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Research Article

The Hamiltonian Structure and Algebrogeometric Solution of a 1 + 1-Dimensional Coupled Equations

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A 1 + 1-dimensional coupled soliton equations are decomposed into two systems of ordinary differential equations. The Abel-Jacobi coordinates are introduced to straighten the flows, from which the algebrogeometric solutions of the coupled 1 + 1-dimensional equations are obtained in terms of the Riemann theta functions.

1. Introduction

The study of explicit solutions for soliton equations is very important in modern mathematics, physics, and other sciences. There are several systematic approaches to obtain explicit solutions of the soliton equations, such as the inverse scattering transformation, the algebrogeometric method, and the Hirota bilinear method (see, e.g., [1–4] and references therein). Algebrogeometric (or quasiperiodic) solutions are very important explicit solutions for soliton equations, recently, based on the nonlinearization technique of Lax pairs and direct method proposed by Cao [5]. This new scheme is further shown to be a very powerful tool, through which algebrogeometric solutions of $(1+1)$-dimensional and $(2+1)$-dimensional continuous and discrete soliton equations can be obtained; see Cao et al. [6, 7], Geng et al. [8, 9], and Dai and Fan [10].

In this paper, we will construct the Hamiltonian structure and search for the algebrogeometric solution of the following coupled 1 + 1-dimensional soliton equations:

$$\begin{aligned} q_t &= r_{xx} - 3q^2 r_x + q_x r^2 + 2qrr_x, \\ r_t &= q_{xx} + 3r^2 r_x - r_x q^2 - 2qrq_x. \end{aligned} \tag{1}$$

Our purpose is to construct the Hamiltonian structure and give the algebrogeometric solutions of the coupled 1 + 1-dimensional soliton equations based on its obtained Lax pairs. The paper is organized as follows. In Section 2, we use Lenard operator pairs to derive another form of the coupled 1 + 1-dimensional soliton equations. In Section 3, based on the trace identity [11, 12], we construct the Hamiltonian structure of the coupled 1 + 1-dimensional soliton equations. In Section 4, based on the Lax pairs of the coupled 1 + 1-dimensional soliton equations, variable separation technique is used to translate the solution of the coupled 1 + 1-dimensional soliton equations to solve ordinary differential equations. In Section 5, a hyperelliptic Riemann surface of genus N and Abel-Jacobi coordinates are defined to straighten the associated flows. Jacobi's inverse problem is discussed, from which the algebrogeometric solutions of the coupled 1 + 1-dimensional soliton equations are constructed in terms of the Riemann theta functions.

2. The Hierarchy and Lax Pairs of the Coupled 1 + 1-Dimensional Soliton Equations

In this section, we introduce the Lenard gradient sequence $\{S_j\}_{j=0,1,2\ldots}$ to derive the hierarchy and its stationary hierarchy associated with (1) by the recursion relation:

$$\begin{gathered} KS_{j-1} = JS_j, \quad j = 1, 2, 3, \ldots. \quad S_j\big|_{(u,v)=0} = 0, \\ S_0 = (q - r, q + r, 1)^T, \end{gathered} \tag{2}$$

where $S_j = (S_j^{(1)}, S_j^{(2)}, S_j^{(3)})$ and operators ($\partial = \partial/\partial x$)

$$K = \begin{pmatrix} \partial & 0 & 0 \\ 0 & \partial & 0 \\ -(q+r) & q-r & \partial \end{pmatrix},$$

$$J = \begin{pmatrix} -2 & 0 & 2(q-r) \\ 0 & 2 & -2(q+r) \\ -(q+r) & q-r & \partial \end{pmatrix}. \tag{3}$$

