Research Article

On Intuitionistic Fuzzy Context-Free Languages

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Taking intuitionistic fuzzy sets as the structures of truth values, we propose the notions of intuitionistic fuzzy context-free grammars (IFCFGs, for short) and pushdown automata with final states (IFPDAs). Then we investigate algebraic characterization of intuitionistic fuzzy recognizable languages including decomposition form and representation theorem. By introducing the generalized subset construction method, we show that IFPDAs are equivalent to their simple form, called intuitionistic fuzzy simple pushdown automata (IF-SPDAs), and then prove that intuitionistic fuzzy recognizable step functions are the same as those accepted by IFPDAs. It follows that intuitionistic fuzzy pushdown automata with empty stack and IFPDAs are equivalent by classical automata theory. Additionally, we introduce the concepts of Chomsky normal form grammar (IFCNF) and Greibach normal form grammar (IFGNF) based on intuitionistic fuzzy sets. The results of our study indicate that intuitionistic fuzzy context-free languages generated by IFCFGs are equivalent to those generated by IFGNFs and IFCNFs, respectively, and they are also equivalent to intuitionistic fuzzy recognizable step functions. Then some operations on the family of intuitionistic fuzzy context-free languages are discussed. Finally, pumping lemma for intuitionistic fuzzy context-free languages is investigated.

1. Introduction

Intuitionistic fuzzy set (IFS) introduced by Atanassov [1–3], which emerges from the simultaneous consideration of the degrees of membership and nonmembership with a degree of hesitancy, has been found to be highly useful in dealing with problems with vagueness and uncertainty. The notion of vague set, proposed by Gau and Buehrer [4], is another generalization of fuzzy sets. However, Burillo and Bustince [5] showed that it is an equivalence of the IFS and studied intuitionistic fuzzy relations. Recently, IFS theory has supported a wealth of important applications in many fields such as fuzzy multiple attribute decision making, fuzzy pattern recognition, medical diagnosis, fuzzy control, and fuzzy optimization [6–10].

In classical theoretical computer science, it is well known that formal languages are very useful in the description of natural languages and programming languages. But they are not powerful in the processing of human languages. For this, Lee and Zadeh [11] introduced the notion of fuzzy languages and gave some characterizations, where fuzzy languages took values in the unit interval [0, 1]. Malik and Mordeson [12–14] studied algebraic properties of fuzzy languages. They stated that fuzzy regular languages can be characterized by fuzzy finite automata, fuzzy regular expressions, and fuzzy regular grammars. Meanwhile, as one of the generators of fuzzy languages, fuzzy automata have been used to solve meaningful issues such as the model of computing with words [15], clinical monitoring [16], neural networks [17], and pattern recognition [18]. Also, fuzzy grammars, automata, and languages tend to the improvement of lexical analysis and simulating fuzzy discrete event dynamical systems and hybrid systems [14, 19].

As is well known, quantum logic was proved by Birkhoff and Von Neumann as a logic of quantum mechanics and
is currently understood as a logic with truth values taken from an orthomodular lattice. To study quantum computation, Ying [20, 21] first proposed automata theory based on quantum logic where quantum automata are defined to be orthomodular lattice-valued generalization of classical automata. More systematic exposition of this theory appeared in [22, 23]. Moore and Crutchfield [24] defined quantum version of pushdown automata and regular and context-free grammars. He showed that the corresponding languages generated by quantum grammars and recognized by quantum automata have satisfactory properties in analogy to their classical counterparts. A basic framework of grammar theory on quantum logic was established by Cheng and Wang [25]. They proved that the set of lattice-valued quantum regular languages generated by lattice-valued quantum regular grammars coincides with that of lattice-valued quantum languages recognized by lattice-valued quantum automata. Then some algebraic properties of automata based on quantum logic were discussed by Qiu [26, 27]. To enhance the processing ability of fuzzy automata, the membership grades were extended to many general algebraic structures. For example, by combining the ideas in [20–23] and the idea in Ying’s another work on topology based on residuated lattice-valued logic [28], Qiu has primarily established automata theory based on complete residuated lattice-valued logic [29–31]. And Li and Pedrycz [32] studied automata theory with membership values in lattice-ordered monoids. They showed that lattice-valued finite automata have more power to recognize fuzzy languages than that of classical fuzzy finite automata. Recently, Li [33] studied automata theory with membership values in lattices, introduced the technique of extended subset construction to prove the equivalence between lattice-valued finite automata and lattice-valued deterministic finite automata, and then presented a minimization algorithm of lattice-valued deterministic finite automata. On the basis of breadth-first and depth-first ways, Jin and Li [34] established a fundamental framework of fuzzy grammars based on lattices, which provided a necessary tool for the analysis of fuzzy automata.

Fuzzy context-free languages, Ass powerful than fuzzy regular languages, have also been studied and can be characterized by fuzzy pushdown automata with two distinct ways and fuzzy context-free grammars, respectively [14, 35]. As a continuation of the work in [29–31], a fundamental framework of fuzzy pushdown automata theory based on complete residuated lattice-valued logic has been established in recent years by Xing et al. [36], and the work generalizes the previous fuzzy automata theory systematically studied by Mordeson and Malik to some extent. The pumping lemma for fuzzy context-free grammar theory in this setting was also investigated by Xing and Qiu [37].

Using the notions of IFSs and fuzzy finite automata, Jun [38, 39] presented the concept of intuitionistic fuzzy finite state machines as a generalization of fuzzy finite state machines, and Zhang and Li [40] discussed intuitionistic fuzzy recognizers, intuitionistic fuzzy finite automata, and intuitionistic fuzzy language. They showed that the languages recognized by intuitionistic fuzzy recognizers are regular, and the intuitionistic fuzzy languages recognized by the intuitionistic fuzzy finite automata and the intuitionistic fuzzy languages recognized by deterministic intuitionistic fuzzy finite automata are equivalent. Recently Chen et al. [41] utilized the intuitionistic fuzzy automata to deal with consumers’ advertising involvement when considering the expression of an IFS characterized by a pair of membership degree and nonmembership degree is similar to human thinking logic with pros and cons. Due to pushdown automata being another kind of important computational models [15] and also motivated by the importance of grammars, languages and models theory [14], it stands to reason that we ought consider the notions of intuitionistic fuzzy pushdown automata, intuitionistic fuzzy context-free grammars, and fuzzy context-free languages because our discussion in this paper will provide a fundamental framework for studying intuitionistic fuzzy set theory on fuzzy pushdown automata and generators as well. How to characterize intuitionistic fuzzy context-free languages and its pumping lemma in this setting becomes open problems; however, there is no research on the algebraic characterization of intuitionistic fuzzy context-free languages. We will try to solve the problems in this paper. Additionally, some examples are given to illustrate the significance of the results. In particular, Example 35 presented in this paper will show that intuitionistic fuzzy pushdown automata have more power than fuzzy pushdown automata when comparing two distinct strings although the degrees of membership of these strings recognized by the underlying fuzzy pushdown automata are equal. Investigating intuitionistic fuzzy context-free languages will reduce the gap between the precision of formal languages and the imprecision of human languages.

The remaining parts of the paper are arranged as follows. Section 2 describes some basic concepts of IFSs. Section 3 gives the definitions of intuitionistic fuzzy pushdown automata with two distinct ways and their languages. It is investigated that, for any intuitionistic fuzzy pushdown automaton with final states (IFPDA, for short), there is a cover, which consists of a collection of classical pushdown automata, equivalent to the IFPDA. By introducing intuitionistic fuzzy recognizable step functions, it is shown that intuitionistic fuzzy pushdown automata with final states and empty stack are intuitionistic fuzzy recognizable step functions, respectively, and conversely any intuitionistic fuzzy recognizable step function can be recognized by an intuitionistic fuzzy pushdown automaton with final states or empty stack. It follows that intuitionistic fuzzy pushdown automata with final states and empty stack are equivalent. Section 4 studies intuitionistic fuzzy context-free grammars (IFCFGs) as a type of generator of intuitionistic fuzzy context-free languages (IFCFLs). The notions of intuitionistic fuzzy Chomsky normal form (IFCNF) and Greibach normal form (IFGNF) are proposed. The results of our study indicate that IFCFLs generated by IFCFGs are equivalent to those generated by IFGNFs and IFCNFs, respectively, and they are also equivalent to intuitionistic fuzzy recognizable step functions. The algebraic properties of IFCFLs are also discussed. Section 5 establishes pumping lemma for IFCFLs. Some examples are then given to illustrate the application of pumping lemma and the significance of IFCFLs. Finally,
conclusions and directions for future work are presented in Section 6.

2. Basic Concepts

Definition 1 (see [40]). An intuitionistic fuzzy set \( A \) in a non-empty set \( X \) is an object having the form:

\[
A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},
\]

where the functions \( \mu_A : X \to [0, 1] \) and \( \nu_A : X \to [0, 1] \) denote the degree of membership (i.e., \( \mu_A(x) \)) and the degree of nonmembership \( (\nu_A(x)) \) of each element \( x \in X \) to the set \( A \), respectively, and the two quantities satisfy the following inequalities:

\[
0 \leq \mu_A(x) + \nu_A(x) \leq 1, \quad \forall x \in X.
\]

For the sake of simplicity, we use the notation \( A = (\mu_A, \nu_A) \) instead of \( A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\} \). An intuitionistic fuzzy set will be abbreviated as an IFS.

Let \( \{A_i \mid i \in I\} \) be a family of IFSs in \( X \). Then the infimum and supremum operations of IFSs are defined as follows:

\[
\bigwedge_{i \in I} A_i = \left\{ x \mid \bigwedge_{i \in I} \mu_{A_i}(x), \bigvee_{i \in I} \nu_{A_i}(x) \mid x \in X \right\},
\]

\[
\bigvee_{i \in I} A_i = \left\{ x \mid \bigvee_{i \in I} \mu_{A_i}(x), \bigwedge_{i \in I} \nu_{A_i}(x) \mid x \in X \right\},
\]

where \( \bigwedge \) and \( \bigvee \) denote supremum and infimum of real numbers in \([0, 1]\), respectively.

