Research Article
Superconvergence Analysis of a Multiscale Finite Element Method for Elliptic Problems with Rapidly Oscillating Coefficients

Xiaofei Guan,1 Xiaoling Wang,2 Cheng Wang,1 and Xian Liu3

1 Department of Mathematics, Tongji University, Shanghai 200092, China
2 Department of Fundamental Subject, Tianjin Institute of Urban Construction, Tianjin 300384, China
3 Department of Geotechnical Engineering, Tongji University, Shanghai 200092, China

Correspondence should be addressed to Cheng Wang; wangcheng@tongji.edu.cn

Received 29 October 2012; Accepted 17 January 2013

Academic Editor: Song Cen

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A new multiscale finite element method is presented for solving the elliptic equations with rapidly oscillating coefficients. The proposed method is based on asymptotic analysis and careful numerical treatments for the boundary corrector terms by virtue of the recovery technique. Under the assumption that the oscillating coefficient is periodic, some superconvergence results are derived, which seem to be never discovered in the previous literature. Finally, some numerical experiments are carried out to demonstrate the efficiency and accuracy of this method, and it is seen that they agree very well with the analytical result.

1. Introduction

In this paper, we consider the following elliptic boundary value problem with rapidly oscillatory coefficients:

\[
L_\varepsilon u^\varepsilon \equiv \frac{\partial}{\partial x_i} \left( a_{ij} \left( \frac{x}{\varepsilon} \right) \frac{\partial u^\varepsilon}{\partial x_j} \right) = f(x), \quad \text{in } \Omega,
\]

where \( \Omega \subset \mathbb{R}^2 \) is a smooth-bounded domain, \( a_{ij}(\xi) : \mathbb{R}^2 \rightarrow \mathbb{R} \) is symmetric and satisfies

\[
\begin{align*}
(1) & \quad \lambda |\xi|^2 \leq a_{ij}(\xi) \xi_i \xi_j \leq \lambda^{-1} |\xi|^2, \quad \forall \xi \in \mathbb{R}^2, \exists \lambda \in (0,1], \\
(2) & \quad a_{ij}(\xi + \xi') = a_{ij}(\xi), \quad \forall \xi \in \mathbb{R}^2, \exists \xi' \in Z^2, \quad 1 \leq i, j \leq n, \\
(3) & \quad \|a_{ij}\|_{L^1(\mathbb{R}^2)} \leq C, \quad \exists C > 0,
\end{align*}
\]

where \( \xi = x/\varepsilon, \varepsilon \) is a small scale parameter. This kind of equation has widely been applied in many areas, such as the behavior of flow in porous media or the thermal and mechanical behavior of composite material structure. In practice, the oscillatory coefficients may span many scales to a great extent. In such cases, the direct accurate numerical computation of the solution becomes difficult because it would require a very fine mesh, and it can easily exceed the limit of today's computer resources because of the requirement of tremendous amount of computer memory and CPU time. Meanwhile, it is desirable to have a numerical method that can solve this equation on a large-scale mesh with capturing the effect of small scales details. Thus, various methods of upsampling or homogenization have been developed.

Based on the homogenization method, there are many discussions [1–4] about the numerical methods of (1). A large amount of examples and applications can also be found in the classical books [5–8], where the formal asymptotic expansions for the limit solution are deduced when \( \varepsilon \) is small enough. In these books, the first-order approximation of these expansions is justified by proving sharp error estimates, from which a general method that allowed us to treat some structures with rapidly oscillatory coefficients is also developed. However, the general method cannot effectively compute the boundary corrector on boundary layer. It should
be noted that the boundary corrector is the important source of error estimates. In [9], He and Cui present a novel finite element method to solve (1) which can effectively compute the boundary corrector even if the boundary layer is very small. The crucial idea is to combine the numerical approximation of the first-order terms of asymptotic expansions with the numerical approximation of the boundary corrector from different meshes exploiting the need for different levels of resolution. The following result (Theorem 2.13 in [9]) can be obtained.

**Lemma 1.** Assume that \( u^\varepsilon \) is the solution of (1) and \( \tilde{u}^{h_0,h_1,h} \) is the finite element solution of (1). For all \( p, 1 < p < +\infty \), there exists a constant \( C \) such that

\[
\left\| \nabla \left( u^\varepsilon - \tilde{u}^{h_0,h_1,h} \right) \right\|_{L^2(\Omega)} + \varepsilon \left\| \nabla \nabla \left( u^\varepsilon - \tilde{u}^{h_0,h_1,h} \right) \right\|_{L^2(\Omega)} \\
\leq C \left[ (h_1 + h_0 + h + \varepsilon) \ln \varepsilon^{1/2} + \varepsilon^{(2p-1)/2p} \right] \left( \left\| u^\varepsilon \right\|_{W^{2,p}(\Omega)} + \left\| u^\varepsilon \right\|_{H^1(\Omega)} \right),
\]

where \( u^\varepsilon \) is the homogenization solution of (1), and \( \text{dist}(x, \partial \Omega) \) is the distance between the point \( x \) and the boundary \( \partial \Omega \).