A direct calculation gives from the recursion relation (2) that

$$S_1 = \begin{pmatrix} -\frac{1}{2}(q+r)(q-r)^2 - \frac{1}{2}(q_x - r_x) \\ -\frac{1}{2}(q+r)^2(q-r) + \frac{1}{2}(q_x + r_x) \\ -\frac{1}{2}(q+r)(q-r) \end{pmatrix},$$

$$S_2 = \begin{pmatrix} \frac{3}{8}(q+r)^2(q-r)^3 + \frac{1}{2}r(q-r)(q_x - r_x) + \frac{1}{2}(q+r)(q-r)(q_x - r_x) \\ +\frac{1}{4}(q_{xx} - r_{xx}) + \frac{1}{4}(q_x + r_x)(q-r)^2 - \frac{1}{2}(q-r)\partial^{-1}(q_x - r_x)r_x \\ \frac{3}{8}(q+r)^3(q-r)^2 + \frac{1}{2}r(q+r)(q_x - r_x) - \frac{1}{2}(q+r)(q-r)(q_x + r_x) \\ -\frac{1}{4}(q_{xx} - r_{xx}) - \frac{1}{4}(q_x - r_x)(q+r)^2 - \frac{1}{2}(q+r)\partial^{-1}(q_x - r_x)r_x \\ \frac{3}{8}(q+r)^2(q-r)^2 + \frac{1}{2}r(q_x - r_x) - \frac{1}{2}\partial^{-1}(q_x - r_x)r_x \end{pmatrix}. \tag{4}$$

Consider the spectral problem

$$\psi_x = U\psi, \qquad U = \begin{pmatrix} \lambda & q+r \\ \lambda(q-r) & -\lambda \end{pmatrix}, \tag{5}$$

and the auxiliary problem

$$\psi_{t_m} = V^{(m)}\psi, \qquad V^{(m)} = \begin{pmatrix} V_{11}^{(m)} & V_{12}^{(m)} \\ V_{21}^{(m)} & -V_{11}^{(m)} \end{pmatrix}, \tag{6}$$

where

$$V_{11}^{(m)} = \sum_{j=0}^{m} S_j^{(3)} \lambda^{m+1-j},$$

$$V_{12}^{(m)} = \sum_{j=0}^{m} S_j^{(2)} \lambda^{m-j}, \tag{7}$$

$$V_{21}^{(m)} = \sum_{j=0}^{m} S_j^{(1)} \lambda^{m+1-j}.$$

Then the compatibility condition of (5) and (6) is $U_{t_m} - V_x^{(m)} + [U, V^{(m)}] = 0$, which is equivalent to the hierarchy of nonlinear evolution equations

$$q_{t_m} = \frac{1}{2}\left(S_{mx}^{(2)} + S_{mx}^{(1)}\right),$$

$$r_{t_m} = \frac{1}{2}\left(S_{mx}^{(2)} - S_{mx}^{(1)}\right). \tag{8}$$

In brief,

$$\left(q_{t_m}, r_{t_m}\right)^T = X_m, \quad m \geq 0, \tag{9}$$

$$X_m = \begin{pmatrix} \partial & \partial \\ \partial & -\partial \end{pmatrix} \begin{pmatrix} S_m^{(2)} \\ S_m^{(1)} \end{pmatrix}. \tag{10}$$

The first two nontrivial equations are

$$q_{t_0} = q_x, \qquad r_{t_0} = r_x,$$

$$q_{t_1} = r_{xx} - 3q^2 r_x + q_x r^2 + 2qrr_x, \tag{11}$$

$$r_{t_1} = q_{xx} + 3r^2 r_x - r_x q^2 - 2qrq_x.$$

The second system is our coupled 1 + 1-dimensional soliton equations (1).

Let $\psi = (\psi_1, \psi_2)^T$ and $\phi = (\phi_1, \phi_2)^T$ be two basic solutions of the spectral equations (5) and (6). We define a matrix W by

$$W = \frac{1}{2}\left(\phi\psi^T + \psi\phi^T\right)\sigma = \begin{pmatrix} f & g \\ h & -f \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{12}$$

in which f, g, and h are three functions. It is easy to calculate by (5) and (6) that

$$W_x = [U, W], \qquad W_{t_m} = \left[V^{(m)}, W\right], \tag{13}$$

which implies that $\partial_x \det W = 0, \partial_{t_m} \det W = 0$. Equation (13) can be written as

