Research Article

Soliton Solutions for Quasilinear Schrödinger Equations

Junheng Qu

Information Science and Department of Mathematics, Science School of Foshan University, Foshan, Guangdong 528000, China

Correspondence should be addressed to Junheng Qu; ququ0315@hotmail.com

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By using a change of variables, we get new equations, whose respective associated functionals are well defined in $H^1(\mathbb{R}^N)$ and satisfy the geometric hypotheses of the mountain pass theorem. Using this fact, we obtain a nontrivial solution.

1. Introduction

We study the existence of solutions for the following quasilinear Schrödinger equations:

$$-\Delta u + V(x)u - \left[\Delta(1+u^2)^{\alpha/2} \right] \frac{\alpha u}{2(1+u^2)^{(2-\alpha)/2}} = u^q + u^p, \quad x \in \mathbb{R}^N, \quad (1)$$

where $V \in C(\mathbb{R}^N, \mathbb{R}^+)$ is bounded and periodic in each variable of x_i , $1 \leq i \leq N$, $N \geq 3$, $T(\alpha) < q+1 < p+1 < \alpha 2^*$, $\alpha 2^* := 2\alpha N/(N-2)$, $\alpha \geq 1$, and here

$$T(\alpha) := \begin{cases} 2\alpha, & \alpha_0 \leq \alpha, \\ 2\alpha_0, & 1 < \alpha < \alpha_0, \\ 12 - 4\sqrt{6}, & \alpha = 1, \end{cases} \quad (2)$$

where α_0 is defined in Lemma 2. These equations are related to existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$iz_t = -\Delta z + W(x)z - h(|z|^2)z - \Delta g(|z|^2)g'(|z|^2)z, \quad x \in \mathbb{R}^N, \quad (3)$$

where W is a given potential and g and h are real functions. Quasilinear equations such as (3) have been accepted as models of several physical phenomena corresponding to various

types of g . The case of $g(s) = s^\alpha$ was used for the superfluid film equation in plasma physics [1]. Besides, (3) also appears in plasma physics and fluid mechanics [2], in dissipative quantum mechanics [3], and in the theory of Heisenberg ferromagnetism and magnons [4, 5]. See also [6, 7] for more physical backgrounds. Equations (3) with $\alpha = 1$ have been studied extensively recently; see [8, 9]. When $g(s) = (1+s)^{\alpha/2}$, then (3) turn into our equations (1) with $h(s) = s^q + s^p$. In particular if we let $\alpha = 1$, that is, $g(s) = (1+s)^{1/2}$, (3) models the self-channeling of a high-power ultrashort laser in matter [10]. In this case, few results are known. In [11], the authors proved global existence and uniqueness of small solutions in transverse space dimensions 2 and 3 and local existence without any smallness condition in transverse space dimension 1. In [12], the authors proved the existence of nontrivial solution. When $\alpha > 1$, although we do not know the physical background of (3), in a mathematical sense, we give the proof of the existence of nontrivial solution.

For (1), the main difficulty is that the energy functional associated to (1) is not well defined in $H^1(\mathbb{R}^N)$. To overcome this difficulty, enlightened by [8, 9], we give a new change of variables. Then we reduce the quasilinear problem (1) to a semilinear one, which we will prove has a nontrivial solutions.

Our main result is the following.

Theorem 1. Assume that $\alpha \geq 1$ and $T(\alpha) < q+1 < p+1 < \alpha 2^*$. Then (1) has a nontrivial solution.

In this paper, C denotes positive (possibly different) constant, $L^p(\mathbb{R}^N)$ denotes the usual Lebesgue space with norm

$|u|_p = (\int_{\mathbb{R}^N} |u|^p dx)^{1/p}$, $1 \leq p < \infty$, and $H^1(\mathbb{R}^N)$ denotes the Sobolev space with norm $\|u\| = (\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx)^{1/2}$.

