Research Article

Three-Dimensional Biorthogonal Divergence-Free and Curl-Free Wavelets with Free-Slip Boundary

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This paper deals with the construction of divergence-free and curl-free wavelets on the unit cube, which satisfies the free-slip boundary conditions. First, interval wavelets adapted to our construction are introduced. Then, we provide the biorthogonal divergence-free and curl-free wavelets with free-slip boundary and simple structure, based on the characterization of corresponding spaces. Moreover, the bases are also stable.

1. Introduction

In recent years, divergence-free and curl-free wavelets are generally studied, due to their potential use in many physical problems [1–5]. Anisotropic divergence-free and curl-free wavelets on the hypercube are firstly constructed in [6, 7], but all these functions only satisfy slip boundary conditions. However, the free-slip boundary is important in many cases, such as the solution of partial differential equations in incompressible fluids and electromagnetism. Inspired by this fact, [8, 9] give the construction of anisotropic divergence-free and curl-free wavelets with free-slip boundary, but the structure is very complicated and the basis functions are not explicit. Recently, based on a simple characterization of 2D divergence-free space, Harouna and Perrier proposed an alternative construction to [8] for divergence-free wavelets in two-dimensional case [10]. Following the similar but non-trivial line, we mainly study the anisotropic 3D divergence-free and curl-free wavelet bases with free-slip boundary in this paper. The traditional understanding that 3D curl-free wavelets are more difficult to construct than divergence-free wavelets is not always right, due to our procedure.

In Section 2, interval wavelets that we will use are introduced. Based on the spaces characterization, 3D biorthogonal divergence-free and curl-free wavelet bases are given in Sections 3 and 4, respectively.

2. Interval Wavelets on $[0, 1]$

In this part, we will introduce the interval wavelets used in the subsequent construction.

The existence of divergence-free and curl-free wavelets on R^d follows from the following fundamental proposition [11].

Proposition 1. Let $(V_j^1(R), \tilde{V}_j^1(R))$ be a biorthogonal MRA of $L^2(R)$, with compactly supported scaling functions $(\phi^1, \tilde{\phi}^1)$ and wavelets $(\psi^1, \tilde{\psi}^1)$, such that $\phi^1, \psi^1 \in C^{1+\varepsilon}$ for $\varepsilon > 0$. Then there exists a biorthogonal MRA $(V_j^0(R), \tilde{V}_j^0(R))$, with associated scaling functions $(\phi^0, \tilde{\phi}^0)$ and wavelets $(\psi^0, \tilde{\psi}^0)$, such that

$$(\phi^1)'(x) = \phi^0(x) - \phi^0(x-1), \quad (\psi^1)' = 4\psi^0. \quad (1)$$

The dual functions verify $\int_x^{x+1} \tilde{\phi}^1(t)dt = \tilde{\phi}^0(x)$ and $(\tilde{\psi}^0)' = -4\tilde{\psi}^1$.

Based on the above proposition, Jouini and Lemarié-Rieusset [12] proved the existence of two one dimensional MRAs of $L^2(0, 1)$ linked by

$$\frac{d}{dx}V_j^1 = V_j^0, \quad \tilde{V}_j^0 = H_0^1(0, 1) \cap \bigcap_0^x \tilde{V}_j^1 = \{f : f' \in \tilde{V}_j^1, f(0) = f(1) = 0\}. \quad (2)$$

In the following, we simply introduce the construction of these spaces. Suppose that φ^1 in Proposition 1 is supported on $[n_{\min}, n_{\max}]$ ($n_{\min}, n_{\max}$ integers) and reproduces polynomials up to degree $r-1$:

$$0 \leq \ell \leq r-1, \quad \frac{x^\ell}{\ell!} = \sum_{k=-\infty}^{+\infty} \tilde{p}_\ell^1(k) \varphi^1(x-k), \quad x \in R, \quad (3)$$

with $\tilde{p}_\ell^1(k) = \langle x^\ell / \ell!, \tilde{\varphi}^1(x-k) \rangle$. Similarly, $\tilde{\varphi}^1$ is supported on $[\tilde{n}_{\min}, \tilde{n}_{\max}]$ and reproduces polynomials up to degree $\tilde{r}-1$.

For j being sufficiently large, the spaces V_j^1 have the structure

$$V_j^1 = \text{span} \left\{ \Phi_{j,\ell}^{1,b} = 2^{j/2} \Phi_\ell^{1,b} (2^j x) \right\}_{\ell=0,\dots,r-1} \oplus V_j^{1,\text{int}} \oplus \text{span} \left\{ \Phi_{j,\ell}^{1,\#} = 2^{j/2} \Phi_\ell^{1,\#} (2^j x) \right\}_{\ell=0,\dots,r-1}, \quad (4)$$

where $V_j^{1,\text{int}} = \text{span} \{ \varphi_{j,k}^1 = 2^{j/2} \varphi^1(2^j x - k) : k = k_b, \dots, 2^j - k_\# \}$ is the space whose supports are included into $[\delta_b/2^j, 1 - \delta_\#/2^j] \subset [0, 1]$ ($\delta_b, \delta_\# \in N$ be two fixed parameters), and $k_b = \delta_b - n_{\min}$, $k_\# = \delta_\# + n_{\max}$. Moreover, $\Phi_\ell^{1,b}$ are the edge scaling functions at the edge 0 being defined by

$$\Phi_\ell^{1,b}(x) = \sum_{k=1-n_{\max}}^{k_b-1} \tilde{p}_\ell^1(k) \varphi^1(x-k) \chi_{[0,+\infty)}. \quad (5)$$

At the edge 1, $\Phi_\ell^{1,b}$ are defined by symmetry using $Tf(x) = f(1-x)$.