$$\begin{aligned} f_x &= -\lambda (q-r) g + (q+r) h, \\ g_x &= -2(q+r) f + 2\lambda g, \\ h_x &= 2\lambda (q-r) f - 2\lambda h, \end{aligned} \tag{14}$$

$$\begin{aligned} f_{t_m} &= h V_{12}^{(m)} - g V_{21}^{(m)}, \\ g_{t_m} &= 2g V_{11}^{(m)} - 2f V_{12}^{(m)}, \\ h_{t_m} &= 2f V_{21}^{(m)} - 2h V_{11}^{(m)}. \end{aligned} \tag{15}$$

We suppose that the functions f, g, and h are finite-order polynomials in λ:

$$f = \sum_{j=0}^{N} f_j \lambda^{N-j+1}, \qquad g = \sum_{j=0}^{N} g_j \lambda^{N-j}, \qquad h = \sum_{j=0}^{N} h_j \lambda^{N-j}. \tag{16}$$

Substituting (16) into (14) yields

$$\begin{aligned} &KG_{j-1} = JG_j \left(j = 1, 2, \ldots, N\right), \qquad JG_0 = 0, \\ &KG_N = 0, \qquad G_j = \left(h_j, g_j, f_j\right)^T. \end{aligned} \tag{17}$$

It is easy to see that (17) implies

$$-(q+r) h_j + (q-r) g_j + f_{jx} = 0 \tag{18}$$

and the equation $JG_0 = 0$ has the general solution

$$G_0 = \alpha_0 S_0, \tag{19}$$

where α_0 is constant of integration. Therefore, if we take (19) as a starting point, then G_j can be determined recursively by relation (17). In fact, noticing $\ker J = \{cS_0 | \forall c \in \mathbb{R}\}$ and acting with the operator $(J^{-1}K)^k$ upon (19), we obtain from (2) and (17) that

$$G_k = \sum_{j=0}^{k} \alpha_j S_{k-j}, \quad k = 0, 1, \ldots, N, \tag{20}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_k$ are integral constants. Substituting (20) into (17) yields a certain stationary evolution equation:

$$\alpha_0 \widehat{X}_N + \alpha_1 \widehat{X}_{N-1} + \cdots + \alpha_N \widehat{X}_0 = 0, \tag{21}$$

where

$$\widehat{X}_j = \begin{pmatrix} \partial & 0 \\ 0 & \partial \end{pmatrix} \begin{pmatrix} S_j^{(1)} \\ S_j^{(2)} \end{pmatrix}. \tag{22}$$

This means that expression (16) is existent.

3. Hamiltonian Structure

Let

$$V = V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & -V_{11} \end{pmatrix}, \tag{23}$$

where

$$\begin{aligned} V_{11} &= \sum_{j\geq 0} S_j^{(3)} \lambda^{-j+1}, \qquad V_{12} = \sum_{j\geq 0} S_j^{(2)} \lambda^{-j}, \\ V_{21} &= \sum_{j\geq 0} S_j^{(1)} \lambda^{-j+1}. \end{aligned} \tag{24}$$

It is easy to calculate

$$\begin{aligned} \operatorname{tr}\left(V \frac{\partial U}{\partial \lambda}\right) &= 2V_{11} + (q-r) V_{12} \\ &= \sum_{j\geq 1} \left(2S_j^{(3)} + (q-r) S_{j-1}^{(2)}\right) \lambda^{-j+1} + 2S_0^{(3)} \lambda, \\ \operatorname{tr}\left(V \frac{\partial U}{\partial q}\right) &= \lambda V_{12} + V_{21} = \sum_{j\geq 0} \left(S_j^{(2)} + S_j^{(1)}\right) \lambda^{-j+1}, \\ \operatorname{tr}\left(V \frac{\partial U}{\partial r}\right) &= -\lambda V_{12} + V_{21} = \sum_{j\geq 0} \left(-S_j^{(2)} + S_j^{(1)}\right) \lambda^{-j+1}. \end{aligned} \tag{25}$$