Unfortunately, the needed CPU time of the method presented in [9] is \( O(\varepsilon^{-n} h^{-m}) \). In this paper, a high-effective finite element method to compute boundary corrector by virtue of the recovery technique is proposed, and some superconvergence results for the multiscale finite element approximation of (1) are obtained. The rest of this paper is organized as follows. In the next section, we present a multiscale finite element method to compute \( u^\varepsilon(x) \). Its convergence analysis are shown in Section 3. Finally, some numerical results conforming our analytical estimates are given in Section 4.

**Notation.** Before closing this section, we would like to fix some notations. First, the Einstein summation is used. Let \( Q = \{ |\xi| < 1, i = 1,2 \} \), and the capital letter \( C \) (with or without subscripts) denotes a positive constant, which is independent of the small parameter \( \varepsilon \) and the mesh size \( h \) (with or without subscripts).

### 2. An Improved Multiscale Finite Element Method

Firstly, we let simply recall the homogenization method described in [5].

#### 2.1. Homogenization Method

Let \( N_k(\xi) (k = 1,2) \) be a 1-periodic function, which satisfies

\[
\frac{\partial}{\partial \xi_i} \left( a_{ij}(\xi) \frac{\partial N_k}{\partial \xi_j} \right) = -\frac{\partial}{\partial \xi_i} a_{jk}, \quad \text{in } \mathbb{R}^2,
\]

\[
\int_Q N_k(\xi) \, d\xi = 0.
\]

Then, the matrix \( \tilde{a} = (\tilde{a}_{ij})_{2 \times 2} \) can be obtained by

\[
\tilde{a}_{ij} = \int_Q \left( a_{ij} + a_k \frac{\partial N_j}{\partial \xi_k} \right) \, d\xi.
\]

The first-order approximation of \( u^\varepsilon(x) \) can be written as

\[
\tilde{u}(x) = u^0(x) + e N_k(\xi) \frac{\partial u^0(x)}{\partial x_k},
\]

where \( u^0(x) \) satisfies the homogenization problem

\[
L_0 u^0(x) := \frac{\partial}{\partial x_i} \left( \tilde{a}_{ij} \frac{\partial u^0}{\partial x_j} \right) = f(x), \quad \text{in } \Omega,
\]

\[
u^0(x) = g(x), \quad \text{on } \partial \Omega.
\]

The boundary corrector term of the homogenization method \( \theta_e \) is defined by

\[
\theta_e = -e N_k \frac{\partial u^0}{\partial x_k}, \quad \text{on } \partial \Omega.
\]

In the next two subsections, we will compute numerically the first-order approximation \( \tilde{u} \) and the boundary corrector term \( \theta_e \), respectively, and furthermore give the multiscale finite element solution of (1).

#### 2.2. Finite Element Approximation of \( \tilde{u} \)

Let \( \mathcal{T}_{h_0} \) be a quasi-uniform triangular partition of \( Q \) with the mesh size \( h_0 \). \( S^{h_0} \) denotes the conforming \( P_1 \) finite element spaces with respect to \( \mathcal{T}_{h_0} \) and \( S_k^{h_0} = S^{h_0} \cap H^1_0(Q) \). The finite element scheme of (4) is to find \( N_k^{h_0} \in S^{h_0} \) such that

\[
\int_Q a_{ij}(\xi) \frac{\partial N_k^{h_0}(\xi)}{\partial \xi_j} \frac{\partial v(\xi)}{\partial \xi_i} \, d\xi = -\int_Q a_{ik}(\xi) \frac{\partial v(\xi)}{\partial \xi_i} \, d\xi, \quad \forall v \in S^{h_0}(Q),
\]

\[
N_k^{h_0}(\xi) \text{ is a } 1\text{-periodic function.}
\]

Then, the numerical approximation \( \tilde{a}_{ij}^{h_0} \) of \( \tilde{a}_{ij} \) can be calculated by

\[
\tilde{a}_{ij}^{h_0} = \int_Q \left( a_{ij}(\xi) + a_k(\xi) \frac{\partial N_k^{h_0}(\xi)}{\partial x_k} \right) \, d\xi.
\]