Similarly, the biorthogonal spaces $\tilde{V}_j^1$ are defined with the same structure as

$$\tilde{V}_j^1 = \text{span} \left\{ \tilde{\Phi}_{j,\ell}^{1,b} \right\}_{\ell=0,\dots,\tilde{r}-1} \oplus \tilde{V}_j^{1,\text{int}} \oplus \text{span} \left\{ \tilde{\Phi}_{j,\ell}^{1,\#} \right\}_{\ell=0,\dots,\tilde{r}-1}. \quad (6)$$

Adjusting the parameters such that

$$\begin{aligned} \Delta_j &= \dim(V_j^1) = \dim(\tilde{V}_j^1) \\ &= 2^j - (\delta_b + \delta_\#) - (n_{\max} - n_{\min}) + 2r + 1. \end{aligned} \quad (7)$$

The last step of the construction is the biorthogonalization process, since the edge scaling functions of V_j^1 and $\tilde{V}_j^1$ are no more biorthogonal. Finally, $(V_j^1, \tilde{V}_j^1)$ form a biorthogonal MRA of $L^2(0, 1)$.

As described in [13], removing the edge scaling functions $\Phi_0^{1,b}$ and $\Phi_0^{1,\#}$ leads to

$$\begin{aligned} V_j^D &= \text{span} \left\{ \Phi_{j,\ell}^{1,b} \right\}_{\ell=1,\dots,r-1} \oplus V_j^{1,\text{int}} \oplus \text{span} \left\{ \Phi_{j,\ell}^{1,\#} \right\}_{\ell=1,\dots,r-1} \\ &=: \text{span} \left\{ \varphi_{j,k}^D : k = 1, \dots, \Delta_j - 2 \right\}. \end{aligned} \quad (8)$$

Similarly, define $\tilde{V}_j^D = \text{span} \left\{ \tilde{\Phi}_{j,\ell}^{1,b} \right\}_{\ell=1,\dots,\tilde{r}-1} \oplus \tilde{V}_j^{1,\text{int}} \oplus \text{span} \left\{ \tilde{\Phi}_{j,\ell}^{1,\#} \right\}_{\ell=1,\dots,\tilde{r}-1}$. After a biorthogonalization process, we finally note that

$$\tilde{V}_j^D = \text{span} \left\{ \tilde{\varphi}_{j,k}^D : k = 1, \dots, \Delta_j - 2 \right\}, \quad (9)$$

and the spaces $(V_j^D, \tilde{V}_j^D)$ form a biorthogonal MRA of $H_0^1(0, 1)$.

The construction of $(V_j^0, \tilde{V}_j^0)$ follows the same structure. Since $(\varphi^1)'(x) = \varphi^0(x) - \varphi^0(x-1)$, φ^0 has compact support $[n_{\min}, n_{\max}-1]$ and reproduces polynomials up to degree $r-2$:

$$0 \leq \ell \leq r-2, \quad \frac{x^\ell}{\ell!} = \sum_{k=-\infty}^{+\infty} \tilde{p}_\ell^0(k) \varphi^0(x-k), \quad (10)$$

with $\tilde{p}_\ell^0(k) = \langle x^\ell / \ell!, \tilde{\varphi}^0(x-k) \rangle$. The scaling function $\tilde{\varphi}^0(x) = \int_x^{x+1} \tilde{\varphi}^1(t) dt$ has support $[\tilde{n}_{\min} - 1, \tilde{n}_{\max}]$ and reproduces polynomials up to degree $\tilde{r}$. Consider

$$\begin{aligned} V_j^0 &= \text{span} \left\{ \Phi_{j,\ell}^{0,b} = 2^{j/2} \Phi_\ell^{0,b} (2^j x) \right\}_{\ell=0,\dots,r-2} \oplus V_j^{0,\text{int}} \\ &\oplus \text{span} \left\{ \Phi_{j,\ell}^{0,\#} = 2^{j/2} \Phi_\ell^{0,\#} (2^j x) \right\}_{\ell=0,\dots,r-2}, \end{aligned} \quad (11)$$

where $V_j^{0,\text{int}} = \text{span} \{ \varphi_{j,k}^0 = 2^{j/2} \varphi^0(2^j x - k) : k = k_b, \dots, 2^j - k_\# + 1 \}$ and supports are included into $[\delta_b/2^j, 1 - \delta_\#/2^j] \subset [0, 1]$. The left edge scaling functions are

$$\Phi_\ell^{0,b}(x) = \sum_{k=2-n_{\max}}^{k_b-1} \tilde{p}_\ell^0(k) \varphi^0(x-k) \chi_{[0,+\infty)}. \quad (12)$$

Biorthogonal spaces $\tilde{V}_j^0$ are similarly defined, but by satisfying vanishing boundary conditions at 0 and 1, then

$$\tilde{V}_j^0 = \text{span} \left\{ \tilde{\Phi}_{j,\ell}^{0,b} \right\}_{\ell=1,\dots,\tilde{r}} \oplus \tilde{V}_j^{0,\text{int}} \oplus \text{span} \left\{ \tilde{\Phi}_{j,\ell}^{0,\#} \right\}_{\ell=1,\dots,\tilde{r}}, \quad (13)$$

with $\tilde{V}_j^{0,\text{int}} = \text{span} \{ \tilde{\varphi}_{j,k}^0 : k = \tilde{k}_b + 1, \dots, 2^j - \tilde{k}_\# \}$ and $\tilde{\Phi}_\ell^{0,b} = \sum_{k=1-\tilde{n}_{\max}}^{\tilde{k}_b} \tilde{p}_\ell^0(k) \tilde{\varphi}^0(x-k) \chi_{[0,+\infty)}$ for $\ell = 1, \dots, \tilde{r}$. It is easy to know $\dim(V_j^0) = \dim(\tilde{V}_j^0) = \Delta_j - 1$.