2. The Change of Variables

We note that the solutions of (1) are the critical points of the following functional:

$$\begin{aligned} I(u) = & \frac{1}{2} \int_{\mathbb{R}^N} \left[1 + \frac{\alpha^2 u^2}{2(1+u^2)^{2-\alpha}} \right] |\nabla u|^2 dx \\ & + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx \\ & - \frac{1}{q+1} \int_{\mathbb{R}^N} u^{q+1} dx - \frac{1}{p+1} \int_{\mathbb{R}^N} u^{p+1} dx. \end{aligned} \quad (4)$$

Since the functional $I(u)$ may not be well defined in the usual Sobolev spaces $H^1(\mathbb{R}^N)$, we make a change of variables as

$$v = G(u) = \int_0^u g(t) dt, \quad (5)$$

where $g(t) = \sqrt{1 + \alpha^2 t^2 / (2(1+t^2)^{2-\alpha})}$. Since $g(t)$ is monotonous with $|t|$, the inverse function $G^{-1}(t)$ of $G(t)$ exists. Then after the change of variables, $I(u)$ can be written by

$$\begin{aligned} J(v) = & \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) |G^{-1}(v)|^2 dx \\ & - \frac{1}{q+1} \int_{\mathbb{R}^N} |G^{-1}(v)|^{q+1} dx \\ & - \frac{1}{p+1} \int_{\mathbb{R}^N} |G^{-1}(v)|^{p+1} dx. \end{aligned} \quad (6)$$

By Lemma 2 listed below, we have $\lim_{t \rightarrow 0} G^{-1}(t)/t = 1$ and $\lim_{t \rightarrow \infty} |G^{-1}(t)|^\alpha / t = \sqrt{2}$ ($\alpha > 1$) or $\sqrt{2/3}$ ($\alpha = 1$), so $J(v)$ is well defined in $H^1(\mathbb{R}^N)$ and $J(v) \in C^1$.

If u is a nontrivial solution of (1), then for all $\phi \in C_0^\infty(\mathbb{R}^N)$ it should satisfy

$$\begin{aligned} \int_{\mathbb{R}^N} [g^2(u) \nabla u \nabla \phi + g(u) g'(u) |\nabla u|^2 \phi + V(x) u \phi \\ - u^q \phi - u^p \phi] dx = 0. \end{aligned} \quad (7)$$

We show that (7) is equivalent to

$$\begin{aligned} J'(v) \psi = & \int_{\mathbb{R}^N} \left[\nabla v \nabla \psi + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \psi \right. \\ & \left. - \frac{|G^{-1}(v)|^q}{g(G^{-1}(v))} \psi - \frac{|G^{-1}(v)|^p}{g(G^{-1}(v))} \psi \right] dx \\ = & 0, \quad \forall \psi \in C_0^\infty(\mathbb{R}^N). \end{aligned} \quad (8)$$

Indeed, if we choose $\phi = (1/g(u))\psi$ in (7), then we get (8). On the other hand, since $u = G^{-1}(v)$, if we let $\psi = g(u)\phi$ in (8), we get (7). Therefore, in order to find the nontrivial solutions of (1), it suffices to study the existence of the nontrivial solutions of the following equations:

$$-\Delta v = -V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} + \frac{|G^{-1}(v)|^q}{g(G^{-1}(v))} + \frac{|G^{-1}(v)|^p}{g(G^{-1}(v))}, \quad x \in \mathbb{R}^N. \quad (9)$$

Before we close this section, we give some properties of the change of variables.

Lemma 2. For all $t > 0$, one has the following:

- (1) $\lim_{t \rightarrow 0} (G^{-1}(t)/t) = 1$,
- (2) (i) if $\alpha > 1$ then $\lim_{t \rightarrow \infty} (|G^{-1}(t)|^\alpha / t) = \sqrt{2}$, and (ii) if $\alpha = 1$ then $\lim_{t \rightarrow \infty} (|G^{-1}(t)| / t) = \sqrt{2/3}$,
- (3) $|G^{-1}(t)| \leq t$,
- (4) (i) if $\alpha_0 \leq \alpha$ then $tg'(t)/g(t) \leq \alpha - 1$, (ii) if $1 < \alpha \leq \alpha_0$ then $tg'(t)/g(t) \leq \alpha_0 - 1$, and (iii) if $\alpha = 1$ then $tg'(t)/g(t) \leq 5 - 2\sqrt{6}$, where $\alpha_0 \approx 1.36$ is a real root of the equation $\alpha^3 - 4\alpha^2 + 8\alpha - 6 = 0$.

Proof. (1) We easily get $\lim_{t \rightarrow 0} (G^{-1}(t)/t) = (G^{-1}(t))'|_{t=0} = 1/g(G^{-1}(0)) = 1$.