For two IFSs \( A = (\mu_A, \nu_A) \) and \( B = (\mu_B, \nu_B) \), we say \( A = B \) if \( \mu_A = \mu_B \) and \( \nu_A = \nu_B \). In addition, if the IFS \( A = (\mu_A, \nu_A) \) in \( X \) satisfies the condition that, for any \( x \in X \), \( \mu_A(x) + \nu_A(x) = 1 \), then \( A \) reduces to a fuzzy set in \( X \). The difference between intuitionistic fuzzy sets and fuzzy sets is whether the sum of the degrees of membership and nonmembership of an element to a set equals one.

An IFR in \( X \times Y \) is an intuitionistic fuzzy subset of \( X \times Y \); that is, it is an expression \( E \) given by

\[
E = \{(x, y, \mu_E(x, y), \nu_E(x, y)) \mid x \in X, y \in Y\},
\]

where the mappings \( \mu_E : X \times Y \to [0, 1] \) and \( \nu_E : X \times Y \to [0, 1] \) satisfy

\[
0 \leq \mu_E(x, y) + \nu_E(x, y) \leq 1, \quad \forall (x, y) \in X \times Y.
\]

An IFBR over \( X \) is an IFS of \( X \times X \). Let \( P = (\mu_P, \nu_P) \) and \( E = (\mu_E, \nu_E) \) be IFRs in \( X \times Y \) and \( Y \times Z \), respectively. Define the composition of IFRs, \( P \circ E = (\mu_{P \circ E}, \nu_{P \circ E}) \) in \( X \times Z \), by

\[
\mu_{P \circ E}(x, z) = \bigvee_{y \in Y} (\mu_P(x, y) \land \mu_E(y, z)),
\]

\[
\nu_{P \circ E}(x, z) = \bigwedge_{y \in Y} (\nu_P(x, y) \lor \nu_E(y, z)),
\]

for all \((x, z) \in X \times Z\).

Furthermore, if \( R \) is an IFBR over \( X \), then its reflexive and transitive closure is \( R^* = \bigcup_{n=0}^{\infty} R^n \), where \( R^{n+1} = R^n \circ R, \ n \geq 0 \), and \( R^0 = \{(\mu_{id}, \nu_{id})\} \), that is,

\[
\mu_{id}(u, v) = \begin{cases} 1, & \text{if } u = v, \\ 0, & \text{if } u \neq v, \end{cases}
\]

\[
\nu_{id}(u, v) = \begin{cases} 0, & \text{if } u = v, \\ 1, & \text{if } u \neq v, \end{cases}
\]

for all \((u, v) \in X \times X\).

Definition 2. Let \( A = (\mu_A, \nu_A) \) be an IFS in \( X \). Then the image set of \( A \), denoted as \( \text{Im}(A) \), is given as

\[
\text{Im}(A) = \text{Im}(\mu_A) \cup \text{Im}(\nu_A),
\]

where \( \text{Im}(\mu_A) = \{\mu_A(x) \mid x \in X\} \) and \( \text{Im}(\nu_A) = \{\nu_A(x) \mid x \in X\} \).

For any \( \lambda, \theta \in [0, 1], \lambda + \theta \leq 1, \) the \((\lambda, \theta)\)-cut set of \( A \) is defined as

\[
A_{(\lambda, \theta)} = \{x \in X \mid \mu_A(x) \geq \lambda, \nu_A(x) \leq \theta\}.
\]

And the support set of \( A \), denoted as \( \text{supp}(A) \), is defined by

\[
\text{supp}(A) = \{x \in X \mid \mu_A(x) > 0, \nu_A(x) < 1\}.
\]

If \( \text{supp}(A) \) is finite, then \( A \) is called a finite IFS in \( X \).

3. Intuitionistic Fuzzy Pushdown Automata

It is well known that any language accepted by a pushdown automaton with final states can be accepted by a certain pushdown automaton with empty stack, and vice versa. As a natural generalization of pushdown automata, we give the notions of intuitionistic fuzzy pushdown automata with final states and empty stack, respectively, and then do research in the algebraic characterization of their intuitionistic fuzzy recognizable languages including decomposition form and representation theorem. Note that \( \Sigma^* \) is the free monoid generated from the set \( \Sigma \) with the operator of concatenation, where the empty string \( \epsilon \) is identified with the identity of \( \Sigma \). And the length of the string \( \omega \in \Sigma^* \) is denoted by \( |\omega| \).

Definition 3. An intuitionistic fuzzy pushdown automaton with final states (IFPDA, for short) is a seven tuple \( \mathcal{A} = (Q, \Sigma, \Gamma, \delta, I, Z_0, F) \), where

(i) \( Q \) is a finite nonempty set of states;
(ii) \( \Sigma \) is a finite nonempty set of input symbols;
(iii) \( \Gamma \) is a finite nonempty set of stack symbols;
(iv) \( \delta = (\mu_\delta, \nu_\delta) \) is a finite IFS in \( Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \times \Gamma^* \);
(v) \( Z_0 \in \Gamma \) is called the start stack symbol;
(vi) \( I = (\mu_I, \nu_I) \) and \( F = (\mu_F, \nu_F) \) are intuitionistic fuzzy subsets in \( \Sigma \), which are called the intuitionistic fuzzy subsets of initial and final states, respectively.
Definition 4. An intuitionistic fuzzy pushdown automaton with empty stack (IFPDA\textsuperscript{0}, for short) is a seven tuple \( M = (Q, \Sigma, \Gamma, \delta, I, Z_0, \emptyset) \), where \( Q, \Sigma, \Gamma, \delta, I \) and \( Z_0 \) are the same as those in IFPDA \( M \), and \( \emptyset \) represents an empty set.

Definition 5. Let \( \mathcal{M} = (Q, \Sigma, \Gamma, \delta, I, Z_0, F) \) be an IFPDA. Define an IFBR on \( Q \times \Sigma^* \times I^* \), \( \tau_{\text{IF}} = (\mu_{\text{IF}}, \nu_{\text{IF}}) \) in the form of

\[
\mu_{\text{IF}}((q, \omega, \beta), (u, \alpha)) = \begin{cases}
\mu_\delta(q, \varepsilon, \beta), (p, \alpha \setminus \text{tail}(\beta)) & \text{if } u = \omega, \text{tail}(\beta) \leq \alpha \\
\mu_\delta(q, \text{head}(\omega), \beta), (p, \alpha \setminus \text{tail}(\beta)) & \text{if } u = \text{tail}(\omega), \text{tail}(\beta) \leq \alpha \\
0 & \text{otherwise}
\end{cases}
\]

\[
\nu_{\text{IF}}((q, \omega, \beta), (p, u, \alpha)) = \begin{cases}
\nu_\delta(q, \varepsilon, \beta), (p, \alpha \setminus \text{tail}(\beta)) & \text{if } u = \omega, \text{tail}(\beta) \leq \alpha \\
\nu_\delta(q, \text{head}(\omega), \beta), (p, \alpha \setminus \text{tail}(\beta)) & \text{if } u = \text{tail}(\omega), \text{tail}(\beta) \leq \alpha \\
0 & \text{otherwise}
\end{cases}
\]

for any \( (q, \omega, \beta), (p, u, \alpha) \in Q \times \Sigma^* \times I^* \). Here, for any nonempty string \( u = x_1 \cdots x_n \), head(\( u \)) = \( x_1 \), tail(\( u \)) = \( x_2 \cdots x_n \), and tail(\( u \)) \( \leq \alpha \). \( \tau_{\text{IF}} \) is the reflexive and transitive closure of \( \tau_{\text{IF}} \). If no confusion, we denote \( \tau \) and \( \tau^+ \) instead of \( \tau_{\text{IF}} \) and \( \tau_{\text{IF}}^+ \), respectively.

Definition 6. Let \( \mathcal{M} = (Q, \Sigma, \Gamma, \delta, I, Z_0, F) \) be an IFPDA. Then we call \( \mathcal{L}(\mathcal{M}) \) an intuitionistic fuzzy language accepted by \( \mathcal{M} \) with final states, where \( \mathcal{L}(\mathcal{M}) = (\mu_{\mathcal{L}(\mathcal{M})}, \nu_{\mathcal{L}(\mathcal{M})}) \), \( \mu_{\mathcal{L}(\mathcal{M})} \), and \( \nu_{\mathcal{L}(\mathcal{M})} \) are functions from \( \Sigma^* \) to the unit interval \([0, 1]\), and

\[
\mu_{\mathcal{L}(\mathcal{M})}(\omega) = \bigvee \{ \mu(q_0) \land \mu_{\text{IF}}((q_0, \omega, \beta), (p, \alpha, r)) \land \mu_\delta(p) | q_0, p \in Q, r \in I^* \},
\]

\[
\nu_{\mathcal{L}(\mathcal{M})}(\omega) = \bigvee \{ \nu(q_0) \lor \nu_{\text{IF}}((q_0, \omega, \beta), (p, \alpha, r)) \lor \nu_\delta(p) | q_0, p \in Q, r \in I^* \}
\]

for any \( \omega \in \Sigma^* \).
\(\mu_{\tilde{\cdot}} ((q_0, \omega, z_0), (p, e, \varepsilon)) \mid q_0, p \in Q\) = \(\bigvee \mu_1(q_0) \wedge \mu_2((q_0, \omega, z_0), (q_1, u_2 \cdots u_n, z_1)) \wedge \mu_3((q_1, u_2 \cdots u_n, z_1), (q_2, u_3 \cdots u_n, z_2)) \wedge \cdots \wedge \mu_n((q_{n-1}, u_n, z_{n-1}, r_{n-1}), (q_n, e, \varepsilon))\) 

\((q_0, q_1, \ldots, q_n) \in Q^n, z_0, \ldots, z_n \in \Gamma\), and \(v_\gamma(\omega) = \bigwedge \{v_1(q_0) \vee v_\gamma_j((q_0, \omega, z_0), (p, e, \varepsilon)) \mid q_0, p \in Q\} = \bigvee \mu_1(q_0) \wedge \mu_2((q_0, \omega, z_0), (q_1, u_2 \cdots u_n, z_1)) \wedge \mu_3((q_1, u_2 \cdots u_n, z_1), (q_2, u_3 \cdots u_n, z_2)) \wedge \cdots \wedge \mu_n((q_{n-1}, u_n, z_{n-1}, r_{n-1}), (q_n, e, \varepsilon))\)

In a similar manner, it is concluded that the following must be true.