In practice, we choose $j \geq j_{\min}$ with

$$\begin{aligned} j_{\min} &> \max \left\{ \log_2 \left[n_{\max} - n_{\min} + \delta_\# + \delta_b + 1 \right], \right. \\ &\quad \left. \log_2 \left[\tilde{n}_{\max} - \tilde{n}_{\min} + \tilde{\delta}_\# + \tilde{\delta}_b + 1 \right] \right\} \end{aligned} \quad (14)$$

to ensure that the supports of edge scaling functions at 0 do not intersect the supports of edge scaling functions at 1.

The construction of wavelet spaces $(W_j^1, \tilde{W}_j^1)$ can be seen from [13]. Moreover, they satisfy the following result.

Proposition 2 (see [12]). *Let $(V_j^1, \tilde{V}_j^1)$ and $(V_j^0, \tilde{V}_j^0)$ be MRAs satisfying $(d/dx)V_j^1 = V_j^0$ and $\tilde{V}_j^0 = H_0^1 \cap \int_0^x \tilde{V}_j^1$; then the wavelet spaces W_j^0 and $\tilde{W}_j^0$ are linked to the biorthogonal wavelet spaces associated to $(V_j^1, \tilde{V}_j^1)$ by*

$$W_j^0 = \frac{d}{dx} W_j^1, \quad \tilde{W}_j^0 = \int_0^x \tilde{W}_j^1. \quad (15)$$

Moreover, let $\{\psi_{j,k}^1\}_{k=1,\dots,2^j}$ and $\{\tilde{\psi}_{j,k}^1\}_{k=1,\dots,2^j}$ be two biorthogonal wavelet bases of W_j^1 and $\tilde{W}_j^1$. Biorthogonal wavelet bases of W_j^0 and $\tilde{W}_j^0$ are directly defined by

$$\psi_{j,k}^0 = 2^{-j} (\psi_{j,k}^1)', \quad \tilde{\psi}_{j,k}^0 = -2^j \int_0^x \tilde{\psi}_{j,k}^1. \quad (16)$$

3. Divergence-Free Wavelets on $[0, 1]^3$

Let $\Omega = [0, 1]^3$ and let $\vec{n}$ be the normal vector; the boundary condition considered in [6] is $\vec{u} \cdot \vec{n} = 0$ on $\Gamma = \bigcup_{k=1}^3 \Gamma_k$ with

$$\Gamma_k = [0, 1]^{k-1} \times \{0\} \times [0, 1]^{3-k}, \quad 1 \leq k \leq 3. \quad (17)$$

It holds that $\vec{u} \cdot \vec{n} = 0$ on Γ if and only if $u_k = 0$ on Γ_k ($1 \leq k \leq 3$). We call it a slip boundary, which is shown in Figure 1.

In this section, we mainly consider the following space with free-slip boundary as Figure 2

$$\mathcal{H}_{\text{div}}(\Omega) = \left\{ \vec{u} \in (L^2(\Omega))^3 : \text{div } \vec{u} = 0, \vec{u} \cdot \vec{n} = 0 \text{ on } \partial\Omega \right\}. \quad (18)$$

For $\vec{u}(x, y, z) = (u_1, u_2, u_3)^T$, the 3D curl-operator is defined as

$$\text{curl } \vec{u} = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)^T. \quad (19)$$

Remark 3. Taking Fourier transform on the both sides of $\text{div } \vec{u} = 0$ leads to the equation

$$\xi_1 \hat{u}_1(\xi) + \xi_2 \hat{u}_2(\xi) + \xi_3 \hat{u}_3(\xi) = 0, \quad \xi = (\xi_1, \xi_2, \xi_3). \quad (20)$$

In $L^2(R^3)$, define the following functions

$$\begin{aligned} \hat{\varphi}_1(\xi) &= \frac{\xi_3 \hat{u}_2 - \xi_2 \hat{u}_3}{i(\xi_1^2 + \xi_2^2 + \xi_3^2)}, & \hat{\varphi}_2(\xi) &= \frac{\xi_1 \hat{u}_3 - \xi_3 \hat{u}_1}{i(\xi_1^2 + \xi_2^2 + \xi_3^2)}, \\ \hat{\varphi}_3(\xi) &= \frac{\xi_2 \hat{u}_1 - \xi_1 \hat{u}_2}{i(\xi_1^2 + \xi_2^2 + \xi_3^2)}. \end{aligned} \quad (21)$$

Then, according to (20), it is easy to verify that

$$\begin{aligned} \hat{u}_1(\xi) &= i(\xi_2 \hat{\varphi}_3(\xi) - \xi_3 \hat{\varphi}_2(\xi)), \\ \hat{u}_2(\xi) &= i(\xi_3 \hat{\varphi}_1(\xi) - \xi_1 \hat{\varphi}_3(\xi)), \\ \hat{u}_3(\xi) &= i(\xi_1 \hat{\varphi}_2(\xi) - \xi_2 \hat{\varphi}_1(\xi)), \end{aligned} \quad (22)$$

which is equivalent to $\vec{u} = \text{curl } \vec{\varphi}$. Therefore, any function $\vec{u} \in (L^2(R^3))^3$ which satisfies $\text{div } \vec{u} = 0$ can be characterized by curl operator as $\vec{u} = \text{curl } \vec{\varphi}$ with $\vec{\varphi} \in (H^1(R^3))^3$. In fact, a similar result holds in 3D nonsmooth domains.