Let \( \mathcal{T}_{h_1} \) be a quasiuniform triangular partition of \( \Omega \) with the mesh size \( h_1 \) and satisfy

\[
\min \frac{S_{k \in E_{h_1}}}{S_{k \in E_{h_1}}} \geq C h_1^2,
\]

where \( S_k \) is the area of the triangular element \( e_k \). \( S_k^{h_0} \) denotes the corresponding conforming \( P_1 \) finite element spaces, and \( S^{h_0} = S^{h_1} \cap H^1_0(Q) \).
The finite element approximation \( u_{h_0, h_1}^k \) of the homogenization problem (7) is to find \( u_{h_0, h_1}^k \in S^h_i \) such that

\[
\int_{\Omega} \frac{\partial u_{h_0, h_1}^k}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx = \int_{\Omega} f(x) v(x) \, dx, \quad \forall v \in S^h_i.
\]

(12)

Furthermore, we turn to the computation of \( \partial u_{h_0, h_1}^k(x) / \partial x_k \) and \( \bar{u}(x) \). Let \( \Sigma^h_i \) be the set of all nodal points of the mesh \( T_h \). Define \( u_{h_0, h_1}^k (k = 1, 2) \) by the following:

\( (A_1) \) for all \( x \in \Sigma^h_i \), the value of \( u_{h_0, h_1}^k \) at the nodal point \( x \) is the average of \( \partial u_{h_0, h_1}^k(x) / \partial x_k \) in all elements including \( x \),

\( (A_2) \) \( u_{h_0, h_1}^k(x) \) is a piecewise linear function in every element.

Therefore, we have a numerical approximation \( \bar{u}_{h_0, h_1}^k(x) \) of \( u(x) \) which is defined by

\[
\bar{u}_{h_0, h_1}^k(x) = u_{h_0, h_1}^k(x) + \varepsilon N_{h}^k(\xi) u_{h_0, h_1}^k(x). \quad (13)
\]

2.3. Finite Element Approximation of \( \theta_\varepsilon(x) \). Let \( m \) be a positive integer satisfying

\[
2^{m-1} \varepsilon < \text{max} \, \text{dist}(x, \partial \Omega) \leq 2^m \varepsilon. \quad (14)
\]

Then, the domain \( \Omega \) can be divided by \( \Omega = \bigcup_{i=0}^m \Omega_i \) and

\[
\Omega_i = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) \leq \varepsilon \}, \quad \text{if} \ i = 0,
\]

\[
\Omega_i = \{ x \in \Omega \mid 2^{i-1} \varepsilon < \text{dist}(x, \partial \Omega) \leq 2^i \varepsilon \}, \quad \text{if} \ 1 \leq i \leq m. \quad (15)
\]

Let \( T_h \) be the regular triangular partition of \( \Omega \) with the mesh size \( h \) and satisfy

\[
(1^t) \ 2^{i-1} \lambda \varepsilon h \leq \sqrt{S_i} \leq 2^{i-1} \lambda^{-1} \varepsilon h, \quad \forall e_i \in T_h,
\]

\[
(2^t) \ |I_i| \leq C \varepsilon h, \quad \forall e_i \in T_h, \quad (16)
\]

where \( \lambda \) and \( C \) are independent of \( \varepsilon \) and \( e_i \).\( S_i \) denotes the area of \( e_i \) and \( I_i \), \( I_i \) are the length of two edges of \( e_i \). Let \( \Sigma^h \) be the set of all nodal points in \( T_h \), \( \Sigma^h \cap \partial \Omega \), and let \( S^h \) be the conforming \( P_i \) finite element spaces with respect to \( T_h \); we define

\[
S^h_i = \{ v \in S^h \mid v(x) = -\varepsilon N_{h}^k \text{ for } v(x) \in \Sigma^h \}, \quad \forall x \in T^h_i,
\]

\[
S^h_0 = \{ v \in S^h \mid v(x) = 0, \forall x \in \partial T^h_i \}. \quad (18)
\]

Then, the finite element approximation of \( \theta_\varepsilon \) is to find \( \bar{\theta}_{\varepsilon}^k \) such that

\[
\int_{\Omega} a_i(\xi) \frac{\partial \bar{\theta}_{\varepsilon}^k}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx = 0, \quad \forall v \in S^h_i. \quad (19)
\]