For (2) if $\alpha > 1$, since $g(t) = \sqrt{1 + \alpha^2 t^2 / (2(1+t^2)^{2-\alpha})} = \sqrt{1 + (\alpha^2 t^2 / (2(1+t^2)))(1+t^2)^{\alpha-1}}$, so $g(t) \sim \sqrt{(\alpha^2/2)t^{2(\alpha-1)}}$ as $t \rightarrow \infty$, then $G(t) = \int_0^t g(s) ds \sim (1/\sqrt{2})t^\alpha$ as $t \rightarrow \infty$. Since $G^{-1}(t)$ is the inverse of $G(t)$, so $G^{-1}(t) \sim (\sqrt{2})^{1/\alpha}$ as $t \rightarrow \infty$, thus we have $\lim_{t \rightarrow \infty} (|G^{-1}(t)|^\alpha / t) = \sqrt{2}$.

When $\alpha = 1$, the result is obvious since $g(t)$ is an increasing bounded function.

For (3), since $[G^{-1}(t) - (1/g(0))t]' = 1/g(G^{-1}(t)) - 1/g(0) \leq 0$, so $G^{-1}(t) \leq (1/g(0))t = t$, which proves (3).

Now we prove (4), since $(t/g(t))g'(t) = t^2 / (2(1+t^2)^2 g^2(t)) = t^2 / (2 + 5t^2 + 3t^4) = 1/(2/t^2 + 5 + 3t^2) \leq 5 - 2\sqrt{6}$, which is (iii). To prove (i), that is,

$$\alpha^2(2-\alpha)t^2 \leq 2(\alpha-1)(1+t^2)^{3-\alpha}, \quad (10)$$

we set $j(t) = 2(\alpha-1)(1+t^2)^{3-\alpha} - \alpha^2(2-\alpha)t^2$, so $j'(t) = 2t[2(\alpha-1)(3-\alpha)(1+t^2)^{2-\alpha} - \alpha^2(2-\alpha)] := 2tk_\alpha(t)$, where $k_\alpha(t) = 2(\alpha-1)(3-\alpha)(1+t^2)^{2-\alpha} - \alpha^2(2-\alpha)$. Then $k'_\alpha(t) = 4(\alpha-1)(3-\alpha)(2-\alpha)t(1+t^2)^{1-\alpha}$. If $\alpha \leq 2$ or $\alpha \geq 3$, we get $k'_\alpha(t) \geq 0$, so $k_\alpha(t) \geq k_\alpha(0)$. We notice that $k_\alpha(0) = \alpha^3 - 4\alpha^2 + 8\alpha - 6$ and $k_\alpha(0)$ is an increasing function with respect to α . By Cardano's formula for cubic equations, we know that $k_\alpha(0)$ has one real root and two complex roots. If we set $\alpha_0 \approx 1.36$ to be the real root of $k_\alpha(0)$, then $k_\alpha(t) \geq k_\alpha(0) \geq 0$ as $\alpha \geq \alpha_0$.

So $j'(t) = 2tk_\alpha(t) \geq 0$. That is $j(t)$ is an increasing function, so $j(t) \geq j(0) = 2(\alpha - 1) > 0$ as $\alpha > 1$. If $2 < \alpha < 3$, we get $k'_\alpha(t) < 0$, so $k_\alpha(t)$ is a decreasing function, but in this case $\lim_{t \rightarrow \infty} k_\alpha(t) = (\alpha - 2)\alpha^2 > 0$, so $k_\alpha(t) \geq 0$ for all $t > 0$. Thus we have the same result as $\alpha \leq 2$ or $\alpha \geq 3$, which proves (i). For (ii), by the definition of α_0 , we have

$$\frac{\alpha_0^2 t^2 (1 + (\alpha_0 - 1)t^2)}{2(1 + t^2)^{3-\alpha_0} + \alpha_0^2 t^2 (1 + t^2)} \leq \alpha_0 - 1, \quad (11)$$

so

$$\begin{aligned} & \alpha_0^2 t^2 (1 + (\alpha_0 - 1)t^2) \\ & \leq 2(\alpha_0 - 1)(1 + t^2)^{3-\alpha_0} + \alpha_0^2 (\alpha_0 - 1)t^2 (1 + t^2). \end{aligned} \quad (12)$$

