Research Article

Existence Results for a $p(x)$ -Kirchhoff-Type Equation without Ambrosetti-Rabinowitz Condition

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We consider the existence and multiplicity of solutions for the $p(x)$ -Kirchhoff-type equations without Ambrosetti-Rabinowitz condition. Using the Mountain Pass Lemma, the Fountain Theorem, and its dual, the existence of solutions and infinitely many solutions were obtained, respectively.

1. Introduction

The Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right| dx \right) \frac{\partial^2 u}{\partial x^2} = 0 \quad (1)$$

was introduced by Kirchhoff [1] in the study of oscillations of stretched strings and plates, where ρ , ρ_0 , h , E , and L are constants. The stationary analogue of the Kirchhoff equation, that is, (1), is as follows:

$$-\left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(x, u). \quad (2)$$

After the excellent work of Lions [2], problem (2) has received more attention; see [3–10] and references therein.

The $p(x)$ -Laplace operator arises from various phenomena, for instance, the image restoration [11], the electro-rheological fluids [12], and the thermoconvective flows of non-Newtonian fluids [13, 14]. The study of the $p(x)$ -Laplace operator is based on the theory of the generalized Lebesgue space $L^{p(x)}(\Omega)$ and the Sobolev space $W^{m,p(x)}(\Omega)$, which we called variable exponent Lebesgue and Sobolev space. We refer the reader to [15–19] for an overview on the variable exponent Sobolev space, and to [20–29] for the study of the $p(x)$ -Laplacian-type equations.

Recently, there has been an increasing interest in studying the Kirchhoff equation involving the $p(x)$ -Laplace operator.

Autuori et al. [30, 31] have dealt with the nonstationary Kirchhoff-type equation involving the $p(x)$ -Laplacian of the form

$$\begin{aligned} u_{tt} - M \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \Delta_{p(x)} u \\ + Q(t, x, u, u_t) + f(x, u) = 0, \\ u_{tt} - M \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \Delta_{p(x)} u \\ + \mu |\nabla u|^{p(x)-2} u + Q(t, x, u, u_t) = f(t, x, u). \end{aligned} \quad (3)$$

In [32–35], applying variational method and Ambrosetti-Rabinowitz (AR) condition, Guowei Dai has studied the existence and multiplicity of solutions for the $p(x)$ -Kirchhoff-type equations with Dirichlet or Neumann boundary condition. In [36], by using (S_+) mapping theory and the Mountain Pass Lemma, Fan has discussed the nonlocal $p(x)$ -Laplacian Dirichlet problem with the nonvariational form

$$\begin{aligned} -A(u) \Delta_{p(x)} u &= B(u) f(x, u), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \quad (4)$$

and the $p(x)$ -Kirchhoff-type equation with the variational form

$$\begin{aligned} & -a \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \Delta_{p(x)} u \\ & = b \left(\int_{\Omega} F(x, u) dx \right) f(x, u), \quad \text{in } \Omega, \\ & u = 0, \quad \text{on } \partial\Omega, \end{aligned} \quad (5)$$

under (AR) condition, where A, B are two functionals defined on $W_0^{1,p(x)}(\Omega)$, and $F(x, t) = \int_0^t f(x, s) ds$.

Motivated by the above works, the purpose of this paper is to study the $p(x)$ -Kirchhoff-type equation

$$\begin{aligned} & - \left(a + b \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \Delta_{p(x)} u = f(x, u), \quad \text{in } \Omega, \\ & u = 0, \quad \text{on } \partial\Omega, \end{aligned} \quad (6)$$

without (AR) condition, where Ω is a smooth bounded domain in $\mathbb{R}^N$, a, b are two positive constants, $\Delta_{p(x)} u = \operatorname{div}(|\nabla u(x)|^{p(x)-2} \nabla u(x))$, $p \in C^{0,\beta}(\overline{\Omega})$ for some $\beta \in (0, 1)$, and

$$1 < p^- := \inf_{\Omega} p(x) \leq p^+ := \sup_{\Omega} p(x) < +\infty. \quad (7)$$

By taking the famous Mountain Pass Lemma, the Fountain Theorem, and its dual, we obtain the existence of solutions and infinitely many solutions for the $p(x)$ -Kirchhoff-type equation (6) under no (AR) condition.

