrm{PVE}\left(\frac{1}{Y}\right) = \frac{r^{N-1/2}}{\sigma\Gamma(N-1/2)}I(a), \tag{C.2}$$

where

$$I(a) = \int_{-\infty}^{\infty} \frac{1}{x}(x-a)^{2N-2} \exp\left(-r(x-a)^2\right) \mathrm{d}x. \tag{C.3}$$

Then, using binomial theorem and term-by-term integration, we get

$$I(a) = a^{2N-2} I_1(a) + \sum_{i=1}^{2N-2} \binom{2N-2}{i} (-a)^{2N-2-i} I_2(a), \tag{C.4}$$

where

$$\begin{aligned} I_1(a) &= \int_{-\infty}^{\infty} \frac{1}{x} \exp\left(-r(x-a)^2\right) \mathrm{d}x, \\ I_2(a) &= \int_{-\infty}^{\infty} x^{i-1} \exp\left(-r(x-a)^2\right) \mathrm{d}x. \end{aligned} \tag{C.5}$$

Following the same argument in the appendix of Quenouille [15], $I_1(a)$ can be derived as

$$I_1(a) = 2\sqrt{\pi r} \int_0^a \exp\left(r\left(x^2 - a^2\right)\right) \mathrm{d}x. \tag{C.6}$$

By the property of a normal distribution, we obtain

$$\begin{aligned} I_2(a) &= \sqrt{\frac{\pi}{r}} E\left\{\left(Z^* + a\right)^{i-1}\right\} \\ &= \sqrt{\frac{\pi}{r}} \sum_{j=0}^{i-1} \binom{i-1}{j} \left(\frac{\mu}{\sigma}\right)^{i-j-1} E\left(Z^{*j}\right). \end{aligned} \tag{C.7}$$

Substituting (C.6) and (C.7) into (C.2) yields the desired result. □

Now, we return to the proof of Theorem 6.

Proof. It is well known that $Y \mid V \sim \mathcal{K}(\mu, \alpha\sigma^2 V, N, r)$. Hence, using Lemma 12, we obtain

$$\begin{aligned} &\mathrm{PVE}\left(\frac{1}{Y}\right) \\ &\quad = \mathrm{PVE}\left(\mathrm{PVE}\left(\frac{1}{Y} \mid V\right)\right) \\ &\quad = \frac{2\sqrt{\pi} r^N \beta^\alpha \mu^{2N-2}}{\Gamma(N-1/2)\Gamma(\alpha)\left(\alpha\sigma^2\right)^N} I_1 \\ &\qquad + \frac{\sqrt{\pi} r^{N-1}}{\beta\Gamma(\alpha)\Gamma(N-1/2)} \\ &\qquad \times \sum_{i=1}^{2N-2} \sum_{j=0}^{i-1} \binom{2N-2}{i} \binom{i-1}{j} \\ &\qquad\qquad \times (-1)^{2N-2-i} \mu^{2N-j-3} E\left(Z^{*j}\right) I_2, \end{aligned} \tag{C.8}$$

where

$$\begin{aligned} I_1 &= \int_0^\mu \int_0^\infty v^{-(N+\alpha+1)} \\ &\qquad \times \exp\left\{-\left(\beta - \frac{r\left(y^2 - \mu^2\right)}{\alpha\sigma^2}\right)\frac{1}{v}\right\} \mathrm{d}v\mathrm{d}x, \end{aligned} \tag{C.9}$$

$$\begin{aligned} I_2 &= \frac{\beta^{\alpha+1}}{\left(\sqrt{\alpha\sigma^2}\right)^{2N-j-2}} \\ &\quad \times \int_0^\infty v^{-(2N-j-2)/2-\alpha-1} \exp\left(-\frac{\beta}{v}\right) \mathrm{d}v. \end{aligned} \tag{C.10}$$

Now, I_1 can be evaluated as

$$\begin{aligned} I_1 &= \Gamma(N+\alpha)\sqrt{\frac{\alpha\sigma^2}{r}} \\ &\quad \times \int_0^{\sqrt{r\mu^2/\alpha\sigma^2}} \frac{\mathrm{d}y}{\left(\beta + r\mu^2/\alpha\sigma^2 - y^2\right)^{N+\alpha}}. \end{aligned} \tag{C.11}$$

The evaluation of this integral is divided into two parts.