2.4. Multiscale Finite Element Approximation of \( u_\varepsilon \). For any \( v \in S^h_0(\Omega) \), we define the linear operator \( R_h \) by

\[
\frac{\partial R_h v(x)}{\partial x_i} \quad \text{is a piecewise function on } T^h_i,
\]

\[
\frac{\partial R_h v(x)}{\partial x_i} = \frac{\partial v(x)}{\partial x_i}, \quad \text{if } x \text{ is the middle point of elements of } T^h_i. \quad (20)
\]

Then, the multiscale finite element approximation \( \bar{u}_{h_0, h_1}^k(x) \) of \( u_\varepsilon(x) \) can be defined by

\[
\bar{u}_{h_0, h_1}^k(x) = u_{h_0, h_1}^k(x) + R_h \bar{\theta}_{\varepsilon}^k(x). \quad (21)
\]

3. Superconvergence Result of \( \bar{u}_{h_0, h_1}^k \)

Firstly, we have the following assumption.

Assumption C1. The functions \( a_{ij} \in W^{1,\infty}(Q) \cap H^2(Q) \), and the homogenization solution \( u_0 \in W^{2,\infty}(\Omega) \cap H^4(\Omega) \).

Then, we introduce the following lemma.

Lemma 2 (see [1, 5]). Let \( \Omega_r = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) \geq r \} \). Assuming that (C1) holds, then there exists \( C \), which is independent of \( \varepsilon \) and \( r \) such that

\[
\| \nabla \theta_\varepsilon \|_{L^2(\Omega)} \leq C \varepsilon^{-1/2} \| u_0 \|_{H^2(\Omega)},
\]

\[
\| \theta_\varepsilon \|_{H^1(\Omega)} \leq C \varepsilon^{-1/2} \| u_0 \|_{H^2(\Omega)},
\]

\[
\| \theta_\varepsilon \|_{H^1(\Omega)} \leq C \varepsilon^{-1/2} \| u_0 \|_{H^2(\Omega)}.
\]

(22)

From Lemma 2, one can easily deduce.

Lemma 3. Assuming that (C1) holds, then there exists \( C \), which is independent of \( \varepsilon \) such that

\[
\| \theta_\varepsilon \|_{H^1(\Omega)} \leq C \varepsilon^{-3/2} \| u_0 \|_{H^2(\Omega)},
\]

\[
\| \theta_\varepsilon \|_{H^1(\Omega)} \leq C \varepsilon^{-3/2} \| u_0 \|_{H^2(\Omega)}.
\]

(23)

Proof. For \( k = 1, 2 \), we define

\[
u_k^\varepsilon(x) = \frac{\partial \theta_\varepsilon(x)}{\partial x_k}.
\]

(25)

Then, the upper bound of \( \| \nu_k^\varepsilon \|_{H^1(\Omega)} \) and \( \| \nu_k^\varepsilon \|_{H^1(\Omega)} \) can be estimated, respectively.

Using the result from (9) and (18), we have

\[
L_i \nu_k^\varepsilon = -\varepsilon \frac{\partial}{\partial x_i} \left( \frac{\partial a_{ij}}{\partial x_k} \frac{\partial \theta_\varepsilon}{\partial x_j} \right), \quad x \in \Omega,
\]

\[
\nu_k^\varepsilon(x) = -\varepsilon \left( \frac{\partial a_{ij}}{\partial x_k} \frac{\partial \theta_\varepsilon}{\partial x_j} \right), \quad x \in \partial \Omega.
\]

(26)
Then, we estimate \( \|v_k^\varepsilon\|_{H^2(\Omega)} \). Obviously, \( v_k^\varepsilon \) can be divided into
\[
v_k^\varepsilon = v_{k,1}^\varepsilon + v_{k,2}^\varepsilon,
\]
where \( v_{k,1}^\varepsilon \) satisfies
\[
L_\varepsilon v_{k,1}^\varepsilon = -\varepsilon^{-1} \frac{\partial}{\partial x_i} \left( \frac{\partial a_{ij}(\xi)}{\partial x_k} \right) \frac{\partial \theta_k^\varepsilon}{\partial x_j}, \quad x \in \Omega,
\]
\[
v_{k,1}^\varepsilon = 0, \quad x \in \partial \Omega,
\]
and \( v_{k,2}^\varepsilon \) satisfies
\[
L_\varepsilon v_{k,2}^\varepsilon = 0, \quad x \in \Omega,
\]
\[
v_{k,2}^\varepsilon = -\varepsilon \left( \frac{\partial}{\partial x_i} \left( N_i(x/\varepsilon) \left( \frac{\partial u_0^0}{\partial x_j} \right) \right) \right), \quad x \in \partial \Omega.
\]