**Proposition 10.** If \(f\) can be accepted by some IFPDA \(\mathcal{M} = (Q, \Sigma, \Gamma, \delta, I, Z_0, F)\), then \(f\) is an IFS in \(\Sigma^\star\), and the image set of \(f\) is finite.

Specially, the IFPDA \(\mathcal{M} = (Q, \Sigma, \Gamma, \delta, I, Z_0, F)\) will be abbreviated as \(\mathcal{M} = (Q, \Sigma, \Gamma, \delta, I, Z_0, F)\), whenever \(\text{Im}(I) \subseteq \{0, 1\}\) and \(\text{supp}(I) = \{q_0\}\). Moreover, if \(\text{Im}(I) \cup \text{Im}(\delta) \subseteq \{0, 1\}\) and \(\text{supp}(I)\) has only one element, then the IFPDA is a classical PDA.

For two IFPDAs \(\mathcal{M}_1\) and \(\mathcal{M}_2\), we say that they are equivalent if they accept the same intuitionistic fuzzy language.

**Proposition 11.** Let \(A\) be an IFS in a nonempty set \(\Sigma^\ast\). Then the following statements are equivalent:

(i) \(A\) can be accepted by an IFPDA \(\mathcal{M} = (Q, \Sigma, \Gamma, \delta, I, Z_0, F)\);

(ii) \(A\) can be accepted by a certain IFPDA \(\mathcal{M}' = (Q', \Sigma, \Gamma', \delta', I', Z_0', F')\), where \(q_0 \in Q'\).

Proof. (i) implies (ii). Construct an IFPDA \(\mathcal{M}' = (Q', \Sigma, \Gamma', \delta', I', Z_0', F')\) as follows: \(Q' = Q \cup \{q_0\}, I' = I \cup \{X_0\}\), where \(q_0 \notin Q, X_0 \notin \Sigma\). Define an IFS \(I'\) in \(Q'\) by

\[
\mu_{I'}(q) = \begin{cases} 
1, & \text{if } q = q_0 \\
0, & \text{if } q \neq q_0.
\end{cases}
\]

Define an IFS \(F'\) in \(Q'\) by

\[
\nu_{I'}(q) = \begin{cases} 
0, & \text{if } q = q_0 \\
1, & \text{if } q \neq q_0.
\end{cases}
\]

There exists especially a simple type of intuitionistic fuzzy pushdown automaton, which is called intuitionistic fuzzy simple pushdown automata. The definition is given as follows.

**Definition 13.** An IFPDA \(\mathcal{M} = (Q, \Sigma, \Gamma, \delta, I, Z_0, F)\) is called an intuitionistic fuzzy simple pushdown automaton (IFSPDA) if the image set of \(\delta\) is contained in the set \([0, 1]\).

Next any IFPDA is proven to be an equivalence of a certain IFSPDA by utilizing the generalized subset construction method. Noting that an IFS requires that the sum of the degrees of membership and nonmembership of an element to a set is no more than the natural number 1. So the proof technique is to some extent different from the technique of
Proposition 14. Let $\mathcal{A}$ be an IFPDA. Then there exists an IFPDA $\mathcal{A}'$ such that $L(\mathcal{A}') = L(\mathcal{A})$.

Proof. Let $\mathcal{A} = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be an IFPDA. Then we construct an IFPDA $\mathcal{A}' = (Q', \Sigma, \Gamma', \delta', q'_0, Z'_0, F')$ as follows:

(i) $Q' = Q \times (L_1 - \{0\}) \times (L_2 - \{1\})$, where $L_1 = X, L_2 = Y, X = \text{Im}(\mu_\delta) \cup \text{Im}(\mu_F)$ and $Y = \text{Im}(v_\delta) \cup \text{Im}(v_F)$;

(ii) $q'_0 = (q_0, 1, 0) \in Q'$;

(iii) $\delta' = (\mu_\gamma, v_\gamma')$ is an IFP in $Q' \times (\Sigma \cup \{e\}) \times \Gamma \times \Gamma'$, where the mappings $\mu_\gamma, v_\gamma : Q' \times (\Sigma \cup \{e\}) \times \Gamma \times \Gamma' \rightarrow \{0, 1\}$ are given as follows.

For any $(q, a, b), (q', c, d) \in Q'$, $\tau \in \Sigma \cup \{e\}, X \in \Gamma$, and $y' \in \Gamma'$, $\mu_\gamma((q, a, b), \tau, (X, (q', c, d), y) = 1$, and $v_\gamma((q, a, b), \tau, (X, (q', c, d), y) = 0$ whenever there exist $a'$ and $b'$ such that $\mu_\gamma((q, a, b), \tau, X, (q', c, d), y') = 1$, and $v_\gamma((q, a, b), \tau, X, (q', c, d), y) = 0$ whenever $\mu_\gamma((q, a, b), \tau, X, (q', c, d), y') = 0$.

(iv) $F' = (\mu_\gamma, v_\gamma')$ is an IF in $Q'$. For any $(q, a, b) \in Q'$,

$$\mu_\gamma((q, a, b)) = \begin{cases} a \land \mu_F(q), & \text{if } 0 \leq a + b \leq 1 \\ 0, & \text{if } a + b > 1 \end{cases}$$

$$v_\gamma((q, a, b)) = \begin{cases} b \lor v_F(q), & \text{if } 0 \leq a + b \leq 1 \\ 1, & \text{if } a + b > 1 \end{cases}$$

Now, it is claimed that for any $\omega = \tau_1 \cdots \tau_r \in \Sigma^*$, $\tau_i \in \Sigma \cup \{e\}, i \in \{1, \ldots, n\}$ and for any $(q_0, a_0, b_0) \in Q'$, $Z, Z_n \in \Gamma^*$, $\mu_\gamma((q_0, a_0, b_0), \epsilon, Z, Z_n) = 1$ and $v_\gamma((q_0, a_0, b_0), \epsilon, Z, Z_n) = 0$ whenever the following condition is satisfied.

(PI) There exist $q_1, \ldots, q_{n-1} \in Q, Z_1, \ldots, Z_{n-1} \in \Gamma$ and $y_1, \ldots, y_{n-1} \in \Gamma^*$ such that $\mu_\gamma((q_0, a_0, b_0), \epsilon, Z_0, Z_0) = 1$ and $v_\gamma((q_0, a_0, b_0), \epsilon, Z_0, Z_0) = 0$ whenever the following condition is satisfied.

It is proved by induction. In fact, if $|\omega| = 0$, then $\omega = \epsilon, \mu_\gamma((q_0, a_0, b_0), \epsilon, Z_0, Z_0) = 1$ and $v_\gamma((q_0, a_0, b_0), \epsilon, Z_0, Z_0) = 0$.

Suppose the result still holds whenever $|\omega| \leq n, n \in N$. If $|\omega| = n + 1, \omega = \tau_1 \cdots \tau_{n+1} = x_{k+1}$ and $x_{k+1} \in \Sigma$, then $|x| = n$ and $x = \tau_1 \cdots \tau_k$.

Next, for any $(q_0, a_0, b_0) \in Q'$, $Z_{k+1}, y_{k+1} \in \Gamma^*$, whenever (PI) is satisfied; that is, there exists a sequence of states $q_1, \ldots, q_k \in Q, Z_1, \ldots, Z_k \in \Gamma$ and $y_1, \ldots, y_k \in \Gamma^*$ such that

$$\begin{align*}
\mu_\gamma((q_k, \tau_{k+1}, \cdots, \tau_{r_k+1}, Z_{k+1})) & = \mu_\gamma(\omega, \epsilon, Z_{k+1}) > 0, \\
v_\gamma((q_k, \tau_{k+1}, \cdots, \tau_{r_k+1}, Z_{k+1})) & = \mu_\gamma(\omega, \epsilon, Z_{k+1}) > 0, \\
v_\gamma((q_k, \tau_{k+1}, \cdots, \tau_{r_k+1}, Z_{k+1}, Z_{k+1})) & = \delta_1 > 0,
\end{align*}$$

where $y_0 = \epsilon, i = 0, \ldots, k - 1$.

Let $a_i = c_1 \land \cdots \land a_i, b_i = d_1 \lor \cdots \lor d_i, i \in \{1, \ldots, k + 1\}$.

Then

$$\begin{align*}
\mu_\gamma((q_0, a_i, b_i), \tau_{i+1}, \cdots, \tau_{r_k+1}, Z_{i+1})) & = \mu_\gamma((q_k, \tau_{k+1}, \cdots, \tau_{r_k+1}, Z_{k+1})) \land \mu_\gamma((q_k, \tau_{k+1}, \cdots, \tau_{r_k+1}, Z_{k+1}, Z_{k+1})) = \delta_1 = \mu_\gamma((q_k, \tau_{k+1}, \cdots, \tau_{r_k+1}, Z_{k+1})) = \delta_1 > 0.
\end{align*}$$

For $a_0 = 0, b_0 = y_0 = \epsilon, i = 0, \ldots, k - 1$.

Hence, for any $\omega = \tau_1 \cdots \tau_r \in \Sigma^*, \tau_i \in \Sigma \cup \{e\}, i \in \{1, \ldots, n\}$, we have $L(\mathcal{A}(\omega)) = \{|\mu_\gamma((q_0, a_0, b_0), \epsilon, Z_0, Z_0), (q_0, a_0, b_0), \epsilon, y) \land \mu_\gamma((q_0, a_0, b_0), \epsilon, Z_0, Z_0) | (q_0, a_0, b_0) \in Q', \epsilon, \gamma \in \Gamma^* \} = \{|q_0, a_0, b_0) | q_0, a_0, b_0) \in Q', \epsilon, \gamma \in \Gamma^* \}$.
Clearly, an IFPDA is a generalization of a classical pushdown automaton (PDA). Next, it will be shown that any IFPDA can be characterized by a collection of pushdown automata. To describe the behavior of a pushdown automaton \( M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F) \), we need to introduce the concept of instantaneous description. An instantaneous description is a three-tuple \((q, \omega, Z) \in Q \times \Sigma^* \times \Gamma^* \), which means that the automaton is in the state \( q \) and has unprocessed input \( \omega \) and stack contents \( Z \). An instantaneous description represents the configuration of a pushdown automaton at a given instant. To introduce the transition in a pushdown automaton in terms of instantaneous descriptions, we define \( \succ \) as a binary relation on \( Q \times \Sigma^* \times \Gamma^* \). We say \((q, \omega, Z) \succ (p, \rho, \sigma)\) if \((q, a, Z)\) contains \((p, \sigma, \rho)\) where \(p, q \in Q\), \(a \in \Sigma \cup \{\epsilon\}\), \(\omega \in \Sigma^*\), \(Z \in \Gamma^*\), and \(y, \sigma, \rho \in \Gamma^*\). Furthermore, we define \( \succ \) as the reflexive and transitive closure of \( \succ \). Then the language accepted by \( M \) with final states is defined as

\[
L(M) = \{ \omega \in \Sigma^* \mid (q_0, \omega, Z_0) \succ (p, \epsilon, \gamma), p \in F, \gamma \in \Gamma^* \}.
\]  

Definition 15. A collection of classical pushdown automata with final states

\[
S = \{ M_{ab} \mid M_{ab} = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F_{ab}), 0 \leq a + b \leq 1, a, b \in [0, 1] \}
\]  

is called a cover if the following conditions hold:

(i) \( a_1 \leq \alpha_2 \) and \( b_2 \leq b_1 \) imply \( F_{a_2,b_2} \subseteq F_{a_1,b_1} \);

(ii) \( F_0 \) is a cover.