2. Preliminary

We recall in this section some definitions and properties of variable exponent Lebesgue-Sobolev space. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$\begin{aligned} & L^{p(x)}(\Omega) \\ & = \left\{ u : u : \Omega \rightarrow \mathbb{R} \text{ is measurable, } \int_{\Omega} |u|^{p(x)} dx < \infty \right\} \end{aligned} \quad (8)$$

with the norm

$$|u|_{L^{p(x)}} = |u|_{p(x)} = \inf \left\{ \sigma > 0 : \int_{\Omega} \left| \frac{u}{\sigma} \right|^{p(x)} dx \leq 1 \right\}. \quad (9)$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\} \quad (10)$$

with the norm

$$\|u\|_{W^{1,p(x)}} = \|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}. \quad (11)$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. $|\nabla u|_{p(x)}$ is an equivalent norm on $W_0^{1,p(x)}(\Omega)$. In this paper we use the notation $\|u\| = |\nabla u|_{p(x)}$ for $u \in W_0^{1,p(x)}(\Omega)$. Define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases} \quad (12)$$

We refer the reader to [36–38] for the elementary properties of the space $W^{1,p(x)}(\Omega)$.

Proposition 1 (see [38]). *Set $\rho(u) = \int_{\Omega} |u(x)|^{p(x)} dx$. For any $u \in L^{p(x)}(\Omega)$, then the following are given:*

- (1) $|u|_{p(x)} = \lambda \Leftrightarrow \rho(u/\lambda) = 1$ if $u \neq 0$;
- (2) $|u|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$;
- (3) $|u|_{p(x)}^- \leq \rho(u) \leq |u|_{p(x)}^{p^+}$ if $|u|_{p(x)} > 1$;
- (4) $|u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^-$ if $|u|_{p(x)} < 1$;
- (5) $\lim_{k \rightarrow +\infty} |u_k|_{p(x)} = 0 \Leftrightarrow \lim_{k \rightarrow +\infty} \rho(u_k) = 0$;
- (6) $\lim_{k \rightarrow +\infty} |u_k|_{p(x)} = +\infty \Leftrightarrow \lim_{k \rightarrow +\infty} \rho(u_k) = +\infty$.

3. Positive Energy Solution

In this section we discuss the existence of weak solutions of (6). For simplicity we write $X = W_0^{1,p(x)}(\Omega)$.

First, we state the assumptions on f as follows.

(f_0) Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, and there exist positive constants c_1, c_2 such that

$$|f(x, t)| \leq c_1 + c_2 |t|^{\alpha(x)-1}, \quad (13)$$

where $\alpha \in C(\overline{\Omega})$ and $1 < \alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

(f'_0) Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, and there exist positive constants c_1, c_2 such that

$$|f(x, t)| \leq c_1 + c_2 |t|^{\alpha(x)-1}, \quad (14)$$

where $\alpha \in C(\overline{\Omega})$ and $p^+ < \alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$; $tf(x, t) \geq 0$ for all $t > 0$.

(f_1) Let $\lim_{t \rightarrow +\infty} (F(x, t)/|t|^{2p^+}) = +\infty$, uniformly for $x \in \overline{\Omega}$, where $F(x, t) = \int_0^t f(x, s) ds$.

(f_2) There exists $\theta \geq 1$ such that $\theta G(x, t) \geq G(x, st)$ for $(x, t) \in \Omega \times \mathbb{R}$ and $s \in [0, 1]$, where

$$G(x, t) = tf(x, t) - 2p^+ F(x, t). \quad (15)$$

(f_3) Let $\lim_{t \rightarrow 0} (F(x, t)/|t|^{p^+}) = 0$, uniformly on $x \in \overline{\Omega}$.