Using the result from Lemma 2 and (28), we have
\[
\|v_k^\varepsilon\|_{H^2(\Omega)} \leq Ce^{-1} \|\theta_k^\varepsilon\|_{H^2(\Omega)} \leq Ce^{-3/2} \|u^0\|_{H^4(\Omega)}.
\]

Let \( v_{k,2,k'} = \partial v_{k,2}/\partial x_{k'} \), and using the result from (29), we have
\[
L_\varepsilon v_{k,2,k'} = 0, \quad x \in \Omega,
\]
\[
v_{k,2,k'} = -\varepsilon \left( \frac{\partial}{\partial x_i} \left( N_i(x/\varepsilon) \left( \frac{\partial u_0^0}{\partial x_j} \right) \right) \right), \quad x \in \partial \Omega.
\]

Combining (30) with (33), we can derive (24) immediately.

Considering the proof of (23), \( v_k^\varepsilon \) can be divided into
\[
v_k^\varepsilon = \overline{v}_{k,1}^\varepsilon + \overline{v}_{k,2}^\varepsilon,
\]
where \( \overline{v}_{k,1}^\varepsilon \) satisfies
\[
L_\varepsilon \overline{v}_{k,1}^\varepsilon = -\varepsilon^{-1} \frac{\partial}{\partial x_i} \left( \frac{\partial a_{ij}(\xi)}{\partial x_k} \right) \frac{\partial \theta_k^\varepsilon}{\partial x_j}, \quad x \in \Omega,
\]
\[
\overline{v}_{k,1}^\varepsilon = 0, \quad x \in \partial \Omega,
\]
and \( \overline{v}_{k,2}^\varepsilon \) satisfies
\[
L_\varepsilon \overline{v}_{k,2}^\varepsilon = 0, \quad x \in \Omega,
\]
\[
\overline{v}_{k,2}^\varepsilon = \frac{\partial \theta_k^\varepsilon}{\partial x_k}, \quad x \in \partial \Omega.
\]

In view of Lemma 2, we have
\[
\|\overline{v}_{k,1}^\varepsilon\|_{H^2(\Omega)} \leq Ce^{-1} \|\theta_k^\varepsilon\|_{H^2(\Omega)}
\]
\[
\leq Ce^{-1} \|u^0\|_{W^{2,\infty}(\Omega)}.
\]

Following the same line of [5] (1992, Theorem 1.2, pages 124–128), we have
\[
\|\overline{v}_{k,2}^\varepsilon\|_{H^2(\Omega)} \leq Ce^{-1} \|\theta_k^\varepsilon\|_{H^2(\Omega)} \leq Ce^{-1/2} \|u^0\|_{W^{2,\infty}(\Omega)} + \|u^0\|_{H^4(\Omega)}.
\]

Assuming that \( \mathcal{T}_h \) is defined as (16), and let \( \theta_h^\varepsilon \) and \( \theta_\varepsilon^l \) be the linear finite element approximation and the linear interpolation of \( \theta_\varepsilon \) with respect to \( \mathcal{T}_h \), respectively. Then, we have the following.

**Lemma 4.** Assuming that (C1) holds, then there exists \( C \) such that
\[
\left\| \nabla \left( \theta_\varepsilon^l - \theta_h^\varepsilon \right) \right\|_{L^2(\Omega)} \leq Ce^{-1/2} \left( \|u^0\|_{W^{2,\infty}(\Omega)} + \|u^0\|_{H^2(\Omega)} \right).
\]
\[
\leq C 2^{\frac{k+1}{2}} \varepsilon^{1/2} h^2 \left( \| u^0 \|_{W^{1,\infty}(\Omega)} + \| u^0 \|_{H^1(\Omega)} \right) \\
\times \| \nabla (\theta^\varepsilon - \theta^h) \|_{L^2(\Omega)} \\
\leq Ch^2 \left( \| u^0 \|_{W^{1,\infty}(\Omega)} + \| u^0 \|_{H^1(\Omega)} \right) \\
\times \| \nabla (\theta^\varepsilon - \theta^h) \|_{L^2(\Omega-\Omega_j)},
\]
(40)
which indicates that
\[
\| \nabla (\theta^\varepsilon - \theta^h) \|_{L^2(\Omega)} \leq Ch^2 \left( \| u^0 \|_{W^{1,\infty}(\Omega)} + \| u^0 \|_{H^1(\Omega)} \right).
\]
Using the result from (44), we have
\[
\| \nabla (\theta^\varepsilon - \theta^h) \|_{\sqrt{\text{dist}(\cdot, \partial \Omega) + \varepsilon}} \leq \left( \frac{\varepsilon}{C} \right) \| u^0 \|_{W^{1,\infty}(\Omega)} + \| u^0 \|_{H^1(\Omega)}.
\]
(41)

Then Lemma 4 can be easily derived.