We add $(\alpha_0 - 1)(\alpha^2 - \alpha_0^2)t^2(1 + t^2)$ to both sides of (12), where $\alpha < \alpha_0$. Then

$$\begin{aligned} & \alpha_0^2 t^2 + \alpha_0^2 (\alpha_0 - 1)t^4 + (\alpha_0 - 1)(\alpha^2 - \alpha_0^2)(t^2 + t^4) \\ & \leq 2(\alpha_0 - 1)(1 + t^2)^{3-\alpha_0} + (\alpha_0 - 1)\alpha^2 t^2 (1 + t^2), \\ & \therefore (\alpha_0^2 + (\alpha_0 - 1)(\alpha^2 - \alpha_0^2))t^2 + (\alpha_0 - 1)\alpha^2 t^4 \\ & \leq (\alpha_0 - 1)[2(1 + t^2)^{3-\alpha} + \alpha^2 t^2 (1 + t^2)]. \end{aligned} \quad (13)$$

We notice that $\alpha_0^2 + (\alpha_0 - 1)(\alpha^2 - \alpha_0^2) \geq \alpha^2$. In fact, $\alpha_0^2 + \alpha_0\alpha^2 - \alpha_0^3 - \alpha^2 + \alpha_0^2 \geq \alpha^2 \Leftrightarrow 2(\alpha_0^2 - \alpha^2) + \alpha_0(\alpha^2 - \alpha_0^2) \geq 0 \Leftrightarrow (\alpha_0^2 - \alpha^2)(2 - \alpha_0) > 0$, and the last inequality is obvious. So

$$\begin{aligned} & \alpha^2 t^2 + (\alpha_0 - 1)\alpha^2 t^4 \\ & \leq (\alpha_0 - 1)[2(1 + t^2)^{3-\alpha} + \alpha^2 t^2 (1 + t^2)] \\ & \therefore \alpha^2 t^2 + (\alpha - 1)\alpha^2 t^4 \\ & \leq (\alpha_0 - 1)[2(1 + t^2)^{3-\alpha} + \alpha^2 t^2 (1 + t^2)], \end{aligned} \quad (14) \quad (15)$$

which implies that $tg'(t)/g(t) \leq \alpha_0 - 1$. $\square$

3. Mountain Pass Geometry

In this section, we establish the geometric hypotheses of the mountain pass theorem.

Lemma 3. *There exist $\rho_0, a_0 > 0$ such that $J(v) \geq a_0$ for all $\|v\| = \rho_0$.*

Proof. Let

$$\begin{aligned} Q(x, t) := & -\frac{1}{2}V(x)|G^{-1}(t)|^2 + \frac{1}{q+1}|G^{-1}(t)|^{q+1} \\ & + \frac{1}{p+1}|G^{-1}(t)|^{p+1}. \end{aligned} \quad (16)$$

Then, by Lemma 2 and $p+1 < \alpha 2^*$, we have

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{Q(x, t)}{t^2} &= \lim_{t \rightarrow 0} \left[-\frac{1}{2}V(x) \left(\frac{G^{-1}(t)}{t} \right)^2 \right. \\ & \quad \left. + \frac{1}{q+1} \left(\frac{G^{-1}(t)}{t} \right)^2 |G^{-1}(t)|^{q-1} \right. \\ & \quad \left. + \frac{1}{p+1} \left(\frac{G^{-1}(t)}{t} \right)^2 |G^{-1}(t)|^{p-1} \right] \\ &= -\frac{1}{2}V(x), \\ \lim_{t \rightarrow \infty} \frac{Q(x, t)}{t^{2^*}} &= \lim_{t \rightarrow \infty} \left[\frac{1}{2}V(x) \left(\frac{|G^{-1}(t)|^\alpha}{t} \right)^{2/\alpha} \frac{1}{t^{2^*-2/\alpha}} \right. \\ & \quad \left. + \frac{1}{q+1} \left(\frac{|G^{-1}(t)|^\alpha}{t} \right)^{(q+1)/\alpha} \frac{1}{t^{2^*-(q+1)/\alpha}} \right. \\ & \quad \left. + \frac{1}{p+1} \left(\frac{|G^{-1}(t)|^\alpha}{t} \right)^{(p+1)/\alpha} \frac{1}{t^{2^*-(p+1)/\alpha}} \right] \\ &= 0. \end{aligned} \quad (17)$$