(f'_3) There exists $\delta > 0$, such that $F(x, t) \leq 0$ for $x \in \overline{\Omega}$, $|t| < \delta$.

(f_4) Let $f(x, t) = -f(x, -t)$ for $x \in \Omega$ and $t \in \mathbb{R}$.

(f_5) Let $\lim_{t \rightarrow 0} (F(x, t)/|t|^q) = 0$, uniformly on $x \in \bar{\Omega}$, where $q \in C(\bar{\Omega})$ satisfies $1 < q(x) < p(x)$ for $x \in \bar{\Omega}$.

Remark 2. Condition (f_2) was first introduced by Jeanjean [39] for the case $p(x) = 2$. Let $f(x, t) = 2p^+|t|^{2p^+-2}t \ln|t|$, then

$$F(x, t) = |t|^{2p^+} \ln|t| - \frac{1}{2p^+}|t|^{2p^+}, \quad G(x, t) = |t|^{2p^+}. \quad (16)$$

It is easy to see that the function f does not satisfy (AR) condition, but it satisfies (f_1)-(f_5) and (f'_3).

Define $I(u) = J(u) - \Phi(u)$, where

$$J(u) = \left(a + \frac{b}{2} \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx, \\ \Phi(u) = \int_{\Omega} F(x, u) du. \quad (17)$$

Then $I \in C^1(X, \mathbb{R})$.

Proposition 3 (see [38]). Assume that (f_0) hold, then the functional $J : X \rightarrow \mathbb{R}$ is sequentially weakly lower semicontinuous, $\Phi : X \rightarrow \mathbb{R}$ is sequentially weakly continuous, and I is sequentially weakly lower semicontinuous.

Proposition 4 (see [37]). Assume that (f_0) hold, and let $u_0 \in W_0^{1,p(x)}(\Omega)$ be a local minimizer (resp., a strictly local minimizer) of I in the $C^1(\bar{\Omega})$ topology. Then u_0 is a local minimizer (resp., a strictly local minimizer) of I in the $W_0^{1,p(x)}(\Omega)$ topology.

Definition 5. We say that $u \in X$ is a weak solution of (6), if

$$\left(a + b \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx \\ = \int_{\Omega} f(x, u) v dx \quad (18)$$

for any $v \in X$.

Definition 6. Let X be a Banach space and $I \in C^1(X, \mathbb{R})$. Given $c \in \mathbb{R}$. we say that I satisfies the Cerami c condition (we denote by $(C)_c$ the condition), if

- (i) any bounded sequence $\{u_n\} \subset X$ such that $I(u_n) \rightarrow c$ and $I'(u_n) \rightarrow 0$ has a convergent subsequence;
- (ii) there exist constants $\delta, R, \beta > 0$ such that

$$\|u\| \|I'(u)\| \geq \beta, \quad \forall u \in I^{-1}[c - \delta, c + \delta], \quad \|u\| \geq R. \quad (19)$$

If $I \in C^1(X, \mathbb{R})$ satisfies $(C)_c$ condition for every $c \in \mathbb{R}$, then we say that I satisfies (C) condition.

Remark 7. Although (PS) condition is stronger than (C) condition, the Deformation Theorem is still valid under (C) condition; we see that the Mountain Pass Lemma, the Fountain Theorem, and its dual are true under (C) condition.

Lemma 8. Assume that conditions (f_0)-(f_2) hold. Then I satisfies (C) condition.

Proof. From [36, Proposition 3.1], I satisfies (i) of (C) condition.