According to the trace identity [11, 12], we have

$$\begin{pmatrix} \dfrac{\delta}{\delta q} \\ \dfrac{\delta}{\delta r} \end{pmatrix} \left(2V_{11} + (q-r) V_{12}\right) = \left(\lambda^{-s} \frac{\partial}{\partial \lambda} \lambda^{s}\right) \begin{pmatrix} \lambda V_{12} + V_{21} \\ -\lambda V_{12} + V_{21} \end{pmatrix}. \tag{26}$$

Comparing the coefficients of λ^{-j+1}, we obtain

$$\begin{pmatrix} \dfrac{\delta}{\delta q} \\ \dfrac{\delta}{\delta r} \end{pmatrix} \left(2S_j^{(3)} + (q-r) S_{j-1}^{(2)}\right) = (-j+2+s) \begin{pmatrix} S_{j-1}^{(2)} + S_{j-1}^{(1)} \\ -S_{j-1}^{(2)} + S_{j-1}^{(1)} \end{pmatrix}, \tag{27}$$

we set $j = 1$ and then get $s = -1$ and

$$\begin{pmatrix} \dfrac{\delta}{\delta q} \\ \dfrac{\delta}{\delta r} \end{pmatrix} \mathscr{H}_j = \begin{pmatrix} S_{j-1}^{(2)} + S_{j-1}^{(1)} \\ -S_{j-1}^{(2)} + S_{j-1}^{(1)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} S_{j-1}^{(2)} \\ S_{j-1}^{(1)} \end{pmatrix}, \tag{28}$$

where $\mathscr{H}_j = (2S_j^{(3)} + (q-r) S_{j-1}^{(2)})/(-j+1)$.

Thus the soliton equation (9) has a Hamiltonian structure:

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_m} = \frac{1}{2} \begin{pmatrix} \partial & \partial \\ \partial & -\partial \end{pmatrix} \begin{pmatrix} S_m^{(2)} \\ S_m^{(1)} \end{pmatrix} = \tilde{J} \begin{pmatrix} \dfrac{\delta}{\delta q} \\ \dfrac{\delta}{\delta r} \end{pmatrix} \mathscr{H}_{m+1}, \quad m \geq 0, \tag{29}$$

where

$$\tilde{J} = \frac{1}{2}\begin{pmatrix} \partial & 0 \\ 0 & -\partial \end{pmatrix}. \tag{30}$$

In speciality, the Hamiltonian structure of (1) is ($m = 1$):

$$\mathscr{H}_2 = \frac{1}{8}(q+r)^2(q-r)^2 - \frac{qq_x - 2rr_x + qr_x}{2} + \frac{\partial^{-1}(q_x - r_x)r_x}{2}. \tag{31}$$

4. Ordinary Differential Equations

In this section, (1) will be decomposed into two systems of solvable ordinary differential equations. Without loss of generality, let $\alpha_0 = 1$. From (2), (17), and (20), we have

$$\begin{gathered} f_0 = 1, \qquad g_0 = q + r, \qquad h_0 = q - r, \\ f_1 = -\frac{1}{2}(q+r)(q-r) + \alpha_1, \\ g_1 = -\frac{1}{2}(q+r)^2(q-r) + \frac{1}{2}(q_x + r_x) + \alpha_1(q+r), \\ h_1 = -\frac{1}{2}(q+r)(q-r)^2 - \frac{1}{2}(q_x - r_x) + \alpha_1(q-r). \end{gathered} \tag{32}$$

By using (16), we can write g and h as the following finite products:

$$\begin{gathered} g = -(q+r)\prod_{j=1}^{N}(\lambda - u_j), \\ h = (q-r)\prod_{j=1}^{N}(\lambda - v_j). \end{gathered} \tag{33}$$

Equation (33) implies by comparing the coefficients of λ^{N-1} that

$$g_1 = -(q+r)\sum_{j=1}^{N} u_j, \qquad h_1 = -(q+r)\sum_{j=1}^{N} v_j. \tag{34}$$

