Research Article

Numerical Solutions of Fractional Integro-differential Equations of Bratu Type by Using CAS Wavelets

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A numerical method based on the CAS wavelets is presented for the fractional integro-differential equations of Bratu type. The CAS wavelets operational matrix of fractional order integration is derived. A truncated CAS wavelets series together with this operational matrix is utilized to reduce the fractional integro-differential equations to a system of algebraic equations. The solution of this system gives the approximation solution for the truncated limited $2^k(2M + 1)$. The convergence and error estimation of CAS wavelets are also given. Two examples are included to demonstrate the validity and applicability of the approach.

1. Introduction

A lot of scientific and engineering problems involving fractional phenomenon are already very large and still growing. One of the main advantages of the fractional phenomenon is that the fractional derivatives and fractional integrals provide an excellent approach to the different kinds of physical fields, such as dispersive transports in amorphous semiconductors, tracer transfer in underground water, and seepage in soil or rocks [1–4]. In recent years, more and more researchers are finding that a variety of dynamical problems exhibit fractional order behavior. This indicates that variable order calculus is an effective mathematical framework to describe the complex dynamical problems [5–7]. Many of the numerical methods using various fractional derivative operators and integral operators for solving fractional differential equations have been proposed. Podlubny [8] used the Laplace transform method to solve the fractional partial differential equations with constant coefficients. Odibat and Momani [9] applied generalized differential transform method to solve the numerical solution of linear partial differential equations of fractional order. Zhang [10] discussed a practical implicit method to solve a class of initial boundary value space-time fractional convection-diffusion equations with variable coefficients. Zhuang et al. [11] proposed explicit and implicit

Euler method for the variable order fractional advection-diffusion equation.

Bratu's problem is also discussed in all kinds of applications, such as chemical reaction theory, the fuel ignition model of the thermal combustion theory, and nanotechnology [12–15]. Both mathematicians and physicists have devoted a lot of effort to Bratu's problem. In [16], Syam and Hamdan presented the Laplace Adomian decomposition method for solving Bratu's problem. Wazwaz [17] proposed the Adomian decomposition method for solving Bratu's problem. Aksoy and Pakdemirli [18] had solved Bratu-type equation of new perturbation iteration solutions.

In this paper, we consider the following fractional integro-differential equations of Bratu type by means of CAS wavelets:

$$D^\alpha u(x) + \lambda \int_0^x k(x, t) e^{u(t)} dt + g(x) = 0, \quad (1)$$

$$n - 1 < \alpha \leq n, \quad 0 \leq x, \quad t \leq 1,$$

with initial condition

$$u^{(j)}(0) = b_j, \quad j = 0, 1, \dots, n - 1. \quad (2)$$

2. Definitions and Properties of Fractional Operator

Here we just recall the most typical definitions which are easy to use in physics. The Caputo fractional differential operator D^α of order $\alpha > 0$ is defined as [8]

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau. \quad (3)$$

The Riemann-Liouville fractional integration of order $\alpha > 0$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad n-1 < \alpha \leq n, \quad (4)$$

where n is a positive integer.

It has the following two basic properties for $n-1 < \alpha \leq n$:

$$\begin{aligned} D^\alpha I^\alpha f(t) &= f(t), \\ I^\alpha D^\alpha f(t) &= f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, \quad t > 0. \end{aligned} \quad (5)$$

3. CAS Wavelets and Some of Their Properties

3.1. CAS Wavelets. The CAS wavelets $\psi_{nm}(t) = \psi(k, n, m, t)$ have four arguments, $n = 1, 2, \dots, 2^k$, where k is any nonnegative integer, m is any integer, and t is the normalized time. The orthonormal CAS wavelets are defined on the interval $[0, 1]$ by [19]

$$\psi_{nm}(t) = \begin{cases} 2^{k/2} \text{CAS}_m(2^k t - n + 1), & \frac{n-1}{2^k} \leq t < \frac{n}{2^k}, \\ 0, & \text{otherwise,} \end{cases} \quad (6)$$

where $\text{CAS}_m(t) = \cos(2m\pi t) + \sin(2m\pi t)$.