Now we check that I satisfies (ii) of (C) condition. Arguing by contradiction, we may assume that, for some $c \in \mathbb{R}$,

$$I(u_n) \rightarrow c, \quad \|u_n\| \rightarrow \infty, \quad \|u_n\| \|I'(u_n)\| \rightarrow 0. \quad (20)$$

Then we have

$$\lim_{n \rightarrow \infty} \left\{ a \int_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{2p^+} \right) |\nabla u|^{p(x)} dx \right. \\ \left. + \frac{b}{2} \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right. \\ \left. \times \int_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{p^+} \right) |\nabla u|^{p(x)} dx \right. \\ \left. + \frac{1}{2p^+} \int_{\Omega} G(x, u) dx \right\} \\ = \lim_{n \rightarrow \infty} \left\{ I(u_n) - \frac{1}{2p^+} \langle I'(u_n), u_n \rangle \right\} = c. \quad (21)$$

Let $v_n = u_n / \|u_n\|$, then up to a subsequence we may assume that

$$v_n \rightharpoonup v \quad \text{in } X, \\ v_n \rightarrow v \quad \text{in } L^{\alpha(x)}(\Omega), \\ v_n(x) \rightarrow v(x) \quad \text{a.e. } x \in \Omega. \quad (22)$$

If $v = 0$, inspired by [13, 14], then we define

$$I(t_n u_n) = \max_{t \in [0,1]} I(t u_n). \quad (23)$$

For any $m > 1/2p^+$, let $w_n = (2mp^+)^{1/p^-} v_n$. Since $w_n \rightarrow 0$ in $L^{\alpha(x)}(\Omega)$ and

$$|F(x, t)| \leq c_5 + c_6 |t|^{\alpha(x)}, \quad (24)$$

by the continuity of $F(x, \cdot)$, $F(\cdot, w_n) \rightarrow 0$ in $L^1(\Omega)$, thus,

$$\lim_{n \rightarrow 0} \int_{\Omega} F(\cdot, w_n) dx = 0. \quad (25)$$

Then for n large enough, $(2mp^+)^{1/p^-} / \|u_n\| \in (0, 1)$ and

$$\begin{aligned}
 I(t_n u_n) &\geq I(w_n) \\
 &= a \int_{\Omega} \frac{1}{p(x)} |\nabla w_n|^{p(x)} dx \\
 &\quad + \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla w_n|^{p(x)} dx \right)^2 - \int_{\Omega} F(x, w_n) dx \\
 &= a \int_{\Omega} \frac{1}{p(x)} \left((2mp^+)^{1/p^-} |\nabla v_n| \right)^{p(x)} dx \\
 &\quad + \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} \left((2mp^+)^{1/p^-} |\nabla v_n| \right)^{p(x)} dx \right)^2 \\
 &\quad - \int_{\Omega} F(x, w_n) dx \\
 &\geq \frac{2ma}{p^+} \int_{\Omega} |\nabla v_n|^{p(x)} dx \\
 &\quad + \frac{2m^2 b}{(p^+)^2} \left(\int_{\Omega} |\nabla v_n|^{p(x)} dx \right)^2 - \int_{\Omega} F(x, w_n) dx \\
 &\geq \frac{2ma}{p^+} + \frac{2m^2 b}{(p^+)^2} - \int_{\Omega} F(x, w_n) dx.
 \end{aligned} \tag{26}$$

That is, $I(t_n u_n) \rightarrow \infty$. From $I(0) = 0$ and $I(u_n) \rightarrow c$, we know that $t_n \in (0, 1)$ and

$$\begin{aligned}
 &a \int_{\Omega} |\nabla t_n u_n|^{p(x)} dx \\
 &\quad + b \left(\int_{\Omega} \frac{1}{p(x)} |\nabla t_n u_n|^{p(x)} dx \right) \int_{\Omega} |\nabla t_n u_n|^{p(x)} dx \\
 &\quad - \int_{\Omega} f(x, t_n u_n) u_n dx \\
 &= \langle I'(t_n u_n), t_n u_n \rangle = t_n \frac{d}{dt} \Big|_{t=t_n} I(tu_n) = 0.
 \end{aligned} \tag{27}$$