Thus, for $\epsilon > 0$ sufficiently small, there exists a constant $C_\epsilon > 0$ such that

$$Q(x, t) \leq \left(-\frac{1}{2}V(x) + \epsilon \right) t^2 + C_\epsilon |t|^{2^*}. \quad (18)$$

Then, we have

$$\begin{aligned} J(v) &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) |G^{-1}(v)|^2 dx \\ & \quad - \frac{1}{q+1} \int_{\mathbb{R}^N} |G^{-1}(v)|^{q+1} dx - \frac{1}{p+1} \\ & \quad \times \int_{\mathbb{R}^N} |G^{-1}(v)|^{p+1} dx \\ & \geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) v^2 dx - \epsilon \\ & \quad \times \int_{\mathbb{R}^N} v^2 dx - C_\epsilon \int_{\mathbb{R}^N} v^{2^*} dx \\ & \geq C\|v\|^2 - C\|v\|^{2^*}. \end{aligned} \quad (19)$$

Thus, by choosing ρ_0 small, we get the result when $\|v\| = \rho_0$. $\square$

Lemma 4. *There exists $v \in H^1(\mathbb{R}^N)$ such that $J(v) < 0$.*

Proof. Given $\phi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ with $\text{supp } \phi := \bar{B}_1$, we will prove that $J(s\phi) \rightarrow -\infty$ as $s \rightarrow \infty$, which will prove

the result if we take $v = s\phi$ with s large enough. By the proof of Lemma 2, we have $G^{-1}(t) \geq Ct^{1/\alpha}$ as $t \geq 1$, so

$$\begin{aligned} J(s\phi) &\leq \frac{1}{2}s^2 \int_{\mathbb{R}^N} |\nabla \phi|^2 dx + \frac{1}{2}s^2 \\ &\quad \times \int_{\mathbb{R}^N} V(x) \phi^2 dx - s^{(p+1)/\alpha} \\ &\quad \times \int_{\{|s\phi| \geq 1\}} \phi^{(p+1)/\alpha} dx \longrightarrow -\infty, \end{aligned} \quad (20)$$

as $s \rightarrow \infty$. Thus, we get the result. $\square$

4. Existence

In consequence of Lemmas 3 and 4 of the Ambrosetti-Rabinowitz mountain pass Theorem [13], see also [14–16], for the constant

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)) > 0, \quad (21)$$

where $\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) \neq 0, J(\gamma(1)) < 0\}$, and there exists a Palais-Smale sequence at level c ; that is, $J(v_n) \rightarrow c$ and $J'(v_n) \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 5. *The Palais-Smale sequence $\{v_n\}$ for J is bounded in $H^1(\mathbb{R}^N)$.*

Proof. Since $\{v_n\} \subset H^1(\mathbb{R}^N)$ satisfies

$$\begin{aligned} J(v_n) &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) |G^{-1}(v_n)|^2 dx \\ &\quad - \frac{1}{p+1} \int_{\mathbb{R}^N} |G^{-1}(v_n)|^{p+1} dx \\ &\quad - \frac{1}{q+1} \int_{\mathbb{R}^N} |G^{-1}(v_n)|^{q+1} dx = c + o(1) \end{aligned} \quad (22)$$

and for any $\psi \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned} J'(v_n) \psi &= \int_{\mathbb{R}^N} \left[\nabla v_n \nabla \psi + V(x) \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} \psi \right. \\ &\quad \left. - \frac{|G^{-1}(v_n)|^q}{g(G^{-1}(v_n))} \psi - \frac{|G^{-1}(v_n)|^p}{g(G^{-1}(v_n))} \psi \right] dx \\ &= o(1) \|\psi\|. \end{aligned} \quad (23)$$

Now, we consider the function $G^{-1}(v_n)g(G^{-1}(v_n))$. Note by Lemma 2 that

$$\begin{aligned} &|\nabla(G^{-1}(v_n)g(G^{-1}(v_n)))| \\ &= \left[1 + \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} g'(G^{-1}(v_n)) \right] |\nabla v_n| \\ &\leq \frac{1}{2} T(\alpha) |\nabla v_n|. \end{aligned} \quad (24)$$