3.2. Function Approximation. A function $u(t) \in L^2[0, 1]$ may be expanded as

$$u(t) = \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} c_{nm} \psi_{nm}(t), \quad (7)$$

where $c_{nm} = \langle u(t), \psi_{nm}(t) \rangle$, in which $\langle \rangle$ denotes the inner product. The series (7) is truncated as

$$u(t) \approx u_{k,M}(t) = \sum_{n=1}^{2^k} \sum_{m=-M}^M c_{nm} \psi_{nm} = C^T \Psi(t), \quad (8)$$

where C and $\Psi(t)$ are two vectors given by

$$\begin{aligned} C &= [c_{1(-M)}, c_{1(-M+1)}, \dots, c_{1(M)}, c_{2(-M)}, \dots, \\ &\quad c_{2(M)}, \dots, c_{2^k(-M)}, \dots, c_{2^k(M)}]^T \\ &= [c_1, c_2, \dots, c_{2^{k(2M+1)}}]^T, \\ \Psi(t) &= [\psi_{1(-M)}, \psi_{1(-M+1)}, \dots, \psi_{1(M)}, \psi_{2(-M)}, \dots, \\ &\quad \psi_{2(M)}, \psi_{2^k(-M)}, \dots, \psi_{2^k(M)}]^T \\ &= [\psi_1, \psi_2, \dots, \psi_{2^{k(2M+1)}}]^T. \end{aligned} \quad (9)$$

3.3. Convergence of the CAS Wavelet Bases. In this section, we indicate that the CAS wavelets expansion of a function $u(t)$, with bounded second derivative, converges uniformly to $u(t)$.

Lemma 1. *If the CAS wavelet expansion of a continuous function $u(t)$ converges uniformly, then the CAS wavelet expansion converges to the function $u(t)$.*

Proof. Let

$$v(t) = \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} c_{nm} \psi_{nm}(t), \quad (10)$$

where $c_{nm} = \langle u(t), \psi_{nm}(t) \rangle$. Multiplying both sides of (11) by $\psi_{pq}(t)$, p , and q are fixed and then integrating termwise, justified by uniformly convergence, on $[0, 1]$, we have

$$\begin{aligned} &\int_0^1 v(t) \psi_{pq}(t) dt \\ &= \int_0^1 \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} c_{nm} \psi_{nm}(t) \psi_{pq}(t) dt \\ &= \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} c_{nm} \int_0^1 \psi_{nm}(t) \psi_{pq}(t) dt = c_{pq}. \end{aligned} \quad (11)$$

Thus $\langle v(t), \psi_{nm}(t) \rangle = c_{nm}$ for $n, m \in \mathbb{Z}$. Consequently u, v have same Fourier expansions with the CAS wavelets basis and therefore $u(t) = v(t)$. $\square$

Theorem 2. *A function $u(t) \in L^2[0, 1]$, with bounded second derivative, say $|u''(t)| \leq \gamma$, can be expanded as an infinite sum of the CAS wavelets and the series converges uniformly to $u(t)$; that is $u(t) = \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} c_{nm} \psi_{nm}(t)$. Furthermore, we have*

$$|u_{k,M}(t) - u(t)| \leq \frac{\gamma}{\pi^2} \sum_{n=2^k}^{\infty} \sum_{m=M+1}^{\infty} \frac{1}{n^{5/2} m^2}, \quad t \in [0, 1]. \quad (12)$$

Proof. From (6), we obtain

$$\begin{aligned} c_{nm} &= \int_0^1 u(t) \psi_{nm}(t) dt \\ &= \int_{(n-1)/2^k}^{n/2^k} 2^{k/2} u(t) \text{CAS}_m(2^k t - n + 1) dt. \end{aligned} \quad (13)$$