For a cover \( S \), its recognized intuitionistic fuzzy language \( f_S = (\mu_{f_S}, \nu_{f_S}) \) in \( \Sigma^* \) is given by

\[
\mu_{f_S}(\omega) = \bigvee \{ a \in [0, 1] \mid M_{ab} \text{ accepts } \omega, M_{ab} \in S \},
\nu_{f_S}(\omega) = \bigwedge \{ b \in [0, 1] \mid M_{ab} \text{ accepts } \omega, M_{ab} \in S \},
\]

for all \( \omega \in \Sigma^* \).

Theorem 16. Let \( f \) be an IFSDA in \( \Sigma^* \). Then \( f \) can be accepted by an IFPDA if and only if \( f \) can be recognized by a cover \( S \).

Proof. If \( f \) can be accepted by an IFPDA, then there exists an IFSDA \( M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F) \) such that \( M \) accepts \( f \) by Proposition 14. Next we construct a cover

\[
S = \{ M_{ab} \mid M_{ab} = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F_{ab}), 0 \leq a + b \leq 1, a, b \in [0, 1] \},
\]

where \( F_{ab} = \{ q \in Q \mid \mu_b(q) \geq a, \nu_b(q) \leq b \} \); the mapping \( \delta^* : Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \rightarrow 2^Q \cdot \ast \) is given by

\[
\delta^* (q, \tau, Z) = \{ (p, \eta) \mid \mu_p(q, \tau, Z, p, \eta) = 1, p \in Q, \eta \in \Gamma^* \},
\]

for all \((q, \tau, Z) \in Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \).

Clearly, the cover \( S \) is well defined.

Next, we will show that \( f \) can be recognized by the cover \( S \). In fact, we have

\[
(q_0, \omega, Z_0) \succ (q, \epsilon, \gamma) \text{ if and only if } \mu_{f_S}((q_0, \omega, Z_0), (q, \epsilon, \gamma)) = 1,
\]

for all \( a, b \in [0, 1] \) with \( a + b \leq 1 \), for all \( \omega \in \Sigma^* \), \( \gamma \in \Gamma^* \).

Then we construct an IFSPDA \( M = (Q, \Sigma, \Gamma, \eta, q_0, Z_0, F) \), where \( \eta = (\mu_\eta, \nu_\eta) \) is an IFSDA in \( Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \times Q \times \Gamma^* \), and the mappings \( \mu_\eta, \nu_\eta : Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \times Q \times \Gamma^* \rightarrow [0, 1] \) are defined as

\[
\mu_\eta(q, \tau, Z, p, \gamma) = \begin{cases} 1, & \text{if } (p, \gamma) \in \delta^* (q, \tau, Z) \\ 0, & \text{otherwise}, \end{cases}
\nu_\eta(q, \tau, Z, p, \gamma) = \begin{cases} 0, & \text{if } (p, \gamma) \in \delta^* (q, \tau, Z) \\ 1, & \text{otherwise}, \end{cases}
\]

for any \((q, \tau, Z, p, \gamma) \in Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \times Q \times \Gamma^* \).

Thus the cover \( S \) is a set of covers which \( M \) accepts.
(q₀, ω, Z₀) >*₀, (q, ε, γ) , γ ∈ Γ*} = \{b ∈ [0, 1] \mid M_{ab} accepts ω, M_{ab} \in S \} = v_f(ω)

Therefore, the IFSPDA \( M \) accepts \( f \).

Theorem 16 shows that every IFPDA is equivalent to a certain cover; however, the cover may have infinite classical pushdown automata elements. Is there a finite cover who is equivalent to the IFPDA? To solve the problem, we introduce the notion of an intuitionistic fuzzy recognizable step function as follows.

**Definition 17.** An IFS \( A \) over \( Σ^* \) is called an intuitionistic fuzzy recognizable step function if there exists a natural number \( n \in N \), recognizable context-free languages \( \mathcal{L}_1, \ldots, \mathcal{L}_n \leq Σ^* \), and \( (q_i, b_i) \in (0,1) \times [0,1) \) with \( 0 \leq a_i + b_i \leq 1 \) for \( i = 1, \ldots, n \) such that

\[
A = (\mu_A, v_A) = \prod_{i=1}^n (a_i, b_i) \cdot 1_{\mathcal{L}_i},
\]

where \( 1_{\mathcal{L}_i} = (\mu_{1_{\mathcal{L}_i}}, v_{1_{\mathcal{L}_i}}) \) represents the intuitionistic characterized function of \( \mathcal{L}_i \), \( i = 1, \ldots, n \), that is,

\[
\mu_{1_{\mathcal{L}_i}}(ω) = \begin{cases} 1, & \text{if } ω \in \mathcal{L}_i \\ 0, & \text{if } ω \notin \mathcal{L}_i \end{cases},
\]

\[
v_{1_{\mathcal{L}_i}}(ω) = \begin{cases} 0, & \text{if } ω \in \mathcal{L}_i \\ 1, & \text{if } ω \notin \mathcal{L}_i \end{cases},
\]

And the equation \((*)\) means that the following equations hold:

\[
\mu_A(ω) = \bigwedge_{i=1}^n a_i \land \mu_{1_{\mathcal{L}_i}}(ω),
\]

\[
v_A(ω) = \bigvee_{i=1}^n b_i \lor v_{1_{\mathcal{L}_i}}(ω),
\]

\[
\forall ω \in Σ^*.
\]

Noting that the family of all the intuitionistic fuzzy recognizable step functions over \( Σ^* \) is denoted by Step*(Σ).

**Proposition 18.** Let \( \mathcal{M}'_1 = (Q_1, Σ, Γ, δ_1, q_{01}, Z_{01}, F_1) \) be an IFPDA. Then the language recognized by \( \mathcal{M}'_1 \) is an intuitionistic fuzzy recognizable step function over \( Σ^* \), that is, \( \mathcal{L}(\mathcal{M}'_1) \in \text{Step}^*(\Sigma) \).

Proof. By Proposition 14, there is an IFSPDA \( \mathcal{M} = (Q, Σ, Γ, \delta, q_0, Z_0, F) \) equivalent to \( \mathcal{M}'_1 \).

Let \( R = \{(μ_F(ω), v_F(ω)) \mid ω ∈ Q \} \setminus \{(0,1)\} = \{(a_i, b_i) \mid i ∈ N_k, N_k = \{1, \ldots, k\}\) \}. Put \( F_i = \{q ∈ Q \mid μ_F(ω) = a_i, v_F(ω) = b_i\} \), for all \( i ∈ N_k \). Then we construct a PDA \( \mathcal{M}_1 = (Q_1, Σ, Γ, δ_1', q_{01}, Z_{01}, F_1) \), where the mapping \( δ' : Q × (Σ \cup \{ε\}) \times Γ \rightarrow 2^{Q \times Γ^*} \) is defined by

\[
δ'(q, τ, X) = \{(p, ω, γ) \mid μ_0(τ, q, X, p, γ) = 1, p ∈ Q, γ ∈ Γ^*\},
\]

for all \( (q, τ, X) ∈ Q × (Σ \cup \{ε\}) × Γ \).

Then \( \mathcal{L}(\mathcal{M}_1) = \mathcal{L}_1 = \{ω ∈ Σ^* \mid (q_0, ω, Z_0) >^*_F (q, ε, γ), q ∈ F_i\}, γ ∈ Γ^*\} \).

Therefore, for any \( ω ∈ Σ^* \), we have \( μ_{\mathcal{L}(\mathcal{M}_1)}(ω) = μ_{\mathcal{L}(\mathcal{M}_1)}(q) = μ_F(ω) = \bigwedge_{i=1}^n a_i \land δ'(q, ε, γ), q ∈ F_i, γ ∈ Γ^*\} \).

\[
\forall ω \in Σ^*.
\]

Therefore, for any \( ω ∈ Σ^* \), we have

\[
\mathcal{L}(\mathcal{M}_1)(ω) = \bigwedge_{i=1}^n a_i \land δ'(q, ε, γ), q ∈ F_i, γ ∈ Γ^*\}
\]

Therefore, for any \( ω ∈ Σ^* \), we have

\[
\mathcal{L}(\mathcal{M}_1)(ω) = \bigwedge_{i=1}^n a_i \land δ'(q, ε, γ), q ∈ F_i, γ ∈ Γ^*\}
\]

Therefore, for any \( ω ∈ Σ^* \), we have

\[
\mathcal{L}(\mathcal{M}_1)(ω) = \bigwedge_{i=1}^n a_i \land δ'(q, ε, γ), q ∈ F_i, γ ∈ Γ^*\}
\]

Therefore, for any \( ω ∈ Σ^* \), we have

\[
\mathcal{L}(\mathcal{M}_1)(ω) = \bigwedge_{i=1}^n a_i \land δ'(q, ε, γ), q ∈ F_i, γ ∈ Γ^*\}
\]

Therefore, for any \( ω ∈ Σ^* \), we have

\[
\mathcal{L}(\mathcal{M}_1)(ω) = \bigwedge_{i=1}^n a_i \land δ'(q, ε, γ), q ∈ F_i, γ ∈ Γ^*\}
\]