Therefore, from (f_2) , we have

$$\begin{aligned}
 &a \int_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{2p^+} \right) |\nabla u_n|^{p(x)} dx \\
 &\quad + \frac{b}{2} \int_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \\
 &\quad \times \int_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{p^+} \right) |\nabla u_n|^{p(x)} dx \\
 &\quad + \frac{1}{2p^+} \int_{\Omega} G(x, u_n) dx
 \end{aligned}$$

$$\begin{aligned}
 &\geq a \int_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{2p^+} \right) |\nabla u_n|^{p(x)} dx \\
 &\quad + \frac{b}{2} \int_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \\
 &\quad \times \int_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{p^+} \right) |\nabla u_n|^{p(x)} dx \\
 &\quad + \frac{1}{2p^+} \int_{\Omega} \frac{G(x, t_n u_n)}{\theta} dx \\
 &\geq \frac{a}{\theta} \int_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{2p^+} \right) t_n^{p(x)} |\nabla u_n|^{p(x)} dx \\
 &\quad + \frac{b}{2\theta} \int_{\Omega} \frac{1}{p(x)} t_n^{p(x)} |\nabla u_n|^{p(x)} dx \\
 &\quad \times \int_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{p^+} \right) t_n^{p(x)} |\nabla u_n|^{p(x)} dx \\
 &\quad + \frac{1}{2p^+} \int_{\Omega} \frac{G(x, t_n u_n)}{\theta} dx \\
 &= \frac{1}{\theta} \left(I(t_n u_n) - \frac{1}{2p^+} \langle I'(t_n u_n), t_n u_n \rangle \right) \\
 &= \frac{1}{\theta} I(t_n u_n) \rightarrow \infty.
 \end{aligned} \tag{28}$$

This contradicts (21).

If $v \neq 0$, from (20), when $\|u_n\| \geq 1$,

$$\frac{a}{p^-} \|u_n\|^{p^+} + \frac{b}{2(p^-)^2} \|u_n\|^{2p^+} - (c + o(1)) \geq \int_{\Omega} F(x, u_n) dx. \tag{29}$$

Then from (f_1) we have

$$\begin{aligned}
 &\frac{a}{p^-} \frac{1}{\|u_n\|^{p^+}} + \frac{b}{2(p^-)^2} - \frac{c + o(1)}{\|u_n\|^{2p^+}} \\
 &\geq \int_{\Omega} \frac{F(x, u_n)}{\|u_n\|^{2p^+}} dx \\
 &= \left(\int_{v_n \neq 0} + \int_{v_n = 0} \right) \frac{F(x, u_n)}{|u_n|^{2p^+}} |v_n|^{2p^+} dx \\
 &= \int_{v_n \neq 0} \frac{F(x, u_n)}{|u_n|^{2p^+}} |v_n|^{2p^+} dx.
 \end{aligned} \tag{30}$$

For $x \in \Theta := \{x \in \Omega : v(x) \neq 0\}$, $|u_n(x)| \rightarrow +\infty$. By (f_1) we have

$$\frac{F(x, u_n)}{|u_n|^{p^+}} |v_n|^{p^+} \rightarrow +\infty. \tag{31}$$

Note that the Lebesgue measure of Θ is positive; using the Fatou Lemma, we have

$$\int_{v_n \neq 0} \frac{F(x, u_n)}{|u_n|^{2p^+}} |v_n|^{2p^+} dx \longrightarrow +\infty. \quad (32)$$

This contradicts (30).

The technique used in this lemma was first introduced by [39, 40]. $\square$

Theorem 9. Assume that conditions (f_0) – (f_2) and (f_3) (or (f'_3)) hold. Then (6) has a nontrivial solution with positive energy.

Proof. From Lemma 8, I satisfies (C) condition. Let us show that the functional I has a Mountain-Pass-type geometry.