Proposition 4 (see [14]). *There is a characterization*

$$\begin{aligned} \mathcal{H}_{\text{div}}(\Omega) \\ = \left\{ \vec{u} = \text{curl } \vec{\varphi} : \vec{\varphi} \in (H^1(\Omega))^3, \vec{\varphi} \times \vec{n} = \vec{0} \text{ on } \partial\Omega \right\}. \end{aligned} \quad (23)$$

Based on Proposition 4, we give the following definition of divergence-free scaling function spaces.

Definition 5. For $j \geq j_{\min}$, the divergence-free scaling function spaces $\vec{V}_j^{\text{div}}$ are defined by

$$\begin{aligned} \vec{V}_j^{\text{div}} &= \text{curl} \left\{ (V_j^0 \otimes V_j^D \otimes V_j^D) \times (V_j^D \otimes V_j^0 \otimes V_j^D) \right. \\ &\quad \left. \times (V_j^D \otimes V_j^D \otimes V_j^0) \right\} \\ &= \text{span} \{ \Phi_{j,k}^{\text{div},1}, \Phi_{j,k}^{\text{div},2}, \Phi_{j,k}^{\text{div},3} \}, \end{aligned} \quad (24)$$

where the divergence-free scaling functions are given by

$$\begin{aligned} \Phi_{j,k}^{\text{div},1} &=: \frac{1}{\sqrt{2}} \text{curl} \left[(\varphi_{j,k_1}^0 \cdot \varphi_{j,k_2}^D \cdot \varphi_{j,k_3}^D, 0, 0)^T \right] \\ &= \frac{1}{\sqrt{2}} \left[\varphi_{j,k_1}^0 \cdot \varphi_{j,k_2}^D \cdot (\varphi_{j,k_3}^D)' \delta_2 - \varphi_{j,k_1}^0 \cdot (\varphi_{j,k_2}^D)' \cdot \varphi_{j,k_3}^D \delta_3 \right], \\ \Phi_{j,k}^{\text{div},2} &=: \frac{1}{\sqrt{2}} \text{curl} \left[(0, \varphi_{j,k_1}^D \cdot \varphi_{j,k_2}^0 \cdot \varphi_{j,k_3}^D, 0)^T \right] \\ &= \frac{1}{\sqrt{2}} \left[(\varphi_{j,k_1}^D)' \cdot \varphi_{j,k_2}^0 \cdot \varphi_{j,k_3}^D \delta_3 - \varphi_{j,k_1}^D \cdot \varphi_{j,k_2}^0 \cdot (\varphi_{j,k_3}^D)' \delta_1 \right], \\ \Phi_{j,k}^{\text{div},3} &=: \frac{1}{\sqrt{2}} \text{curl} \left[(0, 0, \varphi_{j,k_1}^D \cdot \varphi_{j,k_2}^D \cdot \varphi_{j,k_3}^0)^T \right] \\ &= \frac{1}{\sqrt{2}} \left[\varphi_{j,k_1}^D \cdot (\varphi_{j,k_2}^D)' \cdot \varphi_{j,k_3}^0 \delta_1 - (\varphi_{j,k_1}^D)' \cdot \varphi_{j,k_2}^D \cdot \varphi_{j,k_3}^0 \delta_2 \right]. \end{aligned} \quad (25)$$

For proving the consequent main result, we also consider the standard MRA $\vec{V}_j$ of $(L^2(\Omega))^3$:

$$\begin{aligned} \vec{V}_j &= (V_j^1 \otimes V_j^0 \otimes V_j^0) \times (V_j^0 \otimes V_j^1 \otimes V_j^0) \\ &\quad \times (V_j^0 \otimes V_j^0 \otimes V_j^1). \end{aligned} \quad (26)$$

The following conclusion shows that the space $\vec{V}_j$ preserves the divergence-free condition.

Proposition 6. *If $\vec{u} \in (L^2(\Omega))^3$ and $\text{div } \vec{u} = 0$, then $\text{div}[\vec{P}_j \vec{u}] = 0$, where $\vec{P}_j = (p_j^1 \otimes p_j^0 \otimes p_j^0, p_j^0 \otimes p_j^1 \otimes p_j^0, p_j^0 \otimes p_j^0 \otimes p_j^1)$ is the biorthogonal projector on $\vec{V}_j$.*

Proof. Let $\vec{u} = (u_1, u_2, u_3)^T$; then

$$\vec{P}_j \vec{u} = (p_j^1 \otimes p_j^0 \otimes p_j^0 u_1, p_j^0 \otimes p_j^1 \otimes p_j^0 u_2, p_j^0 \otimes p_j^0 \otimes p_j^1 u_3)^T. \quad (27)$$

Therefore, by the fact $d/dx \circ p_j^1 f = p_j^0 \circ (d/dx) f$ in [10], we obtain

$$\begin{aligned} \text{div}[\vec{P}_j \vec{u}] &= \frac{\partial}{\partial x} p_j^1 \otimes p_j^0 \otimes p_j^0 u_1 + \frac{\partial}{\partial y} p_j^0 \otimes p_j^1 \otimes p_j^0 u_2 \\ &\quad + \frac{\partial}{\partial z} p_j^0 \otimes p_j^0 \otimes p_j^1 u_3 \\ &= p_j^0 \otimes p_j^0 \otimes p_j^0 \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} \right) \\ &= p_j^0 \otimes p_j^0 \otimes p_j^0 (\text{div } \vec{u}) = 0. \end{aligned} \quad (28)$$

□

FIGURE 1: Slip boundary condition (divergence).