Furthermore, we can obtain the following lemma.

**Lemma 5.** Assumption that (C1) holds, then there exists C such that

\[
\| \nabla (\theta^\varepsilon - \theta^h) \|_{\sqrt{\text{dist}(\cdot, \partial \Omega) + \varepsilon}} \leq C |\ln \varepsilon| h^2 \left( \| u^0 \|_{W^{1,\infty}(\Omega)} + \| u^0 \|_{H^1(\Omega)} \right).
\]
(42)

**Proof.** Assume that \( \Omega_j \) is defined as Lemma 4. Considering \( \theta^h = \theta^h_{x_0} (\forall x \in \Omega_j) \), and we divide \( \theta^\varepsilon - \theta^h \) into

\[
\theta^\varepsilon - \theta^h = \left( \theta^\varepsilon - \theta^h_{x_0} \right) + \sum_{j=0}^{j-1} \left( \theta^h_{x_0} - \theta^h_{x_{j+1}} \right).
\]
(43)

Then,

\[
\| \theta^\varepsilon - \theta^h \|_{H^1(\Omega-\Omega_j)} \leq \| \theta^\varepsilon - \theta^h_{x_0} \|_{H^1(\Omega-\Omega_j)} \\
+ \sum_{j=0}^{j-1} \| \theta^h_{x_0} - \theta^h_{x_{j+1}} \|_{H^1(\Omega-\Omega_j)} \\
\leq C r_j^{1/2} h^2 \left( \| u^0 \|_{W^{1,\infty}(\Omega)} + \| u^0 \|_{H^1(\Omega)} \right) \\
+ \sum_{j=0}^{j-1} \| \theta^h_{x_0} - \theta^h_{x_{j+1}} \|_{H^1(\Omega-\Omega_j)} \\
\leq C r_j^{1/2} h^2 \left( \| u^0 \|_{W^{1,\infty}(\Omega)} + \| u^0 \|_{H^1(\Omega)} \right) \\
+ C r_j h^2 \left( \| u^0 \|_{W^{1,\infty}(\Omega)} + \| u^0 \|_{H^1(\Omega)} \right) \\
\leq C r_j^{1/2} h^2 \left( \| u^0 \|_{W^{1,\infty}(\Omega)} + \| u^0 \|_{H^1(\Omega)} \right).
\]
(44)

Based on the previous lemmas, the estimate of \( \| \nabla (\theta^\varepsilon - \theta^h_{\Omega_j,h}) \|_{\sqrt{\text{dist}(\cdot, \partial \Omega) + \varepsilon}} \) can be given as follows.

**Lemma 6.** Assumption that (C1) holds, then there exists C such that

\[
\| \nabla (\theta^\varepsilon - \theta^h_{\Omega_j,h}) \|_{\sqrt{\text{dist}(\cdot, \partial \Omega) + \varepsilon}} \leq C \left( h_0 + h_1 + h^2 \right) |\ln \varepsilon| \\
\times \left( \| u^0 \|_{W^{1,\infty}(\Omega)} + \| u^0 \|_{H^1(\Omega)} \right).
\]
(46)

**Proof.** Assuming that \( \theta^h_{\Omega_j,h} \) satisfies

\[
L_x \theta^h_{\Omega_j,h} (x) = 0, \quad x \in \Omega, \\
\theta^h_{\Omega_j,h} (x) = -\varepsilon N^h_x \left( \frac{x}{\varepsilon} \right) u^h_{\Omega_j,h} (x), \quad x \in \partial \Omega,
\]
(47)
and \( \theta^h_{\Omega_j,h} \) is the linear interpolation of \( \theta^h_{\Omega_j,h} \) on \( \mathcal{T}_h \).

Then, we divide \( \theta^\varepsilon - \theta^h_{\Omega_j,h} \) into

\[
\theta^\varepsilon - \theta^h_{\Omega_j,h} = \left( \theta^\varepsilon - \theta^h_{\Omega_j,h} \right) + \left( \theta^h_{\Omega_j,h} - \theta^h_{\Omega_j,h} \right).
\]
(48)