Therefore, for any \( ω ∈ Σ^* \), we have

\[
\mathcal{L}(\mathcal{M}_1)(ω) = \bigwedge_{i=1}^n a_i \land δ'(q, ε, γ), q ∈ F_i, γ ∈ Γ^*\}
\]

Therefore, for any \( ω ∈ Σ^* \), we have

\[
\mathcal{L}(\mathcal{M}_1)(ω) = \bigwedge_{i=1}^n a_i \land δ'(q, ε, γ), q ∈ F_i, γ ∈ Γ^*\}
\]

Therefore, for any \( ω ∈ Σ^* \), we have

\[
\mathcal{L}(\mathcal{M}_1)(ω) = \bigwedge_{i=1}^n a_i \land δ'(q, ε, γ), q ∈ F_i, γ ∈ Γ^*\}
\]

Therefore, for any \( ω ∈ Σ^* \), we have

\[
\mathcal{L}(\mathcal{M}_1)(ω) = \bigwedge_{i=1}^n a_i \land δ'(q, ε, γ), q ∈ F_i, γ ∈ Γ^*\}
\]

Therefore, for any \( ω ∈ Σ^* \), we have

\[
\mathcal{L}(\mathcal{M}_1)(ω) = \bigwedge_{i=1}^n a_i \land δ'(q, ε, γ), q ∈ F_i, γ ∈ Γ^*\}
\]
If \( \omega = \varepsilon \), then \( \mu_{\delta}(q_0, \varepsilon, Z_0), (q_0, \varepsilon, Z_0) \) \( \land \mu_{\delta}(q_0) \lor (V \{ q_0, \varepsilon, Z_0 \} \lor \mu_{\delta}(q_0, \varepsilon, Z_0) \lor \mu_{\delta}(q_0) \mid q_0 \in Q, Z_0 \in \Gamma, i \in N_\Gamma) = \mu_{\delta}(q_0) \lor (V \{ q_0, \varepsilon, Z_0 \} \lor \mu_{\delta}(q_0, \varepsilon, Z_0) \lor \mu_{\delta}(q_0) \mid q_0 \in Q, Z_0 \in \Gamma, i \in N_\Gamma) = \nu_{\delta}\), \(\nu_{\delta}\) is an IFS for \(\{ q_0, \varepsilon, Z_0 \} \). The condition (P2) is the following:

Firstly let us show that, for any \(\tau_1 \cdots \tau_n \in \Gamma^*\), \(\tau_i \in \Sigma \cup \{ \varepsilon \}, i \in \{1, \ldots, n\}, q_i \in Q,\)

\[
\mu_{\delta}(q_i, \omega, X_0, \omega, Z_0) = \begin{cases} 1, & \text{if \(P2\) is satisfied} \\ 0, & \text{otherwise} \end{cases}
\]

\[
\nu_{\delta}(q_i, \omega, X_0, \omega, Z_0) = \begin{cases} 0, & \text{if \(P2\) is satisfied} \\ 1, & \text{otherwise} \end{cases}
\]

where the condition (P2) is the following:

\[
\text{(P2) there exist } q_1, \ldots, q_n \in Q, Z_1, \ldots, Z_n \in \Gamma, 1, \ldots, n - 1 \in \Gamma, \text{ s.t. } a_i = \mu_{\delta}(q_i, \omega, Z_0), (q_i, \tau_i \cdots \tau_n, Z_1 Y_1) \land \mu_{\delta}(q_i, \omega, Z_0, (q_i, \tau_i \cdots \tau_n, Z_1 Y_1)) \land \cdots \land \mu_{\delta}(q_i, \omega, Z_0, (q_i, \tau_i \cdots \tau_n, Z_1 Y_1)) \land \cdots \land \mu_{\delta}(q_i, \omega, Z_0, (q_i, \tau_i \cdots \tau_n, Z_1 Y_1)) \lor 0 \text{ and } b_i = \nu_{\delta}(q_i, \omega, Z_0, (q_i, \tau_i \cdots \tau_n, Z_1 Y_1)) \lor \nu_{\delta}(q_i, \omega, Z_0, (q_i, \tau_i \cdots \tau_n, Z_2 Y_2)) \lor \cdots \lor \nu_{\delta}(q_i, \omega, Z_0, (q_i, \tau_i \cdots \tau_n, Z_{2n} Y_{2n})) = 0. \]

In fact, if (P2) is satisfied, then let \(\mu_{\delta}(q_1, \tau_1 \cdots \tau_n, Z_1 Y_1, (q_1, \tau_1 \cdots \tau_n, Z_{1 Y_1})), (q_1, \tau_1 \cdots \tau_n, Z_{1 Y_1}) = d_1, i = 1, \ldots, n - 2, \text{ where } y_i = \varepsilon. \) We have

\[
\mu_{\delta}(q_1, \tau_1 \cdots \tau_n, Z_1 Y_1, (q_1, \tau_1 \cdots \tau_n, Z_{1 Y_1})), (q_1, \tau_1 \cdots \tau_n, Z_{1 Y_1}) = 1, \quad \nu_{\delta}(q_1, \tau_1 \cdots \tau_n, Z_1 Y_1, (q_1, \tau_1 \cdots \tau_n, Z_{1 Y_1})), (q_1, \tau_1 \cdots \tau_n, Z_{1 Y_1}) = 0.
\]

Hence \(\mu_{\delta}(q_0, \omega, X_0, (q_0, a_0, b_0), \varepsilon, X_0) \geq \mu_{\delta}(q_0, \omega, X_0, (q_0, a_0, b_0), \varepsilon, X_0) = 1, \quad \nu_{\delta}(q_0, \omega, X_0, (q_0, a_0, b_0), \varepsilon, X_0) = 1.

Obviously, \(\delta' = (\mu_{\delta}, \nu_{\delta}) \) is a finite IFS in \(Q' \times (\Sigma \cup \{ \varepsilon \}) \times \Gamma' \times \Gamma', \)

Next, we show \(\mathcal{L}(M') = \mathcal{L}(M).\)
((q₀, a₀, b₀), ε, X₀) = 1 and ψ_x⁺((q₀', ω, X₀), ((q₀, a₀, b₀), ε, X₀)) = 0.

If (P2) is not satisfied, then we assume µ_x⁺((q₀', ω, X₀), (q₀, a₀, b₀), ε, X₀)) > 0.

So there at least exist q₁,...,qₙ₋₁ ∈ Γ, y₁, ..., yₙ₋₁, ∈ Γ, tᵢ, ..., tₙ₋₁ ∈ Γ’ s.t. µ_x⁺((q₀, 1, 0), w₀, Z₀X₀), (q₁, ε₀, d₀), t₁, ..., tₙ₋₁, yₙ₋₁, Y₀X₀) ∧ µ_y⁺((q₁, q₂, d₁), t₂, ..., tₙ₋₁, Z₀Y₀ X₀) ∧ ... ∧ µ_y⁺((qₙ₋₁, 1, 0), w₀, Z₀X₀) ∧ (qₙ, ε₀, d₀), tₙ, yₙ, Y₀X₀) > 0.

Hence aₙ = µ_x⁺((q₀, ω, Z₀), (q₁, t₂, ..., tₙ₋₁, Y₀), (q₂, t₃, ..., tₙ₋₁, Z₀, Y₀)) ∧ µ_y⁺((q₁, q₂, t₃, ..., tₙ₋₁, Y₀), (q₂, t₃, ..., tₙ₋₁, Z₀, Y₀)) ∧ ... ∧ µ_y⁺((qₙ₋₁, t₁, ..., tₙ₋₂, Y₀), (qₙ₋₁, t₁, ..., tₙ₋₂, Y₀), (qₙ₋₁, t₁, ..., tₙ₋₂, Y₀)) > 0

Then the language recognized by (P2) is a type of generator of intuitionistic fuzzy context-free languages, the notion of intuitionistic fuzzy context-free language recognized by an IFPDA (q, a₀, b₀, e, X₀).

Remark 21. Proposition 20 presents an equivalence of an IFPDAs. In particular, due to the underlying truth-valued domain being an IFS, the proof technique used in Proposition 20 is to some extent different from the technique of extended subset construction in [33]. Moreover, Proposition 20 plays an important role in proving the fact that any language recognized by an IFPDAs is an intuitionistic fuzzy recognizable function.

Proposition 22. Let M = (Q, Σ, Γ, δ, q₀, Z₀, 0) be an IFPDAs. Then the language recognized by M is an intuitionistic fuzzy recognizable step function over Σ*, which is 2^(Q × Γ × Σ) ∈ Step⁵(Σ).

Proof. Let M = (Q, Σ, Γ, δ, q₀, Z₀, 0) be an IFPDAs. Then there is a special IFPDAs M' = (Q', Σ, Γ', δ', q₀', X₀, 0) constructed by Proposition 20, which is equivalent to M. For any (a, b) ∈ {L₁ \ {0}} × {L₂ \ {1}} with 0 < a + b ≤ 1, construct a classical PDA with empty stack M_ab = (Q', Σ, Γ', δ', q₀', X₀, 0), where Q', Σ, Γ', q₀', X₀, 0 are the same as those in M', and the function δ_ab⁺ : Q' × Σ × Γ' → 2^(Q' × Σ) is defined by

(i) δ_ab⁺((q₀, 0, 1, 0), ε, X₀) = {{(q₀, 0, 1, 0), Z₀X₀}};

(ii) δ_ab⁺(q, c, d, X₀) = ((q', c ∧ c₁, d ∨ d₁), y) | c₁ = μ_q(q, r, X, q', y) > 0, d₁ = vₚ(q, r, X, q', y) < 1, q' ∈ Q, y ∈ Γ*;

(iii) δ_ab⁺(q, a, b, e, X₀) = ((q, a, b, e)).
Definition 25. An intuitionistic fuzzy grammar is a system $G = (N, T, P, I)$, where

1. $N$ is a finite nonempty alphabet of variables;
2. $T$ is a finite nonempty alphabet of terminals and $N \cup T = \emptyset$;
3. $I$ is an intuitionistic fuzzy set over $N$;
4. $P$ is a finite collection of productions over $T \cup N$, and $P = \{x \to y \mid x \in (N \cup T)^* N (N \cup T)^*, \mu_p(x \to y) > 0, \nu_p(x \to y) < 1\}$, where $\rho = (\mu, \nu)$ is an IFS over $(N \cup T)^* (N \cup T)^*$, $\mu_p(x, y)$, and $\nu_p(x, y)$ mean the membership degree and the nonmembership degree that $x$ will be replaced by $y$, respectively, denoted by $\mu_p(x, y) = \mu_p(x \to y), \nu_p(x, y) = \nu_p(x \to y)$.