FIGURE 2: Free-slip boundary condition (divergence).

Theorem 7. *The divergence-free scaling function spaces $\{\vec{V}_j^{\text{div}}\}_{j \geq j_{\min}}$ is a multiresolution analysis of $\mathcal{H}_{\text{div}}(\Omega)$.*

Proof. Since $\mathcal{H}_{\text{div}}(\Omega) \cap \vec{V}_j$ are a multiresolution analysis of $\mathcal{H}_{\text{div}}(\Omega)$, it is reduced to prove

$$\vec{V}_j^{\text{div}} = \mathcal{H}_{\text{div}}(\Omega) \cap \vec{V}_j. \quad (29)$$

Noting that $(d/dx)V_j^1 = V_j^0$ and $V_j^D \subseteq V_j^1$ from (2) and (8), we know $\vec{V}_j^{\text{div}} \subset \vec{V}_j$. Furthermore, $\vec{V}_j^{\text{div}} \subset \mathcal{H}_{\text{div}}(\Omega)$ by construction. Therefore, $\vec{V}_j^{\text{div}} \subset \mathcal{H}_{\text{div}}(\Omega) \cap \vec{V}_j$.

Conversely, letting $\vec{u} \in \mathcal{H}_{\text{div}}(\Omega) \cap \vec{V}_j$, we are going to prove $\vec{u} \in \vec{V}_j^{\text{div}}$. On the one hand, since $\vec{u} \in \vec{V}_j$, we have $\vec{u} = \vec{P}_j \vec{u}$. On the other hand, since $\vec{u} \in \mathcal{H}_{\text{div}}(\Omega)$, there exists a $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)^T \in (H^1(\Omega))^3$ such that $\vec{u} = \text{curl}(\vec{\varphi})$. Moreover, $\vec{\varphi} \times \vec{n} = \vec{0}$. Thus, $\vec{u} = \vec{P}_j[\text{curl}(\vec{\varphi})]$. Furthermore, we can decompose $\vec{\varphi}$ by isotropic vector wavelets as

$$\vec{\varphi} = \vec{P}_j^D(\vec{\varphi}) + \sum_{j' \geq j} \vec{Q}_{j'}^D(\vec{\varphi}), \quad (30)$$

where $\vec{P}_j^D(\vec{\varphi}) = \sum_k c_{1,k} \varphi_{j,k_1}^0 \varphi_{j,k_2}^D \varphi_{j,k_3}^D \delta_1 + \sum_k c_{2,k} \varphi_{j,k_1}^D \varphi_{j,k_2}^0 \varphi_{j,k_3}^D \delta_2 + \sum_k c_{3,k} \varphi_{j,k_1}^D \varphi_{j,k_2}^D \varphi_{j,k_3}^0 \delta_3$

$$\begin{aligned} & \vec{Q}_{j'}^D(\vec{\varphi}) \\ &= \sum_k \left(d_{j',k}^{1,1} \varphi_{j',k_1}^0 \varphi_{j',k_2}^D \psi_{j',k_3}^D + d_{j',k}^{1,2} \varphi_{j',k_1}^0 \psi_{j',k_2}^D \varphi_{j',k_3}^D \right. \\ & \quad + d_{j',k}^{1,3} \psi_{j',k_1}^0 \varphi_{j',k_2}^D \varphi_{j',k_3}^D + d_{j',k}^{1,4} \varphi_{j',k_1}^0 \psi_{j',k_2}^D \psi_{j',k_3}^D \\ & \quad + d_{j',k}^{1,5} \psi_{j',k_1}^0 \varphi_{j',k_2}^D \psi_{j',k_3}^D + d_{j',k}^{1,6} \varphi_{j',k_1}^0 \psi_{j',k_2}^D \varphi_{j',k_3}^D \\ & \quad \left. + d_{j',k}^{1,7} \psi_{j',k_1}^0 \psi_{j',k_2}^D \psi_{j',k_3}^D \right) \delta_1 \\ & + \sum_k \left(d_{j',k}^{2,1} \varphi_{j',k_1}^D \varphi_{j',k_2}^0 \psi_{j',k_3}^D + d_{j',k}^{2,2} \varphi_{j',k_1}^D \psi_{j',k_2}^0 \varphi_{j',k_3}^D \right. \\ & \quad + d_{j',k}^{2,3} \psi_{j',k_1}^D \varphi_{j',k_2}^0 \varphi_{j',k_3}^D + d_{j',k}^{2,4} \varphi_{j',k_1}^D \psi_{j',k_2}^0 \psi_{j',k_3}^D \\ & \quad + d_{j',k}^{2,5} \psi_{j',k_1}^D \varphi_{j',k_2}^0 \psi_{j',k_3}^D + d_{j',k}^{2,6} \varphi_{j',k_1}^D \psi_{j',k_2}^0 \varphi_{j',k_3}^D \\ & \quad \left. + d_{j',k}^{2,7} \psi_{j',k_1}^D \psi_{j',k_2}^0 \psi_{j',k_3}^D \right) \delta_2 \\ & + \sum_k \left(d_{j',k}^{3,1} \varphi_{j',k_1}^D \varphi_{j',k_2}^D \psi_{j',k_3}^0 + d_{j',k}^{3,2} \varphi_{j',k_1}^D \psi_{j',k_2}^D \varphi_{j',k_3}^0 \right. \\ & \quad + d_{j',k}^{3,3} \psi_{j',k_1}^D \varphi_{j',k_2}^D \varphi_{j',k_3}^0 + d_{j',k}^{3,4} \varphi_{j',k_1}^D \psi_{j',k_2}^D \psi_{j',k_3}^0 \\ & \quad + d_{j',k}^{3,5} \psi_{j',k_1}^D \varphi_{j',k_2}^D \psi_{j',k_3}^0 + d_{j',k}^{3,6} \varphi_{j',k_1}^D \psi_{j',k_2}^D \varphi_{j',k_3}^0 \\ & \quad \left. + d_{j',k}^{3,7} \psi_{j',k_1}^D \psi_{j',k_2}^D \psi_{j',k_3}^0 \right) \delta_3. \end{aligned}$$