Substituting $2^k t - n + 1 = x$ in (13) yields

$$\begin{aligned}
 c_{nm} &= \frac{1}{2^{k/2}} \int_0^1 u\left(\frac{x+n-1}{2^k}\right) \text{CAS}_m(x) dx \\
 &= \frac{1}{2^{k/2}} \int_0^1 u\left(\frac{x+n-1}{2^k}\right) d\left(\frac{\sin(2m\pi x) - \cos(2m\pi x)}{2m\pi}\right) \\
 &= \frac{1}{2^{k/2}} u\left(\frac{x+n-1}{2^k}\right) \frac{\sin(2m\pi x) - \cos(2m\pi x)}{2m\pi} \Big|_0^1 \\
 &\quad - \frac{1}{2^{3k/2} (2m\pi)} \int_0^1 u'\left(\frac{x+n-1}{2^k}\right) \\
 &\quad \times (\sin(2m\pi x) - \cos(2m\pi x)) dx \\
 &= \frac{1}{2^{3k/2} (2m\pi)} \int_0^1 u'\left(\frac{x+n-1}{2^k}\right) d\left(\frac{\text{CAS}_m(x)}{2m\pi}\right) \\
 &= \frac{1}{2^{3k/2} (2m\pi)^2} u'\left(\frac{x+n-1}{2^k}\right) \text{CAS}_m(x) \Big|_0^1 \\
 &\quad - \frac{1}{2^{5k/2} (2m\pi)^2} \int_0^1 u''\left(\frac{x+n-1}{2^k}\right) \text{CAS}_m(x) dx.
 \end{aligned} \tag{14}$$

Thus, we get

$$\begin{aligned}
 |c_{nm}|^2 &= \left| \frac{1}{2^{5k/2} (2m\pi)^2} \int_0^1 u''\left(\frac{x+n-1}{2^k}\right) \text{CAS}_m(x) dx \right|^2 \\
 &\leq \left(\frac{1}{2^{5k/2} (2m\pi)^2} \right)^2 \int_0^1 \left| u''\left(\frac{x+n-1}{2^k}\right) \right|^2 dx \\
 &\quad \times \int_0^1 |\text{CAS}_m(x)|^2 dx \\
 &\leq \left(\frac{\gamma}{2^{5k/2} (2m\pi)^2} \right)^2 \int_0^1 |\text{CAS}_m(x)|^2 dx,
 \end{aligned} \tag{15}$$

from orthonormality of CAS wavelets, we know that $\int_0^1 |\text{CAS}_m(x)|^2 dx = 1$, since $n \leq 2^k$, we have $|c_{nm}| \leq \gamma/(4\pi^2 n^{5/2} m^2)$. Hence the series $\sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} c_{nm}$ is absolutely convergent. On the other hand, we can obtain

$$\begin{aligned}
 &\left| \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} c_{nm} \psi_{nm}(t) \right| \\
 &\leq \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} |c_{nm}| |\psi_{nm}(t)| \leq 2 \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} |c_{nm}| < \infty.
 \end{aligned} \tag{16}$$

Using Lemma 1, the series $\sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} c_{nm} \psi_{nm}(t)$ converges to $u(t)$.

Moreover, we conclude that

$$\begin{aligned}
 |u_{k,M}(t) - u(t)| &\leq \left| 2 \sum_{n=2^k+1}^{\infty} \sum_{m=M+1}^{\infty} c_{nm} \psi_{nm}(t) \right| \\
 &\leq 4 \sum_{n=2^k+1}^{\infty} \sum_{m=M+1}^{\infty} |c_{nm}| \\
 &\leq \frac{\gamma}{\pi^2} \sum_{n=2^k}^{\infty} \sum_{m=M+1}^{\infty} \frac{1}{n^{5/2} m^2}.
 \end{aligned} \tag{17}$$

This completes the proof. $\square$

3.4. Operational Matrix of the Fractional Integration. In this part, we may simply introduce the operational matrix of fractional integration of CAS wavelets; more detailed introduction can be found in [19].

Take the points $t_i = i/2^k(2M+1)$, $i = 0, 1, 2, \dots, 2^k(2M+1) - 1$, then we define

$$\Phi = [\Psi(t_0) \ \Psi(t_1) \ \cdots \ \Psi(t_{2^k(2M+1)-1})]. \tag{18}$$

If J^α is fractional integration operator of CAS wavelets, we can get

$$(J^\alpha \Psi)(t) = P^\alpha \Psi(t), \tag{19}$$

where P^α is called operational matrix of fractional integration of CAS wavelets.

Apart from the CAS wavelets, we consider another basis set of block pulse functions. The set of these functions, over the interval $[0, 1]$, is defined as

$$b_l(t) = \begin{cases} 1, & \frac{i}{l} \leq t < \frac{i+1}{l}, \ i = 0, 1, 2, \dots, l-1 \\ 0, & \text{otherwise,} \end{cases} \tag{20}$$

with a positive integer value for l , we suppose $l = 2^k(2M+1)$ in this paper.