For $\alpha, \beta \in (N \cup T)^*$, if $x \to y \in P$, then $\alpha \beta$ is said to be directly derivable from $\alpha x \beta$, denoted by $\alpha x \beta \Rightarrow \alpha \beta$, and define $\mu_p(\alpha x \beta) \Rightarrow \alpha \beta = \mu_p(x \to y), \nu_p(\alpha x \beta) \Rightarrow \alpha \beta = \nu_p(x \to y)$.

If $\alpha_1, \alpha_2, \ldots, \alpha_m \in (N \cup T)^*$ and $\alpha_1 \Rightarrow \alpha_2 \Rightarrow \cdots \Rightarrow \alpha_m \in P$, then $\alpha_1$ is said to derive $\alpha_m$ in G, or equivalently, $\alpha_1$ is derivable from $\alpha_m$ in G. This is expressed by $\alpha_1 \Rightarrow^*_G \alpha_m$ or simply $\alpha_1 \Rightarrow^* \alpha_m$. The expression $\alpha_1 \Rightarrow \alpha_2 \Rightarrow \cdots \Rightarrow \alpha_m$ is referred to as a derivation chain from $\alpha_1$ to $\alpha_m$.

An intuitionistic fuzzy grammar $G$ generates an intuitionistic fuzzy language $L(G) = (\mu_G, \nu_G)$ in the following manner. For any $\theta = \omega_n \in T^*, n \geq 1$, $\mu_G(\theta) = \bigvee \{\mu(\omega) \wedge \mu(\omega \to \omega_1) \wedge \cdots \wedge \mu(\omega \to \omega_{n-1} \to \omega_n) \mid \alpha \in N, \omega_1, \ldots, \omega_{n-1} \in (N \cup T)^*, \nu_G(\theta) = \bigwedge \{\nu(\omega) \vee \nu(\omega \to \omega_1) \cdots \vee \nu(\omega \to \omega_{n-1} \to \omega_n) \mid \omega \in N, \omega_1, \ldots, \omega_{n-1} \in (N \cup T)^*\}$.

$\mu_G(\theta)$ and $\nu_G(\theta)$ express the membership and nonmembership degree of $\theta$ in the language generated by grammar G, respectively. Obviously, $L(G) = (\mu_G, \nu_G)$ is well defined.

In fact, for any $\theta = \omega_n \in T^*, n \geq 1$, there is the strongest derivation from $\omega_0$ to $\omega_n$, that is $\omega_0 = \omega_0 \Rightarrow \omega_1 \Rightarrow \cdots \Rightarrow \omega_{n-1} \Rightarrow \omega_n$, such that $\mu_G(\theta) = \mu(\omega_0) \wedge \mu(\omega_0 \to \omega_1) \wedge \cdots \wedge \mu(\omega_0 \to \omega_{n-1} \to \omega_n)$.

For $x \in (N \cup T)^*$, $\nu_G(x) = \bigwedge \{\nu(\omega) \vee \nu(\omega \to \omega_1) \cdots \vee \nu(\omega \to \omega_{n-1} \to \omega_n) \mid \omega \in N, \omega_1, \ldots, \omega_{n-1} \in (N \cup T)^*\}$.

Proposition 26. Let $A$ be an IFS over $T^*$. Then the following statements are equivalent:

(i) $A$ is generated by a certain intuitionistic fuzzy grammar $G = (N, T, P, I)$;
(ii) $A$ is generated by a certain intuitionistic fuzzy grammar $G = (N', T', P', S)$.

Proof. (i) implies (ii). Let $A$ be generated by an intuitionistic fuzzy grammar $G = (N, T, P, I)$. Then we construct an intuitionistic fuzzy grammar $G' = (N', T', P', S)$ as follows: $N' = N \cup \{S\}, S \notin T'; T' = T; P' \cup P_1$, where $P_1 = \{S \to q \mid q \in supp(I), \mu(q, S \to q) = \mu(q), \nu(q, S \to q) = \nu(q)\}$. The new set of productions is $P' = P \cup P_1$.

Next we show that $L(G') = L(G)$. In fact, $G' = (N', T', P', I')$, where $I'$ is an IFS over $N', \mu_{I'}(S) = 1, \nu_{I'}(q) = 0 \mu_{I'}(q) = 0$ and $\nu_{I'}(q) = 1$ when $q \in N$.

For any $\theta = \omega_n \in T^*, n \geq 1$, $\mu_{I'}(\theta) = \bigvee \{\mu(\omega) \wedge \mu(\omega \to \omega_1) \wedge \cdots \wedge \mu(\omega \to \omega_{n-1} \to \omega_n) \mid \alpha \in N', \omega_1, \ldots, \omega_{n-1} \in (N' \cup T)^*\} = \bigvee \{\mu(\omega) \wedge \mu(\omega \to \omega_1) \wedge \cdots \wedge \mu(\omega \to \omega_{n-1} \to \omega_n) \mid \omega \in \omega_n \} \in (N' \cup T)^*$.

For any intuitionistic fuzzy grammars $G_1$ and $G_2$, if $L(G_1) = L(G_2)$, then the sets of rules $G_1$ and $G_2$ are said to be equivalent.

For any intuitionistic fuzzy grammar $G = (N, T, P, I)$, if $\text{Im}(I) = \text{Im}(\mu_I) \cup \text{Im}(\nu_I) = \{0, 1\}$ and $\text{supp}(I) = \{S\}$, then $G$ is also written as $G = (N, T, P, S)$. The proof is omitted.
natural numbers $d, l$ such that $μ_G(θ) = \bigvee\{ μ_I(𝑆) \land μ_ρ(𝑆 ⇒ 𝜔_1) \land μ_ρ(𝜔_1 ⇒ 𝜔_2) \land ⋅⋅⋅ \land μ_ρ(𝜔_{n-1} ⇒ 𝜔_n) \}$ and $V_C(θ) = \bigvee\{ V_S(𝑆) \land V_ρ(𝑆 ⇒ 𝜔) \land V_ρ(𝜔) \} \land V_ρ(𝜔_1) \lor ⋅⋅⋅ \lor V_ρ(𝜔_n)$, where $a_1 ∈ Im(μ_I) \land Im(μ_ρ)$ and $b_1 ∈ Im(𝜎) \lor Im(𝜔)$, $i ∈ \{ 1, 2, ⋅⋅⋅, d \}$, $j \in \{ 1, 2, ⋅⋅⋅, l \}$, $k \in \{ 0, 1, 2, ⋅⋅⋅, n \}$.

Let $X = Im(μ_I) \lor Im(μ_ρ)$ and $Y = Im(𝜎) \lor Im(𝜔)$. Then $X$ and $Y$ are finite subsets of the interval $[0,1]$. $(X_1)$ and $(Y_1)$ are also finite by Lemma 8. Since $μ_G(θ) ∈ (X_1)$ and $V_C(θ) ∈ (Y_1)$, for any $θ ∈ T^*$, we have $Im(μ_G) ≤ (X_1)$ and $Im(V_C) ≤ (Y_1)$. Hence $Im(Z(G)) = Im(μ_G) \lor Im(V_C)$ is finite.

Proposition 29. Let $G = (N, T, P, S)$ be an IFCFG. Then $L(G) ∈ Step^2(T)$.

Proof. Let $X = Im(μ_G)$ and $Y = Im(μ_ρ)$. Then $X$ and $Y$ have finite elements because $P$ is a finite collection of productions over $T \cup N$. Suppose $L_1 = X_1$ and $L_2 = Y_1$. Then $L_1$ and $L_2$ are finite by Lemma 8. For any $(a, b) \in (L_1 \setminus \{ 0 \}) \times (L_2 \setminus \{ 1 \})$, $0 ≤ a + b ≤ 1$, we construct a classical context-free grammar $G_{ab} = (N', T, P', S')$ as follows:

$N' = N \times (L_1 \setminus \{ 0 \}) \times (L_2 \setminus \{ 1 \})$, $S' = (S, 1, 0) \in N'$, $P'_{ab}$ consists of the form:

1. $(A, a_1, b_1) → D_1 \cdots D_k$ whenever $μ_I(A → τ_1 \cdots τ_k) > 0$ and $μ_ρ(A → τ_1 \cdots τ_k) < 1$, where

$$D_i = \begin{cases} (τ_i, a_2, b_2), & \text{if } i ∈ N \\ (τ_i), & \text{if } i ∈ T \end{cases}$$

for $i = 1, \ldots, k$; $a_2 = a_1 \land μ_ρ(A → τ_1 \cdots τ_k)$ and $b_2 = b_1 \lor μ_ρ(A → τ_1 \cdots τ_k)$; $(a, b, k) = (A, a_1, b_1)$, \text{for } a ≤ a_1 \land μ_ρ(A → x)$ and $(a, b, k) = (A, a_1, b_1)$, \text{for } b ≥ b_1 \lor μ_ρ(A → x)\land μ_ρ(A → τ_1 \cdots τ_k);

(2) $(A, a_1, b_1) → x$ whenever $a ≤ a_1 \land μ_ρ(A → x)$ and $b ≥ b_1 \lor μ_ρ(A → x)$.

Then $L(G) = \{ ω ∈ T^* | S'⇒^* G_{ab} ω \} = \{ ω ∈ T^* | a ≤ μ_G(S ⇒ u_1) \land μ_ρ(u_1 ⇒ u_2) \land ⋅⋅⋅ \land μ_ρ(u_{n-1} ⇒ u_n) \land ω \geq μ_G(S ⇒ u_1) \lor μ_ρ(u_1 ⇒ u_2) \lor ⋅⋅⋅ \lor μ_ρ(u_{n-1} ⇒ u_n) \lor ω \}$.