(31)

Since $\text{curl}[\varphi_{j',k_1}^0 \varphi_{j',k_2}^D \psi_{j',k_3}^D \delta_1] \in (V_{j'}^D \otimes V_{j'}^0 \otimes W_{j'}^0) \times (V_{j'}^0 \otimes V_{j'}^D \otimes W_{j'}^0) \times (V_{j'}^0 \otimes V_{j'}^0 \otimes W_{j'}^D)$, then

$$\tilde{P}_j [\text{curl}(\varphi_{j',k_1}^0 \varphi_{j',k_2}^D \psi_{j',k_3}^D \delta_1)] = \vec{0}. \quad (32)$$

Similarly, every term in the right sides of (31) satisfies (32). Finally,

$$\tilde{P}_j [\text{curl}(\tilde{Q}_{j'}^D(\tilde{\varphi}))] = \vec{0}. \quad (33)$$

Furthermore, we can obtain

$$\tilde{u} = \tilde{P}_j [\text{curl}(\tilde{\varphi})] = \tilde{P}_j [\text{curl}(\tilde{P}_j^D \tilde{\varphi})] = \text{curl}(\tilde{P}_j^D \tilde{\varphi}). \quad (34)$$

Here, we have used the fact $\text{curl}(\tilde{P}_j^D \tilde{\varphi}) \in \vec{V}_j$ in the last step of (34). By construction, we have $\text{curl}(\tilde{P}_j^D \tilde{\varphi}) \in \vec{V}_j^{\text{div}}$, which means $\tilde{u} \in \vec{V}_j^{\text{div}}$ and the proof is completed. $\square$

Based on the constructive method of vector wavelets and the following decompositions:

$$\begin{aligned} V_j^0 \otimes V_j^D \otimes V_j^D &= (V_{j_{\min}}^0 \oplus_{j_1=j_{\min}}^{j-1} W_{j_1}^0) \\ &\quad \otimes (V_{j_{\min}}^D \oplus_{j_2=j_{\min}}^{j-1} W_{j_2}^D) \\ &\quad \otimes (V_{j_{\min}}^D \oplus_{j_3=j_{\min}}^{j-1} W_{j_3}^D) \\ &= (V_{j_{\min}}^0 \otimes V_{j_{\min}}^D \otimes V_{j_{\min}}^D) \\ &\quad \oplus_{j_3=j_{\min}}^{j-1} (V_{j_{\min}}^0 \otimes V_{j_{\min}}^D \otimes W_{j_3}^D) \\ &\quad \oplus_{j_2=j_{\min}}^{j-1} (V_{j_{\min}}^0 \otimes W_{j_2}^D \otimes V_{j_{\min}}^D) \\ &\quad \oplus_{j_1=j_{\min}}^{j-1} (W_{j_1}^0 \otimes V_{j_{\min}}^D \otimes V_{j_{\min}}^D) \\ &\quad \oplus_{j_2,j_3=j_{\min}}^{j-1} (V_{j_{\min}}^0 \otimes W_{j_2}^D \otimes W_{j_3}^D) \\ &\quad \oplus_{j_1,j_3=j_{\min}}^{j-1} (W_{j_1}^0 \otimes V_{j_{\min}}^D \otimes W_{j_3}^D) \\ &\quad \oplus_{j_1,j_2=j_{\min}}^{j-1} (W_{j_1}^0 \otimes W_{j_2}^D \otimes V_{j_{\min}}^D) \\ &\quad \oplus_{j_1,j_2,j_3=j_{\min}}^{j-1} (W_{j_1}^0 \otimes W_{j_2}^D \otimes W_{j_3}^D), \\ V_j^D \otimes V_j^0 \otimes V_j^D &= (V_{j_{\min}}^D \otimes V_{j_{\min}}^0 \otimes V_{j_{\min}}^D) \\ &\quad \oplus_{j_3=j_{\min}}^{j-1} (V_{j_{\min}}^D \otimes V_{j_{\min}}^0 \otimes W_{j_3}^D) \\ &\quad \oplus_{j_2=j_{\min}}^{j-1} (V_{j_{\min}}^D \otimes W_{j_2}^0 \otimes V_{j_{\min}}^D) \\ &\quad \oplus_{j_1=j_{\min}}^{j-1} (W_{j_1}^D \otimes V_{j_{\min}}^0 \otimes V_{j_{\min}}^D) \\ &\quad \oplus_{j_2,j_3=j_{\min}}^{j-1} (V_{j_{\min}}^D \otimes W_{j_2}^0 \otimes W_{j_3}^D) \\ &\quad \oplus_{j_1,j_3=j_{\min}}^{j-1} (W_{j_1}^D \otimes V_{j_{\min}}^0 \otimes W_{j_3}^D) \\ &\quad \oplus_{j_1,j_2=j_{\min}}^{j-1} (W_{j_1}^D \otimes W_{j_2}^0 \otimes V_{j_{\min}}^D) \\ &\quad \oplus_{j_1,j_2,j_3=j_{\min}}^{j-1} (W_{j_1}^D \otimes W_{j_2}^0 \otimes W_{j_3}^D), \end{aligned}$$