Thus from (32) and (34), we obtain

$$\begin{gathered} -\frac{1}{2}(q+r)^2(q-r) + \frac{1}{2}(q_x + r_x) + \alpha_1(q+r) \\ = -(q+r)\sum_{j=1}^{N} u_j, \\ -\frac{1}{2}(q+r)(q-r)^2 - \frac{1}{2}(q_x - r_x) + \alpha_1(q-r) \\ = -(q-r)\sum_{j=1}^{N} v_j. \end{gathered} \tag{35}$$

Let us consider the function $\det W$ which is a $(2N+2)$th-order polynomial in λ with constant coefficients of the x-flow and t_m-flow:

$$-\det W = f^2 + gh = \prod_{j=1}^{2N+2}(\lambda - \lambda_j) = R(\lambda). \tag{36}$$

Substituting (16) into (36) and comparing the coefficients of λ^{2N+1} yield

$$2f_0 f_1 + g_0 h_0 = -\sum_{j=1}^{2N+2}\lambda_j, \tag{37}$$

which together with (32) gives

$$\alpha_1 = -\frac{1}{2}\sum_{j=1}^{2N+2}\lambda_j. \tag{38}$$

From (36) we see that

$$f|_{\lambda=u_k} = \sqrt{R(u_k)}, \qquad f|_{\lambda=v_k} = \sqrt{R(v_k)}. \tag{39}$$

Again by using (14) and (33), we obtain

$$\begin{gathered} g_x|_{\lambda=u_k} = -(q+r)u_{kx}\prod_{j=1, j\neq k}^{N}(u_k - u_j) = -2(q+r)f|_{\lambda=u_k}, \\ h_x|_{\lambda=v_k} = -(q-r)v_{kx}\prod_{j=1, j\neq k}^{N}(v_k - v_j) = 2v_k(q-r)f|_{\lambda=v_k}, \end{gathered} \tag{40}$$

which together with (39) gives

$$\begin{gathered} u_{kx} = \frac{2\sqrt{R(u_k)}}{\prod_{j=1, j\neq k}^{N}(u_k - u_j)}, \qquad 1 \le k \le N, \\ v_{k,x} = \frac{-2\sqrt{R(v_k)}}{\prod_{j=1, j\neq k}^{N}(v_k - v_j)}, \qquad 1 \le k \le N. \end{gathered} \tag{41}$$

In a way similar to the above expression, by using (6) ($m = 1, t_1 = t$), (15), and (39), we arrive at the evolution of $\{u_k\}$ and $\{v_k\}$ along the t_m-flow:

$$u_{k,t} = \frac{2f|_{\lambda=u_k} V_{12}^{(1)}|_{\lambda=u_k}}{(q+r)\prod_{j=1,j\neq k}^{N}(u_k-u_j)} = \frac{2\sqrt{R(u_k)}\left[u_k - (1/2)(q+r)(q-r) + (1/2)\partial\ln(q-r)\right]}{\prod_{j=1,j\neq k}^{N}(u_k-u_j)}, \tag{42}$$

$$\begin{aligned} v_{k,t} &= -\frac{2f|_{\lambda=v_k} V_{21}^{(1)}|_{\lambda=v_k}}{(q-r)\prod_{j=1,j\neq k}^{N}(v_k-v_j)} \\ &= \left(-2\sqrt{R(v_k)}\left[v_k^2 - \frac{1}{2}(q+r)(q-r)v_k + \frac{1}{2}\partial\ln(q-r)v_k\right]\right)\left(\prod_{j=1,j\neq k}^{N}(v_k-v_j)\right)^{-1}. \end{aligned} \tag{43}$$

Therefore, if the $(2N+2)$ distinct parameters $\lambda_1, \lambda_2, \ldots, \lambda_{2N+2}$ are given and let $u_k(x,t)$ and $v_k(x,t)$ be distinct solutions of ordinary differential equations (41), (42), and (43), then (q,r) determined by (35) is a solution of the coupled 1 + 1-dimensional equations (1).