Combining Lemma 2, we have $G^{-1}(v_n)g(G^{-1}(v_n)) \in H^1(\mathbb{R}^N)$. Thus, since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, by choosing $\psi = G^{-1}(v_n)g(G^{-1}(v_n))$ in (23), we deduce that

$$\begin{aligned} o(1) \|v_n\| &= J'(v_n) G^{-1}(v_n) g(G^{-1}(v_n)) \\ &= \int_{\mathbb{R}^N} \left[\left(1 + \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} g'(G^{-1}(v_n)) \right) |\nabla v_n|^2 \right. \\ &\quad \left. + V(x) |G^{-1}(v_n)|^2 - |G^{-1}(v_n)|^{q+1} \right. \\ &\quad \left. - |G^{-1}(v_n)|^{p+1} \right] dx \\ &\leq \int_{\mathbb{R}^N} \left[\frac{1}{2} T(\alpha) |\nabla v_n|^2 + V(x) |G^{-1}(v_n)|^2 \right. \\ &\quad \left. - |G^{-1}(v_n)|^{q+1} - |G^{-1}(v_n)|^{p+1} \right] dx. \end{aligned} \quad (25)$$

Therefore, by (22) and (25), we have

$$\begin{aligned} &(q+1) J(v_n) - J'(v_n) G^{-1}(v_n) g(G^{-1}(v_n)) \\ &\geq \left(\frac{q+1}{2} - \frac{1}{2} T(\alpha) \right) \\ &\quad \times \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{q-1}{2} \int_{\mathbb{R}^N} V(x) |G^{-1}(v_n)|^2 dx \\ &\quad + \left(1 - \frac{q+1}{p+1} \right) \int_{\mathbb{R}^N} |G^{-1}(v_n)|^{p+1} dx \\ &\geq \left(\frac{q+1}{2} - \frac{1}{2} T(\alpha) \right) \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{q-1}{2} \\ &\quad \times \int_{\mathbb{R}^N} V(x) |G^{-1}(v_n)|^2 dx. \end{aligned} \quad (26)$$

Combining (22) and (26), we get $\int_{\mathbb{R}^N} |G^{-1}(v_n)|^{p+1} dx$ is bounded. To verify that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$ we start splitting

$$\begin{aligned} \int_{\mathbb{R}^N} V(x) v_n^2 dx &= \int_{\{x: |v_n(x)| > 1\}} V(x) v_n^2 dx \\ &\quad + \int_{\{x: |v_n(x)| \leq 1\}} V(x) v_n^2 dx. \end{aligned} \quad (27)$$

By the proof of Lemma 2, we have $G^{-1}(t) \geq Ct^{1/\alpha}$, for all $t > 1$ and $G(1) = \int_0^1 g(t) dt \geq \int_0^1 dt = 1$. Therefore

$$\begin{aligned} &\int_{\{x: |v_n(x)| > 1\}} V(x) v_n^2 dx \\ &\leq C \int_{\{x: |v_n(x)| > 1\}} |G^{-1}(v_n)|^{2\alpha} dx \\ &\leq C \int_{\{x: |v_n(x)| > 1\}} |G^{-1}(v_n)|^{p+1} dx \leq C. \end{aligned} \quad (28)$$

Since $g(t)$ is increasing and $G(t) = \int_0^t g(s)ds \leq g(t)t$, we have

$$\begin{aligned} & \int_{\{x: |v_n(x)| \leq 1\}} V(x) |G^{-1}(v_n)|^2 dx \\ & \geq \frac{1}{g^2(G^{-1}(1))} \int_{\{x: |v_n(x)| \leq 1\}} V(x) v_n^2 dx. \end{aligned} \quad (29)$$

Hence $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, and this proves Lemma 5. $\square$

Now we give the completion of the proof of Theorem 1.