$$\begin{aligned} V_j^D \otimes V_j^D \otimes V_j^0 &= (V_{j_{\min}}^D \otimes V_{j_{\min}}^D \otimes V_{j_{\min}}^0) \\ &\quad \oplus_{j_3=j_{\min}}^{j-1} (V_{j_{\min}}^D \otimes V_{j_{\min}}^D \otimes W_{j_3}^0) \\ &\quad \oplus_{j_2=j_{\min}}^{j-1} (V_{j_{\min}}^D \otimes W_{j_2}^D \otimes V_{j_{\min}}^0) \\ &\quad \oplus_{j_1=j_{\min}}^{j-1} (W_{j_1}^D \otimes V_{j_{\min}}^D \otimes V_{j_{\min}}^0) \\ &\quad \oplus_{j_2,j_3=j_{\min}}^{j-1} (V_{j_{\min}}^D \otimes W_{j_2}^D \otimes W_{j_3}^0) \\ &\quad \oplus_{j_1,j_3=j_{\min}}^{j-1} (W_{j_1}^D \otimes V_{j_{\min}}^D \otimes W_{j_3}^0) \\ &\quad \oplus_{j_1,j_2=j_{\min}}^{j-1} (W_{j_1}^D \otimes W_{j_2}^D \otimes V_{j_{\min}}^0) \\ &\quad \oplus_{j_1,j_2,j_3=j_{\min}}^{j-1} (W_{j_1}^D \otimes W_{j_2}^D \otimes W_{j_3}^0), \end{aligned} \quad (35)$$

we can give the definition of anisotropic divergence-free wavelets as follows.

Definition 8. For j_1, j_2 , and $j_3 \geq j_{\min}$, the anisotropic divergence-free wavelets are defined by

$$\begin{aligned} \Psi_{j,k}^{\text{div},(1,1)} &= \frac{1}{\sqrt{4^{j_3} + 1}} \text{curl} [\varphi_{j_{\min},k_1}^0 \cdot \varphi_{j_{\min},k_2}^D \cdot \psi_{j_3,k_3}^D \delta_1] \\ \Psi_{j,k}^{\text{div},(1,2)} &= \frac{1}{\sqrt{4^{j_2} + 1}} \text{curl} [\varphi_{j_{\min},k_1}^0 \cdot \psi_{j_2,k_2}^D \cdot \varphi_{j_{\min},k_3}^D \delta_1] \\ \Psi_{j,k}^{\text{div},(1,3)} &= \frac{1}{\sqrt{2}} \text{curl} [\psi_{j_1,k_1}^0 \cdot \varphi_{j_{\min},k_2}^D \cdot \varphi_{j_{\min},k_3}^D \delta_1] \\ \Psi_{j,k}^{\text{div},(1,4)} &= \frac{1}{\sqrt{4^{j_2} + 4^{j_3}}} \text{curl} [\varphi_{j_{\min},k_1}^0 \cdot \psi_{j_2,k_2}^D \cdot \psi_{j_3,k_3}^D \delta_1] \\ \Psi_{j,k}^{\text{div},(1,5)} &= \frac{1}{\sqrt{4^{j_3} + 1}} \text{curl} [\psi_{j_1,k_1}^0 \cdot \varphi_{j_{\min},k_2}^D \cdot \psi_{j_3,k_3}^D \delta_1] \\ \Psi_{j,k}^{\text{div},(1,6)} &= \frac{1}{\sqrt{4^{j_2} + 1}} \text{curl} [\psi_{j_1,k_1}^0 \cdot \psi_{j_2,k_2}^D \cdot \varphi_{j_{\min},k_3}^D \delta_1] \\ \Psi_{j,k}^{\text{div},(1,7)} &= \frac{1}{\sqrt{4^{j_2} + 4^{j_3}}} \text{curl} [\psi_{j_1,k_1}^0 \cdot \psi_{j_2,k_2}^D \cdot \psi_{j_3,k_3}^D \delta_1], \\ \Psi_{j,k}^{\text{div},(2,1)} &= \frac{1}{\sqrt{4^{j_3} + 1}} \text{curl} [\varphi_{j_{\min},k_1}^D \cdot \varphi_{j_{\min},k_2}^0 \cdot \psi_{j_3,k_3}^D \delta_2] \\ \Psi_{j,k}^{\text{div},(2,2)} &= \frac{1}{\sqrt{2}} \text{curl} [\varphi_{j_{\min},k_1}^D \cdot \psi_{j_2,k_2}^0 \cdot \varphi_{j_{\min},k_3}^D \delta_2] \\ \Psi_{j,k}^{\text{div},(2,3)} &= \frac{1}{\sqrt{4^{j_1} + 1}} \text{curl} [\psi_{j_1,k_1}^D \cdot \varphi_{j_{\min},k_2}^0 \cdot \varphi_{j_{\min},k_3}^D \delta_2] \\ \Psi_{j,k}^{\text{div},(2,4)} &= \frac{1}{\sqrt{4^{j_3} + 1}} \text{curl} [\varphi_{j_{\min},k_1}^D \cdot \psi_{j_2,k_2}^0 \cdot \psi_{j_3,k_3}^D \delta_2] \\ \Psi_{j,k}^{\text{div},(2,5)} &= \frac{1}{\sqrt{4^{j_1} + 4^{j_3}}} \text{curl} [\psi_{j_1,k_1}^D \cdot \varphi_{j_{\min},k_2}^0 \cdot \psi_{j_3,k_3}^D \delta_2] \\ \Psi_{j,k}^{\text{div},(2,6)} &= \frac{1}{\sqrt{4^{j_1} + 1}} \text{curl} [\psi_{j_1,k_1}^D \cdot \psi_{j_2,k_2}^0 \cdot \varphi_{j_{\min},k_3}^D \delta_2] \end{aligned}$$