5. Algebrogeometric Solutions

In this section, we will give the algebrogeometric solutions of the coupled 1 + 1-dimensional equation (1). To this end, we first introduce the Riemann surface Γ of the hyperelliptic curve

$$\Gamma : \zeta^2 = R(\lambda), \qquad R(\lambda) = \prod_{j=1}^{2N+2}(\lambda - \lambda_j), \tag{44}$$

with genus N on Γ. On Γ there are two infinite points ∞_1 and ∞_2, which are not branch points of Γ. We equip Γ with a canonical basis of cycles: $a_1, a_2, \ldots, a_N; b_1, b_2, \ldots, b_N$ which are independent and have intersection numbers as follows:

$$\begin{aligned} a_i \circ a_j &= 0, \qquad b_i \circ b_j = 0, \\ a_i \circ b_j &= \delta_{ij}, \quad i, j = 1, 2, \ldots, N. \end{aligned} \tag{45}$$

We will choose the following set as our basis:

$$\widetilde{\omega}_l = \frac{\lambda^{l-1}d\lambda}{\sqrt{R(\lambda)}}, \quad l = 1, 2, \ldots, N, \tag{46}$$

which are linearly independent of each other on Γ, and let

$$A_{ij} = \int_{a_j}\widetilde{\omega}_i, \qquad B_{ij} = \int_{b_j}\widetilde{\omega}_i. \tag{47}$$

It is possible to show that the matrices $A = (A_{ij})$ and $B = (B_{ij})$ are $N \times N$ invertible matrices [13, 14]. Now we define the matrices C and τ by $C = (C_{ij}) = A^{-1}$, $\tau = (\tau_{ij}) = A^{-1}B$. Then the matrix τ can be shown to be symmetric ($\tau_{ij} = \tau_{ji}$) and it has a positive-definite imaginary part ($\operatorname{Im}\tau > 0$). If we normalize $\widetilde{\omega}_j$ into the new basis ω_j

$$\omega_j = \sum_{l=1}^{N} C_{jl}\widetilde{\omega}_l, \quad l = 1, 2, \ldots, N, \tag{48}$$

then we have

$$\begin{aligned} \int_{a_j}\omega_j &= \sum_{l=1}^{N} C_{jl}\int_{a_j}\widetilde{\omega}_l = \sum_{l=1}^{N} C_{jl}A_{li} = \delta_{ji}, \\ \int_{b_j}\omega_i &= \sum_{l=1}^{N} C_{jl}\int_{b_j}\overline{\omega}_l = \sum_{l=1}^{N} C_{jl}B_{li} = \tau_{ji}. \end{aligned} \tag{49}$$

Now we introduce the Abel-Jacobi coordinates as follows:

$$\rho_j^{(1)}(x,t) = \sum_{k=1}^{N}\int_{p_0}^{p(u_k(x,t))}\omega_j = \sum_{k=1}^{N}\sum_{l=1}^{N}\int_{\lambda(p_0)}^{u_k} C_{jl}\frac{\lambda^{l-1}d\lambda}{\sqrt{R(\lambda)}}, \tag{50}$$

$$\rho_j^{(2)}(x,t) = \sum_{k=1}^{N}\int_{p_0}^{p(v_k(x,t))}\omega_j = \sum_{k=1}^{N}\sum_{l=1}^{N}\int_{\lambda(p_0)}^{v_k(x,t)} C_{jl}\frac{\lambda^{l-1}d\lambda}{\sqrt{R(\lambda)}}, \tag{51}$$

where $p(u_k(x,t)) = (u_k, \sqrt{R(u_k)})$, $p(v_k(x,t)) = (v_k, \sqrt{R(v_k)})$, and $\lambda(p_0)$ is the local coordinate of p_0. From (42) and (50), we get

$$\partial_x\rho_j^{(1)} = \sum_{k=1}^{N}\sum_{l=1}^{N} C_{jl}\frac{u_k^{l-1}u_{kx}}{\sqrt{R(u_k)}} = \sum_{k=1}^{N}\sum_{l=1}^{N}\frac{C_{jl}u_k^{l-1}}{\prod_{j=1,j\neq k}^{N}(u_k-u_j)}, \tag{52}$$

which implies

$$\partial_x\rho_j^{(1)} = 2C_{jN} = \Omega_j^{(1)}, \quad j = 1, 2, \ldots, N. \tag{53}$$