Let $B_l(t) = [b_0(t), b_1(t), \dots, b_{l-1}(t)]^T$, then the block pulse functions operational matrix of fractional integration F^α is given by [20]

$$(J^\alpha B_l)(t) = F^\alpha B_l(t), \tag{21}$$

where

$$F^\alpha = \frac{1}{l^\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{l-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{l-2} \\ 0 & 0 & 1 & \cdots & \xi_{l-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{22}$$

$$\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}.$$

There is a relation between the block pulse functions and CAS wavelets:

$$\Psi(t) = \Phi B_l(t). \tag{23}$$

Therefore we can derive P^α easily by using (19), (21), and (23):

$$P^\alpha = \Phi F^\alpha \Phi^{-1}. \tag{24}$$

4. Numerical Solution of (1)-(2)

Consider the nonlinear fractional integrodifferential equations of Bratu type

$$D^\alpha u(x) + \lambda \int_0^x k(x, t) e^{u(t)} dt + g(x) = 0, \quad (25)$$

$$u^{(j)}(0) = b_j, \quad 0 \leq x, t \leq 1, \quad n-1 < \alpha \leq n,$$

where λ is arbitrary parameter, $k(x, t) \in L^2([0, 1] \times [0, 1])$, and $g(x)$ is a known function.

Let $D^\alpha u(x) \approx C^T \Psi(x)$, $g(x) \approx G^T \Psi(x)$, $k(x, t) \approx \Psi^T(x) K \Psi(t)$, and $e^{u(t)} \approx \sum_{i=0}^l [u(t)]^i / i!$, then

$$\begin{aligned} u(x) &\approx C^T P^\alpha \Psi(x) + \sum_{k=0}^{n-1} u^{(k)}(0) \frac{x^k}{k!} \\ &= C^T P^\alpha \Psi(x) + Q \Psi(x) = (C^T P^\alpha + Q) \Psi(x), \end{aligned} \quad (26)$$

where coefficient Q is known and can be obtained by using the initial conditions.

Substituting (23) into (26), we obtain

$$u(x) \approx (C^T P^\alpha + Q) \Psi(x) = (C^T P^\alpha + Q) \Phi B_l(x). \quad (27)$$

Define $(C^T P^\alpha + Q) \Phi = [a_1, a_2, \dots, a_m] = A$, then $u(x) \approx AB_l(x)$.

Applying the properties of BPFs, we have

$$[u(x)]^i \approx [a_1^i, a_2^i, \dots, a_m^i] B_l(x) = A_i B_l(x). \quad (28)$$

By substituting the above expanded forms into (1), we get

$$\begin{aligned} D^\alpha u(x) + \lambda \int_0^x k(x, t) e^{u(t)} dt + g(x) &\approx C^T \Psi(x) + \lambda \int_0^x \Psi^T(x) K \Psi(t) \sum_{j=0}^l \frac{[u(t)]^j}{j!} dt + G^T \Psi(x) \\ &\approx C^T \Psi(x) + \lambda \Psi^T(x) K \\ &\quad \times \int_0^x \Phi B_l(t) \sum_{j=0}^l \frac{B_j^T(t) A_j^T}{j!} dt + G^T \Psi(x) \\ &\approx C^T \Psi(x) + \lambda \Psi^T(x) K \Phi \sum_{j=0}^l \frac{1}{j!} \int_0^x \tilde{A}_j^T B_l(t) dt + G^T \Psi(x) \\ &\approx C^T \Psi(x) + \lambda \Psi^T(x) K \Phi \sum_{j=0}^l \frac{1}{j!} \tilde{A}_j^T F^1 B_l(x) + G^T \Psi(x) = 0, \end{aligned} \quad (29)$$

where $\tilde{A}_j^T$ is the product operational matrix of A_j^T .

Putting the collocation points $\{x_i\}_{i=0}^{2^k(2M+1)-1}$ into (29), (29) will be

$$C^T \Psi(x_i) + \lambda \Psi^T(x_i) K \Phi \sum_{j=0}^l \frac{1}{j!} \tilde{A}_j^T F^1 B_l(x_i) + G^T \Psi(x_i) = 0. \quad (30)$$

Solving the nonlinear algebraic equations (30) by using Newton iteration method, we obtain the vector C^T , and then we get the approximate solution $u(x) = (C^T P^\alpha + Q) \Psi(x)$.