Firstly, considering the first item of the right-hand side of (48) and assuming that \( \theta^h_{\Omega_j,h} \) satisfies the problem

\[
L_x \theta^h_{\Omega_j,h} (x) = 0, \quad x \in \Omega, \\
\theta^h_{\Omega_j,h} (x) = -\varepsilon N^h_x \left( \frac{x}{\varepsilon} \right) u^h_{\Omega_j,h} (x), \quad x \in \partial \Omega,
\]
(49)
we have
\[
\| \nabla (\theta_{\epsilon} - \theta_{\epsilon}^h) \|_{L^2(\Omega)} \\
\leq C \| \nabla (\theta_{\epsilon} - \theta_{\epsilon}^h) \|_{L^2(\Omega)} \\
\leq C \left( \| \nabla (\theta_{\epsilon} - \theta_{\epsilon}^h) \|_{L^2(\Omega)} + \| \nabla (\theta_{\epsilon}^h - \theta_{\epsilon}^h) \|_{L^2(\Omega)} \right).
\]
(50)

Using the same method of Lemma 2, we have
\[
\| \theta_{\epsilon} - \theta_{\epsilon}^h \|_{L^\infty(\Omega)} \leq C ( h_0 + h_1 ),
\]
(51)
\[
\| \nabla (\theta_{\epsilon} - \theta_{\epsilon}^h) \|_{L^2(\Omega)} \leq C \epsilon^{1/4} ( h_0 + h_1 ).
\]
(52)

Then, we have
\[
\begin{align*}
\| \epsilon N_k^{h_0} \frac{x}{\epsilon} \cdot \nabla \theta_{\epsilon}^h - \theta_{\epsilon}^h \|_{W^{2,\infty}(\Omega)} \\
\leq C \epsilon ( h_0 + h_1 ) \| u_0^0 \|_{W^{2,\infty}(\Omega)} ,
\end{align*}
\]
(53)
\[
\| \nabla (\theta_{\epsilon}^h - \theta_{\epsilon}^h) \|_{L^2(\Omega)} \leq C \epsilon^{1/2} ( h_0 + h_1 ) \| u_0^0 \|_{W^{2,\infty}(\Omega)}.
\]
(54)

Combining (52) with (54), we have
\[
\| \nabla (\theta_{\epsilon} - \theta_{\epsilon}^h) \|_{L^2(\Omega)} \leq C \epsilon^{1/2} ( h_0 + h_1 ) \| u_0^0 \|_{W^{2,\infty}(\Omega)}.
\]
(55)

Next, considering the second item of the right-hand side of (48), \( \theta_{\epsilon}^h - \theta_{\epsilon}^h \) can be divided into
\[
\begin{align*}
\theta_{\epsilon}^h - \theta_{\epsilon}^h &= (\theta_{\epsilon}^h - \theta_{\epsilon}^h) \\
&+ (\theta_{\epsilon}^h - \theta_{\epsilon}^h).
\end{align*}
\]
(56)

Similarly, we have
\[
\begin{align*}
\| \nabla (\theta_{\epsilon}^h - \theta_{\epsilon}^h) \|_{L^2(\Omega)} &\leq C \epsilon^{1/2} ( h_0 + h_1 ) \| u_0^0 \|_{W^{2,\infty}(\Omega)} ,
\end{align*}
\]
(57)
\[
\| \nabla (\theta_{\epsilon}^h - \theta_{\epsilon}^h) \|_{L^2(\Omega)} \leq C \epsilon^{1/2} ( h_0 + h_1 ) \| u_0^0 \|_{W^{2,\infty}(\Omega)}.
\]
(58)

Combining Lemma 6 with (55)–(58), we have (46).

Next, using the extrapolation technique [10], we are in a position to estimate
\[
\theta_{\epsilon} (x) - R_h \frac{4\theta_{\epsilon}^h - 3\theta_{\epsilon}^h}{2} (x) \leq \left| \theta_{\epsilon} (x) - R_h \frac{4\theta_{\epsilon}^h - 3\theta_{\epsilon}^h}{2} (x) \right|.
\]
(59)

\[
\theta_{\epsilon} (x) - R_h \frac{4\theta_{\epsilon}^h - 3\theta_{\epsilon}^h}{2} (x) \leq C \epsilon \left| \ln \epsilon \right|^{1/2} \| u_0 \|_{W^{2,\infty}(\Omega)}.
\]
(60)

**Lemma 7.** Assuming that (C1) holds, then there exists \( C \) such that
\[
\| \nabla (\theta_{\epsilon} - R_h (4\theta_{\epsilon}^h - \theta_{\epsilon}^h) / 3) \|_{L^2(\Omega)} \leq C ( h_0 + h_1 + \epsilon ) \| u_0^0 \|_{W^{2,\infty}(\Omega)} + \| u_0^0 \|_{H^1(\Omega)}.
\]
(61)

**Proof.** Let \( \theta_{\epsilon}^h \) be defined as above. Assuming that \( \theta_{\epsilon}^{1/2} \) is the linear interpolation of \( \theta_{\epsilon} \) on \( \mathcal{T}_{h/2} \), we have
\[
\| \nabla (\theta_{\epsilon} - R_h (4\theta_{\epsilon}^h - \theta_{\epsilon}^h) / 3) \|_{L^2(\Omega)} \leq C \epsilon \left| \ln \epsilon \right|^{1/2} \| u_0 \|_{W^{2,\infty}(\Omega)} + \| u_0^0 \|_{H^1(\Omega)}.
\]
(62)

Then, (61) can be easily derived.