Note that $I(0) = 0$. By (f_3) , there exists $\delta > 0$, and for any $u \in X$ with $|u|_{L^\infty(\Omega)} < \delta$,

$$\begin{aligned} I(u) &= a \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \\ &\quad + \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \int_{\Omega} F(x, u) dx \\ &\geq \frac{a}{p^+} \|u\|^{p^+} + \frac{b}{(p^+)^2} \|u\|^{2p^+} - \int_{\Omega} F(x, u) dx > 0. \end{aligned} \quad (33)$$

This shows that 0 is a strictly local minimizer of I in the $C(\overline{\Omega})$ topology, and hence 0 is a strictly local minimizer of I in the $C^1(\overline{\Omega})$ topology. By [37, Theorem 1.1], 0 is a strictly local minimizer of I in the $W_0^{1,p(x)}(\Omega)$ topology. Thus there exists $r > 0$ such that $I(u) > 0$ for every $u \in X \setminus \{0\}$ with $\|u\| \leq r$.

We claim that $\inf_{\|u\|=r} I(u) > 0$. To prove this claim, arguing by contradiction, assume that there exists a sequence $\{u_n\} \subset X$ with $\|u_n\| = r$ such that $I(u_n) \rightarrow 0$ as $n \rightarrow \infty$. We may assume that $u_n \rightharpoonup u_0$ in X . Since I is sequentially weakly lower semicontinuous, we have that $I(u_0) = 0$, and hence $u_0 = 0$. Since Φ is sequentially weakly continuous, then we have that $\Phi(u_n) \rightarrow \Phi(0) = 0$, and hence $J(u_n) = I(u_n) + \Phi(u_n) \rightarrow 0$. It follows from this that $u_n \rightarrow 0$ in X which contradicts with $\|u_n\| = r$.

Let $y \in X$ with $y > 0$ in Ω and $\|y\| = 1$. By (f_0) and (f_1) , for $s \geq 1$ we have

$$\begin{aligned} I(sy) &= a \int_{\Omega} \frac{1}{p(x)} |\nabla sy|^{p(x)} dx \\ &\quad + b \left(\int_{\Omega} \frac{1}{p(x)} |\nabla sy|^{p(x)} dx \right)^2 - \int_{\Omega} F(x, sy) dx \\ &\leq \frac{a}{p^-} s^{p^+} + \frac{b}{(p^-)^2} s^{2p^+} \\ &\quad - c_1 s^{2p^+} \int_{\Omega} |y|^{2p^+} dx + c_2 \longrightarrow -\infty \quad \text{as } s \longrightarrow +\infty. \end{aligned} \quad (34)$$

We set $e = sy$. Then for s large, we obtain

$$\|e\| > r, \quad I(e) < 0. \quad (35)$$

Hence by the famous Mountain Pass Lemma, problem (6) has a nontrivial weak solution with positive energy. $\square$

4. Infinitely Many Solutions

Since X is a reflexive and separable Banach space, then there exists $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$\begin{aligned} X &= \overline{\text{span} \{e_j : j = 1, 2, \dots\}}, \\ X^* &= \overline{\text{span} \{e_j^* : j = 1, 2, \dots\}}, \end{aligned} \quad (36)$$

$$\langle e_i, e_j^* \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For convenience, we write $X_j = \text{span}\{e_j\}$, $Y_k = \oplus_{j=1}^k X_j$, $Z_k = \overline{\oplus_{j=k}^{\infty} X_j}$.

Lemma 10 (see [21]). If $\alpha \in C(\overline{\Omega})$, $1 < \alpha(x) < p^*$ for any $x \in \overline{\Omega}$, denote

$$\beta_k = \sup \{ |u|_{\alpha(x)} : \|u\| = 1, u \in Z_k \}. \quad (37)$$

Then $\lim_{k \rightarrow +\infty} \beta_k = 0$.

Proposition 11 (Fountain Theorem). Assume that $I \in C^1(X, \mathbb{R})$ is an even functional. If, for any $k \in \mathbb{N}$, there exists $\rho_k > r_k > 0$ such that

- (A_1) $a_k = \max_{u \in Y_k, \|u\|=\rho_k} I(u) \leq 0$,
- (A_2) $b_k = \inf_{u \in Z_k, \|u\|=r_k} I(u) \rightarrow +\infty$ as $k \rightarrow \infty$,
- (A_3) I satisfies $(C)_c$ condition for every $c > 0$, then I has an unbounded sequence of critical values.