With the help of the following equality

$$\sum_{k=1}^{N}\frac{u_k^{l-1}}{\prod_{j=1,j\neq k}^{N}(u_k-u_j)} = \delta_{lN}, \quad l = 1, 2, \ldots, N, \tag{54}$$

in a similar way, we obtain from (50), (51), (41), (42), and (43) that

$$\begin{aligned} \partial_t\rho_j^{(1)} &= 2\sum_{l=1}^{N} C_{jl}\left(\sum_{i=1}^{N} u_i\right) - C_{jN}(q+r)(q-r) \\ &\quad + C_{jN}\partial\ln(q-r) = \Omega_j^{(2)}, \\ \partial_x\rho_j^{(2)} &= -\Omega_j^{(1)}, \quad j = 1, 2, \ldots, N, \\ \partial_t\rho_j^{(2)} &= -\Omega_j^{(2)}, \quad j = 1, 2, \ldots, N. \end{aligned} \tag{55}$$

On the basis of these results, we obtain the following:

$$\rho_j^{(1)}(x,t) = \Omega_j^{(1)}x + \Omega_j^{(2)}t + \gamma_j^{(1)},$$
$$\rho_j^{(2)}(x,t) = -\Omega_j^{(1)}x - \Omega_j^{(2)}t + \gamma_j^{(2)}, \tag{56}$$

where $\gamma_j^{(i)}(i = 1, 2)$ are constants and

$$\gamma_j^{(1)} = \sum_{k=1}^{N}\int_{p_0}^{p(\tilde{u}_k(0,0))}\omega_j, \qquad \gamma_j^{(2)} = \sum_{k=1}^{N}\int_{p_0}^{p(\tilde{v}_k(0,0))}\omega_j,$$
$$\rho^{(1)} = \left(\rho_1^{(1)}, \rho_2^{(1)}, \ldots, \rho_N^{(1)}\right)^T,$$
$$\rho^{(2)} = \left(\rho_1^{(2)}, \rho_2^{(2)}, \ldots, \rho_N^{(2)}\right)^T, \tag{57}$$
$$\Omega^{(m)} = \left(\Omega_1^{(m)}, \Omega_2^{(m)}, \ldots, \Omega_N^{(m)}\right)^T,$$
$$\gamma^{(m)} = \left(\gamma_1^{(m)}, \gamma_2^{(m)}, \ldots, \gamma_N^{(m)}\right)^T, \quad m = 1, 2.$$

Now we introduce the Abel map $\mathscr{A}(p)$:

$$\mathscr{A}(p) = \int_{p_0}^{p}\omega, \quad \omega = (\omega_1, \omega_2, \ldots, \omega_N)^T,$$
$$\mathscr{A}\left(\sum_k p_k\right) = \sum n_k \mathscr{A}(p_k) \tag{58}$$

and Abel-Jacobi coordinates:

$$\rho^{(1)} = \mathscr{A}\left(\sum_{k=1}^{N} p(u_k)\right) = \sum_{k=1}^{N}\int_{p_0}^{p(u_k)}\omega,$$
$$\rho^{(2)} = \mathscr{A}\left(\sum_{k=1}^{N} p(v_k)\right) = \sum_{k=1}^{N}\int_{p_0}^{p(v_k)}\omega. \tag{59}$$