Next, we turn to estimate \( \| \nabla w^\epsilon / \sqrt{\text{dist}(\cdot, \partial \Omega) + \epsilon} \|_{L^2(\Omega)} \).

**Lemma 8.** Assuming that (C1) holds, then there exists \( C \) such that
\[
\| \nabla w^\epsilon / \sqrt{\text{dist}(\cdot, \partial \Omega) + \epsilon} \|_{L^2(\Omega)} \leq C \epsilon \left| \ln \epsilon \right|^{1/2} \| u_0 \|_{W^{2,\infty}(\Omega)}.
\]
(63)

**Proof.** Following the same line of [5] and \( w^\epsilon = u^\epsilon - u - \theta_\epsilon \), there exists \( C \) such that
\[
\| \nabla w^\epsilon / \sqrt{\text{dist}(\cdot, \partial \Omega) + \epsilon} \|_{L^2(\Omega)} \leq C \epsilon \left| \ln \epsilon \right|^{1/2} \| u_0 \|_{W^{2,\infty}(\Omega)}.
\]
(64)
Finally, noting the definitions of $\overline{u}, \overline{u}^{h_0, h_1}, \theta_\varepsilon, R_\varepsilon, \omega_\varepsilon$, and $\omega_\varepsilon$, and combining Lemma 3 with Lemmas 7-8 and Lemma 2.4 in [9], we have

$$
\| \nabla \left( u^* - \overline{u}^{h_0, h_1} \right) \|_{L^2(\Omega)} \leq \frac{V \left( \nabla \left( u - \overline{u}^{h_0} \right) \right)}{\text{dist} \left( , \partial \Omega \right) + \varepsilon} + \frac{V \left( \nabla \left( \theta_\varepsilon - \overline{\theta}_\varepsilon \right) \right)}{\text{dist} \left( , \partial \Omega \right) + \varepsilon} + \frac{\nabla \omega_\varepsilon}{\text{dist} \left( , \partial \Omega \right) + \varepsilon}
$$

$$
\leq C \left[ (h_1 + h_0 + \varepsilon + h^2) \right] |\ln \varepsilon| \times \left( \| u^0 \|_{W^{2, \infty}(\Omega)} + \| u^0 \|_{H^1(\Omega)} \right).
$$

Combining the above lemmas, we can conclude the following result.

**Theorem 9.** Assuming that (C1) holds, then there exists $C$ such that

$$
\| \nabla \left( u^* - \overline{u}^{h_0, h_1} \right) \|_{L^2(\Omega)} \leq C \left[ (h_1 + h_0 + \varepsilon + h^2) \right] |\ln \varepsilon| \times \left( \| u^0 \|_{W^{2, \infty}(\Omega)} + \| u^0 \|_{H^1(\Omega)} \right).
$$

### 4. Numerical Example

In this section, some numerical results will be shown. In order to show the numerical accuracy of the method presented in this paper, the exact solution of problem (1) should firstly be obtained. However, it is very difficult to find them out. Then, the exact solution will be replaced by the finite element solution in a fine mesh with the mesh size $1/256$.

It should not be confused that $u^*$ denotes the finite element solution of (1) in a fine mesh, and $\overline{u}^{h_0, h_1}$, obtained by the multiscale finite element scheme presented in the above section, is the multiscale finite element solution of problem (1). Some numerical results will be presented by solving the following model problem:

$$
a_{11} = 0.3 + 2x_1 (1 - x_1),
$$

$$
a_{12} = a_{21} = x_1 (1 - x_1) x_2 (1 - x_2),
$$

$$
a_{22} = 0.1 + 2x_2 (1 - x_2),
$$

### Acknowledgments

The authors would like to thank the referees for their valuable suggestions and corrections, which contribute significantly to the improvement of the paper. This research was financially supported by the National Basic Research Program of China (973 Program: 2011CB013800), and the National Natural Science Foundation of China, with Grant nos. 11126132, 50101311, 50908167, and 11101311. The financial support of research is gratefully acknowledged.

### References


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