5. Existence of Uniqueness

Theorem 3 (uniqueness theorem). *Equation (1) has a unique solution whenever $0 < \beta < 1$, where*

$$\beta = \frac{\lambda M L}{\Gamma(\alpha) \alpha (\alpha + 1)}, \quad (31)$$

$$M = \sup_{0 \leq x, t \leq 1} |k(x, t)| < \infty.$$

Proof. Suppose $t \in [0, 1]$, then $u(t)$ is bounded. Therefore the nonlinear term $e^{u(t)}$ in (1) is Lipschitz continuous with $|e^u - e^{u_*}| \leq L|u - u_*|$, $L > 0$.

Let u and u^* be two different solutions of (1), then we can get

$$D^\alpha u(x) = -\lambda \int_0^x k(x, t) e^{u(t)} dt - g(x), \quad (32)$$

$$D^\alpha u_*(x) = -\lambda \int_0^x k(x, t) e^{u_*(t)} dt - g(x).$$

Using Riemann-Liouville fractional integration, we have

$$\begin{aligned} I^\alpha D^\alpha u(x) &= -\frac{1}{\Gamma(\alpha)} \int_0^x (x - \xi)^{\alpha-1} g(\xi) d\xi \\ &\quad - \frac{\lambda}{\Gamma(\alpha)} \int_0^x \int_0^\xi (x - \xi)^{\alpha-1} k(\xi, t) e^{u(t)} dt d\xi, \\ I^\alpha D^\alpha u_*(x) &= -\frac{1}{\Gamma(\alpha)} \int_0^x (x - \xi)^{\alpha-1} g(\xi) d\xi \\ &\quad - \frac{\lambda}{\Gamma(\alpha)} \int_0^x \int_0^\xi (x - \xi)^{\alpha-1} k(\xi, t) e^{u_*(t)} dt d\xi. \end{aligned} \quad (33)$$

Because $I^\alpha D^\alpha x(t) = x(t) - \sum_{k=0}^{n-1} (t^k/k!) x^{(k)}(0^+)$, so (33) can transform as

$$\begin{aligned} u(x) - \sum_{j=0}^{n-1} \frac{x^j}{j!} u^{(j)}(0) &= -\frac{1}{\Gamma(\alpha)} \int_0^x (x - \xi)^{\alpha-1} g(\xi) d\xi \\ &\quad - \frac{\lambda}{\Gamma(\alpha)} \int_0^x \int_0^\xi (x - \xi)^{\alpha-1} k(\xi, t) e^{u(t)} dt d\xi, \end{aligned} \quad (34)$$

$$\begin{aligned} u_*(x) - \sum_{j=0}^{n-1} \frac{x^j}{j!} u_*^{(j)}(0) &= -\frac{1}{\Gamma(\alpha)} \int_0^x (x - \xi)^{\alpha-1} g(\xi) d\xi \\ &\quad - \frac{\lambda}{\Gamma(\alpha)} \int_0^x \int_0^\xi (x - \xi)^{\alpha-1} k(\xi, t) e^{u_*(t)} dt d\xi. \end{aligned}$$

TABLE 1: Absolute errors for different values of k, M .

t	$k = 1, M = 1$	$k = 2, M = 1$	$k = 3, M = 1$
0	0.05345058698637	0.02590432861120	0.00591634971861
1/6	0.06527407677339	0.03850276248440	0.00861046012104
2/6	0.05142870790491	0.01273940964104	0.00925569874104
3/6	0.02376570358066	0.00782417705008	0.00174295063041
4/6	0.03786855145819	0.00811001255098	0.00314673017636
5/6	0.08725888105360	0.04440645099374	0.03718748499037