According to the Riemann theorem [13, 14], there exists a Riemann constant vector $M \in \mathbb{C}^N$ such that the function

$$F^{(m)}(\lambda) = \theta\left(\mathscr{A}(p(\lambda)) - \rho^{(m)} - M^{(m)}\right), \quad m = 1, 2, \tag{60}$$

has exactly N zeros at $u_1, u_2, \ldots, u_N$ for $m = 1$ or $v_1, v_2, \ldots, v_N$ for $m = 2$. To make the function single valued, the surface Γ is cut along all a_k, b_k to form a simple connected region, whose boundary is denoted by γ. By [13, 14], the integrals

$$I(\Gamma) = \frac{1}{2\pi i}\int_{\gamma}\lambda d\ln F^{(m)}, \quad m = 1, 2, \tag{61}$$

are constants independent of $\rho^{(1)}$ and $\rho^{(2)}$ with

$$I = I(\Gamma) = \sum_{j=1}^{N}\int_{a_j}\lambda\omega_j. \tag{62}$$

By the residue theorem, we have

$$\sum_{j=1}^{N}u_j = I - \sum_{s=1}^{2}\mathrm{Res}_{\lambda=\infty_s}\lambda d\ln F^{(1)}(\lambda),$$
$$\sum_{j=1}^{N}v_j = I - \sum_{s=1}^{2}\mathrm{Res}_{\lambda=\infty_s}\lambda d\ln F^{(2)}(\lambda). \tag{63}$$

Here we need only to compute the residues in (63). In a way similar to calculations in [10], we arrive at

$$\mathrm{Res}_{\lambda=\infty_s}\lambda d\ln F^{(m)}(\lambda) = (-1)^{s+m}\partial\ln\theta_s^{(m)},$$
$$m = 1, 2; \ s = 1, 2, \tag{64}$$

where

$$\theta_s^{(1)} = \theta\left(\Omega^{(1)}x + \Omega^{(2)}t + \xi_s\right),$$
$$\theta_s^{(2)} = \theta\left(-\Omega^{(1)}x - \Omega^{(2)}t + \eta_s\right), \tag{65}$$

and ξ_s and η_s are constants. Thus from (63) and (64), we arrive at

$$\sum_{j=1}^{N}u_j = I - \partial\ln\frac{\theta_1^{(1)}}{\theta_2^{(1)}}, \qquad \sum_{j=1}^{N}v_j = I - \partial\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}}. \tag{66}$$

Substituting (66) into (35), we get an algebrogeometric solution for the coupled 1 + 1-dimensional soliton equations (1):

$$\begin{aligned} q = {} & \frac{A(t)}{2}\exp\left(-\partial^{-1}\sum_{j=1}^{2N+2}\lambda_j + 2\partial^{-1}I - 2\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}}\right. \\ & \left.-\partial^{-1}\exp\left(-2\ln\frac{\theta_2^{(2)}\theta_2^{(1)}}{\theta_1^{(2)}\theta_1^{(1)}}\right)\right) \\ & + \frac{A(t)}{2}\exp\left(\partial^{-1}\sum_{j=1}^{2N+2}\lambda_j + 2\partial^{-1}I - 2\ln\frac{\theta_1^{(2)}}{\theta_2^{(1)}}\right. \\ & \left.+\partial^{-1}\exp\left(-2\ln\frac{\theta_2^{(2)}\theta_2^{(1)}}{\theta_1^{(2)}\theta_1^{(1)}}\right)\right), \end{aligned} \tag{67}$$

$$\begin{aligned} r = {} & \frac{A(t)}{2}\exp\left(\partial^{-1}\sum_{j=1}^{2N+2}\lambda_j + 2\partial^{-1}I - 2\ln\frac{\theta_1^{(2)}}{\theta_2^{(1)}}\right. \\ & \left.+\partial^{-1}\exp\left(-2\ln\frac{\theta_2^{(2)}\theta_2^{(1)}}{\theta_1^{(2)}\theta_1^{(1)}}\right)\right) \\ & - \frac{A(t)}{2}\exp\left(-\partial^{-1}\sum_{j=1}^{2N+2}\lambda_j + 2\partial^{-1}I - 2\ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}}\right. \\ & \left.-\partial^{-1}\exp\left(-2\ln\frac{\theta_2^{(2)}\theta_2^{(1)}}{\theta_1^{(2)}\theta_1^{(1)}}\right)\right), \end{aligned} \tag{68}$$

where $A(t)$ is arbitrary complex functions about variable t.

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