(i) For $N + \alpha - 1/2 \in \mathbb{N}$, making use of formula 2.271 (6) in Gradshteyn and Ryzhik [16], we have

$$\begin{aligned} &\int_0^{\sqrt{r\mu^2/\alpha\sigma^2}} \frac{\mathrm{d}y}{\left(\beta + \left(r\mu^2/\alpha\sigma^2\right) - y^2\right)^{N+\alpha}} \\ &\quad = \left(\frac{\alpha\sigma^2}{\alpha\beta\sigma^2 + r\mu^2}\right)^{N+\alpha-1/2} \\ &\qquad \cdot \sum_{k=0}^{N+\alpha-3/2} \frac{\binom{N+\alpha-3/2}{k}}{2k+1} \left(\frac{r\mu^2}{\alpha\beta\sigma^2}\right)^{k+1/2}. \end{aligned} \tag{C.12}$$

(ii) For $N + \alpha - 1 \in \mathbb{N}$, by using formulas 2.171 (4), 2.103 (4) in Gradshteyn and Ryzhik [16] and after some algebraic manipulations, we get

$$\begin{aligned} &\int_0^{\sqrt{r\mu^2/\alpha\sigma^2}} \frac{\mathrm{d}y}{\left(\beta + \left(r\mu^2/\alpha\sigma^2\right) - y^2\right)^{N+\alpha}} \\ &\quad = \frac{\{2(N+\alpha-1)\}!}{\{(N+\alpha-1)!\}^2} \left(\frac{\alpha\sigma^2}{4\left(\alpha\beta\sigma^2 + r\mu^2\right)}\right)^{N+\alpha-1} \end{aligned}$$

$$\times \left\{ \frac{\sqrt{r\alpha\sigma^2\mu^2}}{\alpha\beta\sigma^2 + r\mu^2} \times \sum_{k=1}^{N+\alpha-1} 2^{2k-1} \frac{(k-1)!k!}{(2k)!} \times \left(1 + \frac{r\mu^2}{\alpha\beta\sigma^2}\right)^k + \int_0^{\sqrt{r\mu^2/\alpha\sigma^2}} \frac{\mathrm{d}y}{(\beta + (r\mu^2/\alpha\sigma^2) - y^2)} \right\}, \tag{C.13}$$

$$\int_0^{\sqrt{r\mu^2/\alpha\sigma^2}} \frac{\mathrm{d}y}{(\beta + r\mu^2/\alpha\sigma^2 - y^2)} = \sqrt{\frac{\alpha\sigma^2}{\alpha\beta\sigma^2 + r\mu^2}} \log\left(\sqrt{1 + \frac{r\mu^2}{\alpha\beta\sigma^2}} + \sqrt{\frac{r\mu^2}{\alpha\beta\sigma^2}}\right), \tag{C.14}$$

respectively. Finally, I_2 is easy to evaluate as

$$I_2 = \frac{\beta^{j/2-N+2}\Gamma(N + \alpha - j/2 - 1)}{(\alpha\sigma^2)^{N-j/2-1}}. \tag{C.15}$$

Consequently, combining (C.8)–(C.15) yields the desired result. □

D. Proof of Corollary 8

Proof. Since a Student's t-distribution approaches a normal distribution as $\nu \to \infty$, we have $\mathrm{PVE}(1/Z) = \lim_{\nu\to\infty}\mathrm{PVE}(1/X)$, where $Z \sim \mathcal{N}(\mu, \sigma^2)$ and $X \sim t(\mu, \sigma^2, \nu)$. In addition, Quenouille [15] showed that

$$\mathrm{PVE}\left(\frac{1}{Z}\right) = \frac{\sqrt{2}}{\sigma}\mathscr{D}\left(\frac{\mu}{\sqrt{2}\sigma}\right). \tag{D.1}$$

Hence, by restricting limit to odd degrees in Lemma 1 and using L'Hospital rule, we have

$$\lim_{\nu\to\infty}\mathrm{PVE}\left(\frac{1}{X}\right)\Big|_{\nu \text{ is odd}} = \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \times \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{\mu^{2k+1}}{(2k+1)(2n+1)^k(\sigma^2)^{k+1}}, \tag{D.2}$$

where $\nu = 2n + 1$. Comparing (D.1) and (D.2), one can obtain

$$\int_0^z \exp\left(t^2\right)\mathrm{d}t = \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{2}{2n+1}\right)^k\frac{z^{2k+1}}{2k+1} = z\lim_{n\to\infty}{}_2F_1\left(\frac{1}{2}, -n; \frac{3}{2}; \frac{-2z^2}{2n+1}\right). \tag{D.3}$$

Thus, substituting (D.3) into (23) yields the desired result (24). □

E. Proof of Corollary 9

Proof. It is well known that Dawson's integral is a special case of the confluent hypergeometric function. That is,

$$\mathscr{D}(z) = z\,{}_1F_1\left(1; \frac{3}{2}; -z^2\right). \tag{E.1}$$

By using formula 9.212 (1) in Gradshteyn and Ryzhik [16] and comparing (24) in Corollary 8 and (E.1), (27) can be obtained directly. □

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