Then we have

$$\begin{aligned}
& |u - u_*| \\
&= \left| -\frac{\lambda}{\Gamma(\alpha)} \int_0^x \int_0^\xi (x - \xi)^{\alpha-1} k(\xi, t) e^{u(t)} dt d\xi \right. \\
&\quad \left. + \frac{\lambda}{\Gamma(\alpha)} \int_0^x \int_0^\xi (x - \xi)^{\alpha-1} k(\xi, t) e^{u_*(t)} dt d\xi \right| \\
&= \left| -\frac{\lambda}{\Gamma(\alpha)} \int_0^x \int_0^\xi (x - \xi)^{\alpha-1} k(\xi, t) [e^{u(t)} - e^{u_*(t)}] dt d\xi \right| \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^x \int_0^\xi (x - \xi)^{\alpha-1} k(\xi, t) |e^{u(t)} - e^{u_*(t)}| dt d\xi \\
&\leq \frac{\lambda ML}{\Gamma(\alpha)} \int_0^x \int_0^\xi (x - \xi)^{\alpha-1} |u - u_*| dt d\xi \\
&= \frac{\lambda ML}{\Gamma(\alpha)} \int_0^x (x - \xi)^{\alpha-1} |u - u_*| \xi d\xi \\
&= \frac{\lambda ML}{\Gamma(\alpha)} |u - u_*| \int_0^x \xi (x - \xi)^{\alpha-1} d\xi \\
&= \frac{\lambda ML}{\Gamma(\alpha)} |u - u_*| \left(\frac{x^{\alpha+1}}{\alpha} - \frac{x^{\alpha+1}}{\alpha+1} \right) \\
&= \frac{\lambda ML}{\Gamma(\alpha)} |u - u_*| x^{\alpha+1} \frac{1}{\alpha(\alpha+1)} \\
&\leq \frac{\lambda ML}{\Gamma(\alpha) \alpha(\alpha+1)} |u - u_*|.
\end{aligned} \tag{35}$$

Therefore $|u - u_*|(1 - (\lambda ML/\Gamma(\alpha)\alpha(\alpha+1))) \leq 0$.

This implies that $|u - u_*|(1 - \beta) \leq 0$, where $\beta = \lambda ML/\Gamma(\alpha)\alpha(\alpha+1)$.

As $0 < \beta < 1$, $|u - u_*| = 0$; implying $u = u_*$, we can prove (1) has the uniqueness solution. $\square$

6. Numerical Examples

To show the efficiency and the accuracy of the proposed method, we consider the following two examples.

Example 1. Consider the following equation:

$$D^{0.5} u(x) + \int_0^x x \left(t - \frac{1}{2} \right) e^{u(t)} dt + g(x) = 0, \tag{36}$$

FIGURE 1: The comparison between numerical and exact solution for $k = 1, M = 1$.

with this condition $u(0) = 0$ and $g(x) = -(\Gamma(3)/\Gamma(2.5))x^{1.5} + (\Gamma(2)/\Gamma(1.5))x^{0.5} - (x/2)(e^{x^2-x} - 1)$. The exact solution of this problem is $u(x) = x^2 - x$. Table 1 shows the absolute errors of the approximate solutions and the exact solution. The comparison between the numerical solutions and the exact solution for different k is shown in Figures 1, 2, and 3.

Taking a closer look at Table 1 and Figures 1–3, with k increasing, we find that the approximate solutions converge to the exact solution.

Example 2. Consider this equation

$$D^\alpha u(x) + 3 \int_0^x (x - t) e^{u(t)} dt + g(x) = 0, \quad 0 < x, t < 1, \tag{37}$$

such that the initial conditions $u(0) = 0$, $g(x) = -(1/2)x^3 - (1/x)$. The exact solution of the problem for $\alpha = 1$ is given by $u(x) = \ln x$. The comparison of numerical results for $\alpha = 0.5$, $\alpha = 0.75$, and $\alpha = 1$ and the exact solution for $\alpha = 1$ are shown in Figure 4. From Figure 4, we can see clearly that the numerical solutions are in very good agreement with the exact solution when $\alpha = 1$.

FIGURE 2: The comparison between numerical and exact solution for $k = 2, M = 1$.

FIGURE 3: The comparison between numerical and exact solution for $k = 3, M = 1$.

It is evident from Figure 4 that, as α close to 1, the numerical solution of the CAS wavelets converges to the exact solution; that is, the solution of fractional integrodifferential equation approaches to the solution of integer order integrodifferential equation.

7. Conclusion

In this paper, a numerical method is presented by numerical solutions of fractional integrodifferential equations of Bratu type. Taking full advantage of the definition of Caputo type fractional derivative and the properties of CAS wavelet, we

FIGURE 4: Numerical solution and exact solution of $\alpha = 1$.

transform the initial problem into a nonlinear algebraic system equation. By solving the nonlinear system, numerical solutions are obtained. The convergence analysis of CAS wavelets and the uniqueness theorem of this equation are proposed. The numerical results show that the approximation is in very good coincidence with the exact solution.

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