Proof. First, we will prove that $J'(v) = 0$. That is, v is a weak solution of (9). To prove this, it suffices to show that

$$\begin{aligned} J'(v) \psi &= \int_{\mathbb{R}^N} \left[\nabla v \nabla \psi + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \psi \right. \\ & \quad \left. - \frac{|G^{-1}(v)|^q}{g(G^{-1}(v))} \psi - \frac{|G^{-1}(v)|^p}{g(G^{-1}(v))} \psi \right] dx = 0, \\ & \quad \forall \psi \in C_0^\infty(\mathbb{R}^N). \end{aligned} \quad (30)$$

From Lemma 5, $\{v_n\}$ is a bounded Palais-Smale sequence, and there exists $v \in H^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$. By the Lebesgue dominated theorem, we have

$$\begin{aligned} & J'(v_n) \psi - J'(v) \psi \\ &= \int_{\mathbb{R}^N} (\nabla v_n - \nabla v) \nabla \psi dx \\ & \quad + \int_{\mathbb{R}^N} V(x) \left[\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v)}{g(G^{-1}(v))} \right] \psi dx \\ & \quad - \int_{\mathbb{R}^N} \left[\frac{|G^{-1}(v_n)|^q}{g(G^{-1}(v_n))} - \frac{|G^{-1}(v)|^q}{g(G^{-1}(v))} \right] \psi dx \\ & \quad - \int_{\mathbb{R}^N} \left[\frac{|G^{-1}(v_n)|^p}{g(G^{-1}(v_n))} - \frac{|G^{-1}(v)|^p}{g(G^{-1}(v))} \right] \psi dx \rightarrow 0. \end{aligned} \quad (31)$$

Hence, $J'(v) = 0$. That is, v is a weak solution of (1).

Next, in order to complete the proof of Theorem 1, we must show that v is nontrivial. By contradiction, we assume $v = 0$. To prove this, we claim that, for all $R > 0$,

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} v_n^2 dx = 0 \quad (32)$$

cannot occur. Suppose by contradiction that (32) occurs; that is, $\{v_n\}$ vanishes. Then by the Lions compactness lemma [16], $v_n \rightarrow 0$ in $L^r(\mathbb{R}^N)$ for any $r \in (2, 2^*)$. By the proof of Lemma 2, we get $G^{-1}(t) : (\sqrt{2}t)^{1/\alpha}$ as $t \rightarrow \infty$, so there exists

a suitable constant C such that $G^{-1}(t) \leq Ct^{1/\alpha}$. In addition, since $G(t) \leq g(t)t$, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{|G^{-1}(v_n)|^p}{g(G^{-1}(v_n))} v_n dx \\ & \leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |G^{-1}(v_n)|^{p+1} dx \\ & \leq \lim_{n \rightarrow \infty} C \int_{\mathbb{R}^N} v_n^{(p+1)/\alpha} dx = 0, \\ & \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |G^{-1}(v_n)|^{q+1} dx \\ & = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |G^{-1}(v_n)|^{p+1} dx = 0, \end{aligned} \quad (33)$$

and $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|G^{-1}(v_n)|^q / g(G^{-1}(v_n))) v_n dx = 0$ is obvious since $q < p$, which implies that

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} J'(v_n) v_n \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + V(x) \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n \right. \\ & \quad \left. - \frac{|G^{-1}(v_n)|^q}{g(G^{-1}(v_n))} v_n - \frac{|G^{-1}(v_n)|^p}{g(G^{-1}(v_n))} v_n \right] dx \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + V(x) \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n \right] dx. \end{aligned} \quad (34)$$

Then,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx = 0, \quad (35)$$

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n dx = 0. \quad (36)$$

On the other hand, by (25), we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V(x) |G^{-1}(v_n)|^2 dx = 0. \quad (37)$$

Combining (35) and (37), we get a contradiction since $J(v_n) \rightarrow c > 0$. Thus, $\{v_n\}$ does not vanish and there exist $k, R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n \rightarrow \infty} \int_{B_R(y_n)} v_n^2 dx \geq k > 0. \quad (38)$$

Define $\tilde{v}_n(x) = v_n(x + y_n)$. We may assume that the components of $\{y_n\}$ are integer multiples of the periods of $V(x)$. Since $\{v_n\}$ is a Palais-Smale sequence for J and $V(x)$ is periodic in x_i , $1 \leq i \leq N$, $\{\tilde{v}_n\}$ is also a Palais-Smale sequence for J with $J'(\tilde{v}) = 0$ if $\tilde{v}_n \rightharpoonup \tilde{v}$ in $H^1(\mathbb{R}^N)$. Since $\{\tilde{v}_n\}$ does not vanish, we have that $\tilde{v} \neq 0$ is a nontrivial solution of (9). $\square$

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