Proposition 12 (Dual Fountain Theorem). Assume that $I \in C^1(X, \mathbb{R})$ is an even functional. If, for any $k \geq k_0$, there exists $\rho_k > r_k > 0$ such that

- (B_1) $a_k = \inf_{u \in Z_k, \|u\|=\rho_k} I(u) \geq 0$,
- (B_2) $b_k = \max_{u \in Y_k, \|u\|=r_k} I(u) < 0$,
- (B_3) $d_k = \inf_{u \in Z_k, \|u\|\leq\rho_k} I(u) \rightarrow 0$ as $k \rightarrow \infty$,
- (B_4) I satisfies $(c)_c^*$ condition for every $c \in [d_{k_0,0}]$, then I has a sequence of negative critical values converging to 0.

Theorem 13. Assume that the conditions (f'_0) , (f_1) – (f_4) hold. Then (6) has infinitely many solutions $\{u_k\}$ such that $I(u_k) \rightarrow \infty$ as $k \rightarrow \infty$.

Proof. By conditions (f'_0) , (f_1) , and (f_3) , for any $\varepsilon > 0$, there exists C_ε such that

$$F(x, u) \geq C_\varepsilon |u|^{2p^+} - \varepsilon |u|^{p^+}, \quad \forall (x, u) \in \Omega \times \mathbb{R}. \quad (38)$$

For $u \in Y_k$, when $\|u\| > 1$,

$$\begin{aligned} I(u) &= a \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \\ &\quad + \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \int_{\Omega} F(x, u) dx \\ &\leq \frac{a}{p^-} \|u\|^{p^+} + \frac{b}{2(p^-)^2} \|u\|^{2p^+} \\ &\quad - C_{\varepsilon} |u|_{2p^+}^{2p^+} + \varepsilon |u|_{p^+}^{p^+} \longrightarrow -\infty \quad \text{as } \|u\| \longrightarrow +\infty. \end{aligned} \quad (39)$$

Then for some $\rho_k > 0$ large enough,

$$a_k := \max_{u \in Y_k, \|u\| = \rho_k} I(u) \leq 0. \quad (40)$$

On the other hand, by (f'_0) and (f_3) , there exists $C_{\varepsilon} > 0$ such that

$$|F(x, u)| \leq \varepsilon |u|^{p^+} + C_{\varepsilon} |u|^{\alpha(x)}, \quad \forall (x, u) \in \Omega \times \mathbb{R}. \quad (41)$$

Let $\beta_k := \sup_{u \in Z_k, \|u\| = \rho_k} |u|_{\alpha^-}$. From Lemma 10, $\beta_k \rightarrow 0$ as $k \rightarrow \infty$. For $u \in Z_k$, when $\|u\| \leq 1$ and ε small enough,

$$\begin{aligned} I(u) &= a \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \\ &\quad + \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \int_{\Omega} F(x, u) dx \\ &\geq \frac{a}{p^+} \|u\|^{p^-} + \frac{b}{2(p^+)^2} \|u\|^{2p^-} - C_{\varepsilon} |u|_{\alpha^-}^{\alpha^-} - \varepsilon |u|_{p^+}^{p^+} \\ &\geq \frac{a}{2p^+} \|u\|^{p^-} - c |u|_{\alpha^-}^{\alpha^-} \\ &\geq \frac{a}{2p^+} \|u\|^{p^-} - c \beta_k \|u\|^{\alpha^-}. \end{aligned} \quad (42)$$

If we choose $r_k := (a/4cp^+ \beta_k^{\alpha^-})^{1/(\alpha^- - p^-)} \rightarrow \infty$ as $k \rightarrow \infty$, then, for $u \in Z_k$ with $\|u\| = r_k$,

$$I(u) \geq \frac{a}{4p^+} \left(\frac{a}{4cp^+ \beta_k^{\alpha^-}} \right)^{p^-/(\alpha^- - p^-)} := \bar{b}_k, \quad (43)$$

which implies that $b_k := \inf_{u \in Z_k, \|u\| = r_k} I(u) \geq \bar{b}_k \rightarrow +\infty$ as $k \rightarrow \infty$. $\square$

Theorem 14. Assume that conditions (f'_0) , (f_1) , (f_2) , (f_4) , and (f_5) hold. Then (6) has infinitely many solutions $\{u_k\}$ such that $I(u_k) \rightarrow 0$ as $k \rightarrow \infty$.