Next it suffices to show that $L(G) = \{ (a, b, k) ∈ G_{ab} | a = 0 \land k = 0 \}$, that is, $μ_G(ω) = \bigvee\{ a \in L_1 \} \land μ_ρ(ω)$ and $V_C(ω) = \bigvee\{ b \in L_2 \} \lor μ_ρ(ω)$, for all $ω ∈ T^*$.

Suppose $μ_G(ω) = a_0 > 0$. Then there exist $u'_1, \ldots, u'_{n-1} ∈ (N \cup T)^*$ such that $a_k = μ_G(S ⇒ u'_1) \land μ_ρ(u'_1 ⇒ u'_2) \land ⋅⋅⋅ \land μ_ρ(u'_{n-1} ⇒ u_n) \land ω$. Put $c = μ_G(S ⇒ u'_1) \lor μ_ρ(u'_1 ⇒ u'_2) \lor ⋅⋅⋅ \lor μ_ρ(u'_{n-1} ⇒ u_n) \lor ω$. Then $c < 1$ and $ω ∈ L(G)$. Therefore, $μ_G(ω) ≤ \bigvee\{ a \in L_1 \} \land μ_ρ(ω)$ and $V_C(ω) ≥ \bigvee\{ b \in L_2 \} \lor μ_ρ(ω)$, for all $ω ∈ T^*$.
Proof. (1) implies (5). Let $A \in \text{Step}^G(\Sigma)$. Then suppose $A = \bigcup_{i=1}^{k} (a_i, b_i) \cdot L_i$, where $a_i, b_i \in [0, 1]$, $a_i + b_i \leq 1$, $i \in N_k$, and $L_1, \ldots, L_k \in \Sigma^*$ are classical context-free languages.

If supp($A$) = $\{e\}$, then we construct an IFCFG $G = (N, \Sigma, P, I)$ as follows: $N = \{S\}$, $P = \{S \rightarrow e \mid \mu_p(S \rightarrow e) = \mu_p(e), v_p(S \rightarrow e) = v_p(e)\}$. Clearly, $\mathcal{L}(G) = A$.

If supp($A$) $\neq \{e\}$, then there is a Chomsky normal form grammar $G_i = (N_i, \Sigma, P_i, S_i)$ such that $\mathcal{L}(G_i) = \mathcal{L}_i \setminus \{e\}$, for any $\mathcal{L}_i$ with $\mathcal{L}_i \setminus e \neq \emptyset$ whenever $i \neq j$. Then we construct an IFSCFG $G = (N, \Sigma, P, I)$ according to the method constructed by Theorem 30, where $N = \bigcup_{i=1}^{k} N_i$, $P = \bigcup_{i=1}^{k} P_i$ and $I = (\mu_i, v_i)$ is an IFS over $N$, and the mappings $\mu_i : N \rightarrow [0, 1]$ are defined by $\mu_i(S_i) = a_i$, $\nu_i(S_i) = b_i$; $\mu_i(q) = 0$ and $v_i(q) = 1$ whenever $q \in N \setminus N_i, i = 1, \ldots, k$. Next, we construct an IFCFG $G^* = (N^*, \Sigma^*, P^*, S^*)$ as follows:

(i) $N^* = N \cup \{\varepsilon\}, S \notin N^*$;
(ii) $P^* = P \cup P^\alpha$, where $P^\alpha$ has the productions in the form of

$$(E1) \quad S \rightarrow e \text{ with } \mu_p(S \rightarrow e) = \mu_p(e), v_p(S \rightarrow e) = \mu_p(e) \text{ whenever } e \in \text{supp}(A);$$

$$(E2) \quad S \rightarrow BC \text{ with } \mu_p(S \rightarrow BC) = \mu_p(S) \text{ and } v_p(S \rightarrow BC) = v_p(S) \text{ whenever } S \in \text{supp}(I) \text{ and } S \rightarrow BC \in P_i, i = 1, \ldots, k;$$

$$(E3) \quad S \rightarrow \alpha \text{ with } \mu_p(S \rightarrow \alpha) = \bigvee \{\nu_{ij}(S_{ij}) \mid S_{ij} \in \text{supp}(I), S_{ij} \rightarrow \alpha \in P_i, i = 1, \ldots, k\} \text{ whenever } S_{ij} \in \text{supp}(I) \text{ and } S_{ij} \rightarrow \alpha \in P_i, i = 1, \ldots, k.$$

Then we have $\mathcal{L}(G^*) = (\mu_{G^*}, v_{G^*}) = A$. In fact, for any $\omega \in \Sigma^*$, if $\omega = e$, then $\mu_{G^*}(\omega) = \mu_p(e)$ and $v_{G^*}(\omega) = v_p(e)$; if $\omega \neq e$, then $\mu_{G^*}(\omega) = \bigvee \{\mu_p(S_{ij}) \rightarrow \omega \mid S_{ij} \in \text{supp}(I), S_{ij} \rightarrow \alpha \in P_i, i = 1, \ldots, k\}$ and $v_{G^*}(\omega) = \bigvee \{v_p(S_{ij}) \mid S_{ij} \in \text{supp}(I), S_{ij} \rightarrow \alpha \in P_i, i = 1, \ldots, k\}$.

We have $\mathcal{L}(G^*) = \mathcal{L}(G_1, \ldots, G_k) \subseteq \mathcal{L}(G) = A$.

Theorem 32. (1) The family $\text{Step}^\omega(\Sigma)$ is closed under the operations of union, scalar product, reversal, concatenation, and Kleene closure. That is, $A \cup B, (\lambda, \theta) A, A^\ast, AB, A^* \in \text{Step}^\omega(\Sigma)$, for any $A, B \in \text{Step}^\omega(\Sigma), \lambda, \theta \in [0, 1], 0 \leq \lambda + \theta \leq 1$.

(2) $h : \Sigma^* \rightarrow \Sigma^*$ is a homomorphism. If $A \in \text{Step}^\omega(\Sigma_2)$, then $h^{-1}(A) = A \cap h \in \text{Step}^\omega(\Sigma_1)$. (3) Let $h : \Sigma^* \rightarrow \Sigma^*$ be a homomorphism. If $h$ satisfies, for $\tau \in \Sigma, h(\tau) \neq \varepsilon$, and $g = (\mu_g(v_g)) \in \text{Step}^\omega(\Sigma_2)$, then $h(g) = (\mu_g(h(g))) \in \text{Step}^\omega(\Sigma_1)$, where $h_{\mu_g}(g) = \left(\bigvee \{\mu_\gamma(\alpha) \mid h(\alpha) = \alpha, \alpha \in \Sigma_1\}\right)$.

Proof. (1) Let $A, B \in \text{Step}^\omega(\Sigma)$. By Definition 17, we can assume $A = (\mu_A, v_A) = \bigcup_{i=1}^{k} (a_i, b_i) \cdot \xi_i, B = (\mu_B, v_B) = \bigcup_{j=1}^{k} (c_j, d_j) \cdot \eta_j$, where all $\xi_i$ and $\eta_j$ are classical context-free languages, $0 \leq a_i + b_i \leq 1, 0 \leq c_j + d_j \leq 1, a_i, b_i, c_j, d_j \in [0, 1], i \in N_k, j \in N_j$. With respect to the union, we have $A \cup B = (\mu_{A \cup B}, v_{A \cup B}) = (\mu_A \vee \mu_B, v_A \vee v_B)$, $A^\ast = (\mu_A \rightarrow^\ast, v_A \rightarrow^\ast)$, $AB = (\mu_{A \rightarrow \rightarrow^\ast}, v_{A \rightarrow \rightarrow^\ast})$.

With respect to the scalar product, for each $(\lambda, \theta) A = (\lambda \mu_A, \theta v_A), (\lambda \land \lambda \mu_A) = \lambda \land \lambda \mu_A = \lambda \land (\mu_A \land \mu_a) = \bigvee_{i=1}^{k} (\lambda \land \mu_A \land \mu_a), (\theta \lor \lambda v_A) = \theta \lor \lambda v_A = \theta \lor (\bigvee_{i=1}^{k} \lambda v_A) = (\bigvee_{i=1}^{k} \theta \lor \lambda v_A) = (\bigvee_{i=1}^{k} \theta \lor \lambda v_A), (\lambda \land \lambda v_A) = \lambda \land \lambda v_A = \lambda \land (\bigvee_{i=1}^{k} \lambda v_A) = (\bigvee_{i=1}^{k} \lambda v_A) = \bigvee_{i=1}^{k} (\lambda \land \theta v_A) = \theta \lor \lambda v_A$.

By Definition 17, $(\lambda, \theta) A \in \text{Step}^\omega(\Sigma)$. For the reversal operation $\mathcal{L}^{-1}(\xi_i) = \mathcal{L}(\xi_i \setminus \varepsilon) \in \text{IFCFG}(\Sigma)$. If $\mathcal{L}^{-1}(\xi_i)$ is a context-free language, then $\mathcal{L}^{-1}(\xi_i) \subseteq \text{IFCFG}(\Sigma)$. If $\mathcal{L}^{-1}(\xi_i)$ is a context-free language, then $\mathcal{L}^{-1}(\xi_i) \subseteq \text{IFCFG}(\Sigma)$.
For the Kleene closure, \( A^* = (\mu_A, \nu_A) \) is defined by
\[
\mu_A(\omega) = \bigvee \{ \mu_A(\omega_1) \land \cdots \land \mu_A(\omega_k) : k \geq 1, \omega = \omega_1 \cdots \omega_k \}
\]
for any \( \omega \in \Sigma^* \). Since \( A \in \text{Step}^2(\Sigma) \), we assume that the IFSPDA \( \mathcal{A} = (Q, \Sigma, \delta, q_0, Z_0, F) \) accepts \( A \) by Theorem 19. Let \( R = \{ (\mu_A(\omega), \nu_A(\omega)) : \omega \in \Sigma \} \). Then \( A = \bigcup_{\omega \in \Sigma} (a_1, b_1) \cdot j_\omega \), where \( \mathcal{L}(\omega) \) is a PDA of the form \( \mathcal{P} = (Q, \Gamma, \delta, \gamma, q_0, Z_0, F_Z) \), where \( \delta(q, \gamma, z_0) = (q', \gamma') \) if \( z_0 \in \Gamma \). Hence \( \mathcal{P} \) is a context-free language. It is easily verified that \( \mathcal{P}(A) = \bigcup_{\omega \in \Sigma} (a_1, b_1) \cdot j_{\omega} \). Since \( \mathcal{P} \) is context-free, \( A \in \text{Step}^2(\Sigma) \).