Proof. By conditions (f'_0) , (f_1) , and (f_5) , for any $\varepsilon > 0$, there exists C_{ε} such that

$$F(x, u) \geq C_{\varepsilon} |u|^{2p^+} - \varepsilon |u|^{q^+}, \quad \forall (x, u) \in \Omega \times \mathbb{R}. \quad (44)$$

For $u \in Y_k$, when $\|u\|$ is large enough,

$$\begin{aligned} I(u) &= a \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \\ &\quad + \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \int_{\Omega} F(x, u) dx \\ &\leq \frac{a}{p^-} \|u\|^{p^+} + \frac{b}{(p^-)^2} \|u\|^{2p^+} \\ &\quad - C_{\varepsilon} |u|_{2p^+}^{2p^+} + \varepsilon |u|_{q^+}^{q^+} \longrightarrow -\infty \\ &\quad \text{as } \|u\| \longrightarrow +\infty. \end{aligned} \quad (45)$$

Then for some $r_k > 0$ large enough,

$$b_k := \max_{u \in Y_k, \|u\| = r_k} I(u) < 0. \quad (46)$$

On the other hand, by (f_5) , there exists $C_{\varepsilon} > 0$ such that

$$|F(x, u)| \leq \varepsilon |u|^{q^-} + C_{\varepsilon} |u|^{\alpha(x)}, \quad \forall (x, u) \in \Omega \times \mathbb{R}. \quad (47)$$

Let $\beta_k := \sup_{u \in Z_k, \|u\| = r_k} |u|_{q^-}$, then $\beta_k \rightarrow 0$ as $k \rightarrow \infty$. For $u \in Z_k$, when $\|u\|$ and ε small enough,

$$\begin{aligned} I(u) &= a \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \\ &\quad + \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \int_{\Omega} F(x, u) dx \\ &\geq \frac{a}{p^+} \|u\|^{p^+} - c C_{\varepsilon} |u|_{\alpha^-}^{\alpha^-} - c \varepsilon |u|_{q^+}^{q^+} \\ &\geq \frac{a}{2p^+} \|u\|^{p^+} - c |u|_{q^+}^{q^+} \\ &\geq \frac{a}{2p^+} \|u\|^{p^+} - c \beta_k^{q^+} \|u\|^{q^+}. \end{aligned} \quad (48)$$

If we choose $\rho_k := (4cp^+ \beta_k^{q^+}/a)^{1/(p^+ - q^+)} \rightarrow 0$ as $k \rightarrow \infty$, then, for $u \in Z_k$ with $\|u\| = \rho_k$,

$$I(u) \geq c \beta_k^{q^+} \left(\frac{4cp^+ \beta_k^{q^+}}{a} \right)^{q^+/(p^+ - q^+)} := \bar{a}_k, \quad (49)$$

which implies that $a_k := \inf_{u \in Z_k, \|u\| = \rho_k} I(u) \geq \bar{a}_k \rightarrow 0$ as $k \rightarrow \infty$.

Furthermore, if $u \in Z_k$ with $\|u\| \leq \rho_k$, then

$$I(u) \geq -c \beta_k^{q^-} \rho_k^{q^-} \rightarrow 0 \quad \text{as } k \rightarrow \infty, \quad (50)$$

which implies that $d_k = \inf_{u \in Z_k, \|u\| \leq \rho_k} I(u) \rightarrow 0$ as $k \rightarrow \infty$. $\square$

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