5. Pumping Lemma for IFCFLs

In this section, we mainly discuss the pumping lemma for IFCFLs, which will become a powerful tool for proving a certain intuitionistic fuzzy language non-context-free.

Theorem 33. Let \( A = (\mu_A, \nu_A) \) be an IFCFL over \( \Sigma^* \). Then there exists a finite natural number \( n \) such that for any \( \omega \in \Sigma^* \) with \( n \leq |\omega| \), there are \( u, v, w, x, y, u_1, v_1, w_1, x_1, y_1 \in \Sigma \) such that \( \omega = uvwxy = u_1v_1w_1x_1y_1 \), \( |vwx| \leq n \), \( |v_1x_1y_1| \leq n \), and \( \mu_A(u_1v_1w_1x_1y_1) \geq \mu_A(uvwx) \geq \mu_A(u_1v_1w_1x_1y_1) \) for all \( i \geq 0 \).

Proof. Let \( A = (\mu_A, \nu_A) \) be an IFCFL over \( \Sigma^* \). Then there is an IFCNF G = \( (N, T, P, S) \) which accepts \( A \). According to Proposition 29, \( \mathcal{L}(G) = \bigcup_{(a, b) \in \Sigma} (a, b) \cdot 1 \), where \( a, b \in \Sigma \). Then \( \mathcal{P} \) is context-free, \( A \in \text{Step}^2(\Sigma) \).

Next, let us look at an example to negate an intuitionistic fuzzy language to be an IFCFL.

Example 34. Let \( A = (\mu_A, \nu_A) \) be an IFS over \( T^* \). The mappings \( \mu_A, \nu_A : T^* \rightarrow [0, 1] \) are defined by
\[
\mu_A(z) = \begin{cases} 0.5, & \text{if } z = d^ib^jc^k \text{ (}i < j < k\text{)}, \\ 0, & \text{otherwise,} \end{cases}
\]
\[
\nu_A(z) = \begin{cases} 0.3, & \text{if } z = d^ib^jc^k \text{ (}i < j < k\text{)}, \\ 1, & \text{otherwise,} \end{cases}
\]

where \( i, j, k \) are natural numbers.

Suppose \( A \) is an IFCFL. Then there exists a certain IFCNF G such that \( \mathcal{L}(G) = A \). For constant \( n \), put \( z = a^n b^{n+1} c^{n+2} \). Hence, \( \mu_A(z) = \mu_{\mathcal{L}(G)}(z) = 0.5 \) and \( \nu_A(z) = \nu_{\mathcal{L}(G)}(z) = 0.3 \). Let \( z = uvwx \), where \( |vwx| \leq n \) and \( |uv| \geq 1 \). If \( uvw \) does not have \( c \)'s, then \( uvw^i \) has at least \( n + 2a \) or \( b \); if \( uvw \) has at least a \( c \), then it has not an \( a \) since \( |uv| \leq n \). And so \( uvw \) has \( a \)'s, but no more than \( 2n + 2b \) and \( c \)'s in total, that is, \( |uvw| \leq n + 2n + 2 \). Therefore, it is impossible that \( uvw \) has more \( b \)'s than \( a \)'s and also has more \( c \)'s than \( b \)'s. By calculation, we have \( \mu_{\mathcal{L}(G)}(uvw) = 0 \) and \( \nu_{\mathcal{L}(G)}(uvw) = 1 \). No matter how \( z \) is broken into \( uvw \), we have a contradiction with Theorem 33. Therefore, \( A \) is not an IFCFL.

The following example will show that intuitionistic fuzzy pushdown automata have more power than fuzzy pushdown automata when comparing two distinct strings although the degrees of membership of these strings recognized by the underlying fuzzy pushdown automata are equal.

Example 35. Let $\Sigma = \{0, 1\}$. Then $L = \{\omega \omega^{-1} \mid \omega \in \Sigma^*\} \subseteq \Sigma^*$ is clearly a context-free language but not a regular language by classical automata theory, where $\omega^{-1}$ represents the reversal of the string $\omega$. Given an IFPDA $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, I, Z_0, F)$ and a fuzzy pushdown automaton $\mathcal{N} = (Q, \Sigma, \Gamma, \sigma_0, Z_0, \sigma_1)$. Put $Q = \{q_0, q_1, q_2\}, \Gamma = \{Z_0, 0, 1\}$, an IFS $\delta = (\mu_\delta, \nu_\delta)$ is defined by

$$
\mu_\delta(q_0, 0, Z_0, q_0, 0Z_0) = 0.7, \nu_\delta(q_0, 0, Z_0, q_0, 0Z_0) = 0.2,
$$

$$
\mu_\delta(q_0, 1, Z_0, q_0, 1Z_0) = 0.6, \nu_\delta(q_0, 1, Z_0, q_0, 1Z_0) = 0.3,
$$

$$
\mu_\delta(q_0, 0, 0, Z_0, q_0, 0) = 0.3, \nu_\delta(q_0, 0, 0, Z_0, q_0, 0) = 0.6,
$$

$$
\mu_\delta(q_0, 0, 1, q_0, 01) = 0.3, \nu_\delta(q_0, 0, 1, q_0, 01) = 0.5,
$$

$$
\mu_\delta(q_0, 1, 0, q_0, 10) = 0.5, \nu_\delta(q_0, 1, 0, q_0, 10) = 0.4,
$$

$$
\mu_\delta(q_1, 0, 0, q_1, e) = 0.5, \nu_\delta(q_1, 0, 0, q_1, e) = 0.3,
$$

$$
\mu_\delta(q_1, 1, 0, q_1, e) = 0.5, \nu_\delta(q_1, 1, 0, q_1, e) = 0.35,
$$

$$
\mu_\delta(q_1, 1, Z_0, q_1, Z_0) = 1, \nu_\delta(q_1, 1, Z_0, q_1, Z_0) = 0,
$$

$$
\mu_\delta(q_1, 0, 0, q_1, 0) = 1, \nu_\delta(q_1, 0, 0, q_1, 0) = 0,
$$

$$
\mu_\delta(q_1, 1, q_1, 1) = 1, \nu_\delta(q_1, 1, q_1, 1) = 0,
$$

$$
\mu_\delta(q_1, e, Z_0, q_2, Z_0) = 1, \nu_\delta(q_1, e, Z_0, q_2, Z_0) = 0.
$$

Otherwise $\mu_\delta(q, r, Z, p, y) = 0$ and $\nu_\delta(q, r, Z, p, y) = 1$ for $(q, r, Z, p, y) \in Q \times (\Sigma \cup \{e\}) \times \Gamma \times Q \times \Gamma^*$.

The IFS $I = (\mu_I, \nu_I)$ and $F = (\mu_F, \nu_F)$ in $Q$ are defined by $\mu_I(q_0) = 1, \nu_I(q_0) = 0, \mu_I(q_1) = \mu_\delta(q_1) = 1, \nu_I(q_1) = \nu_\delta(q_1) = 0, \mu_F(q_0) = \mu_I(q_0) = 0$ and $\nu_F(q_0) = \nu_I(q_0) = 1$.

And set $\eta = \mu_\delta, \sigma_0 = \mu_\delta$, and $\sigma_1 = \mu_F$.

By computing with the strings, 010, 1110, 1010, 001100, and 101100e $\Sigma^*$, we have

$$
\mu_{\mathcal{F}}(010) = f_\mathcal{F}(010) = 0.6, \nu_{\mathcal{F}}(010) = 0.3,
$$

$$
\mu_{\mathcal{F}}(111) = f_\mathcal{F}(111) = 0.5, \nu_{\mathcal{F}}(111) = 0.35,
$$

$$
\mu_{\mathcal{F}}(01110) = f_\mathcal{F}(01110) = 0.5, \nu_{\mathcal{F}}(01110) = 0.4,
$$

$$
\mu_{\mathcal{F}}(10101) = f_\mathcal{F}(10101) = 0.3, \nu_{\mathcal{F}}(10101) = 0.5,
$$

$$
\mu_{\mathcal{F}}(0011100) = f_\mathcal{F}(0011100) = 0.6, \nu_{\mathcal{F}}(0011100) = 0.3,
$$

$$
\mu_{\mathcal{F}}(1011101) = f_\mathcal{F}(1011101) = 0.3, \nu_{\mathcal{F}}(1011101) = 0.5.
$$

This implies that $\mu_{\mathcal{F}}(01100)$ is the worst because the degree of nonmembership of $\nu_{\mathcal{F}}(01100)$ is smaller than the $\nu_{\mathcal{F}}(01100)$'s. Comparing the above five strings, 010 is the best and 001100 is the worst.

6. Conclusions

Taking intuitionistic fuzzy sets as the structures of truth values, we have investigated intuitionistic fuzzy context-free languages and established pumping lemma for the underlying languages. Firstly, the notions of intuitionistic fuzzy push-down automata (IFPDAs) and their recognizable languages are introduced and discussed in detail. Using the generalized subset construction method, we show that IFPDAs are equivalent to IFSPDAs and then prove that intuitionistic fuzzy step functions are the same as those accepted by IFPDAs. Furthermore, we have presented algebraic characterization of intuitionistic fuzzy recognizable languages including decomposition form and representation theorem. It follows that the languages accepted by IFPDAs are equivalent to those accepted by IFPDAs by classical automata theory. Secondly, we have introduced the notions of IFCFGs, IFCNFs, and IFGNFs. It is shown that they are equivalent in the sense that they generate the same classes of intuitionistic fuzzy context-free languages (IFCFGs). In particular, IFCFGs are proven to be an equivalence of IFPDAs as well. Then some operations on the family of IFCFLs are discussed. Finally pumping lemma for IFCFLs has been established. Thus, together with [38–40], we have more systematically established intuitionistic fuzzy automata theory as a generalization of fuzzy automata theory.

As mentioned in Section 1, IFS and fuzzy automata theory have supported a wealth of important applications in many fields. The next step is to consider the potential application of IFPDAs and IFCFLs such as in model checking and clinical monitoring. Additionally, many related researches in theories, such as IFPDAs based on the composition of t-norm and t-conorm and the minimal algorithm of IFPDAs, will be studied in the future.

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