$$\begin{aligned}
\Psi_{j,k}^{\text{div},(2,7)} &= \frac{1}{\sqrt{4^{j_1} + 4^{j_3}}} \text{curl} [\psi_{j_1,k_1}^D \cdot \psi_{j_2,k_2}^0 \cdot \psi_{j_3,k_3}^D \delta_2], \\
\Psi_{j,k}^{\text{div},(3,1)} &= \frac{1}{\sqrt{2}} \text{curl} [\varphi_{j_{\min},k_1}^D \cdot \varphi_{j_{\min},k_2}^D \cdot \psi_{j_3,k_3}^0 \delta_3] \\
\Psi_{j,k}^{\text{div},(3,2)} &= \frac{1}{\sqrt{4^{j_2} + 1}} \text{curl} [\varphi_{j_{\min},k_1}^D \cdot \psi_{j_2,k_2}^D \cdot \varphi_{j_{\min},k_3}^0 \delta_3] \\
\Psi_{j,k}^{\text{div},(3,3)} &= \frac{1}{\sqrt{4^{j_1} + 1}} \text{curl} [\psi_{j_1,k_1}^D \cdot \varphi_{j_{\min},k_2}^D \cdot \varphi_{j_{\min},k_3}^0 \delta_3] \\
\Psi_{j,k}^{\text{div},(3,4)} &= \frac{1}{\sqrt{4^{j_2} + 1}} \text{curl} [\varphi_{j_{\min},k_1}^D \cdot \psi_{j_2,k_2}^D \cdot \psi_{j_3,k_3}^0 \delta_3] \\
\Psi_{j,k}^{\text{div},(3,5)} &= \frac{1}{\sqrt{4^{j_1} + 1}} \text{curl} [\psi_{j_1,k_1}^D \cdot \varphi_{j_{\min},k_2}^D \cdot \psi_{j_3,k_3}^0 \delta_3] \\
\Psi_{j,k}^{\text{div},(3,6)} &= \frac{1}{\sqrt{4^{j_1} + 4^{j_2}}} \text{curl} [\psi_{j_1,k_1}^D \cdot \psi_{j_2,k_2}^D \cdot \varphi_{j_{\min},k_3}^0 \delta_3] \\
\Psi_{j,k}^{\text{div},(3,7)} &= \frac{1}{\sqrt{4^{j_1} + 4^{j_2}}} \text{curl} [\psi_{j_1,k_1}^D \cdot \psi_{j_2,k_2}^D \cdot \psi_{j_3,k_3}^0 \delta_3].
\end{aligned} \tag{36}$$

Remark 9. The coefficients before the operator “curl” are used to guarantee the biorthogonality in the following construction of dual wavelets.

Proposition 10. *Defining the wavelet spaces*

$$\vec{W}_j^{\text{div},(\varepsilon,n)} = \text{span} \{ \Psi_{j,k}^{\text{div},(\varepsilon,n)} \}, \quad \varepsilon = 1, 2, 3, \quad n = 1, 2, \dots, 7; \tag{37}$$

$$\text{then } \vec{V}_j^{\text{div}} = \vec{V}_{j_{\min}}^{\text{div}} \oplus_{j_{\min} \leq j_1, j_2, j_3 \leq j-1} (\oplus_{\varepsilon=1,2,3, n=1,2,\dots,7} \vec{W}_j^{\text{div},(\varepsilon,n)}).$$

Proof. It can be easily obtained from (35) and Definition 8. $\square$

Definition 11. Biorthogonal divergence-free scaling functions and wavelets are defined by

$$\begin{aligned}
\vec{\Phi}_{j,k}^{\text{div},1} &= \frac{1}{\sqrt{2}} [\bar{\varphi}_{j,k_1}^0 \cdot \bar{\varphi}_{j,k_2}^D \cdot \bar{\gamma}_{j,k_3} \delta_2 - \bar{\varphi}_{j,k_1}^0 \cdot \bar{\gamma}_{j,k_2} \cdot \bar{\varphi}_{j,k_3}^D \delta_3] \\
\vec{\Phi}_{j,k}^{\text{div},2} &= \frac{1}{\sqrt{2}} [\bar{\gamma}_{j,k_1} \cdot \bar{\varphi}_{j,k_2}^0 \cdot \bar{\varphi}_{j,k_3}^D \delta_3 - \bar{\varphi}_{j,k_1}^D \cdot \bar{\varphi}_{j,k_2}^0 \cdot \bar{\gamma}_{j,k_3} \delta_1] \\
\vec{\Phi}_{j,k}^{\text{div},3} &= \frac{1}{\sqrt{2}} [\bar{\varphi}_{j,k_1}^D \cdot \bar{\gamma}_{j,k_2} \cdot \bar{\varphi}_{j,k_3}^0 \delta_1 - \bar{\gamma}_{j,k_1} \cdot \bar{\varphi}_{j,k_2}^D \cdot \bar{\varphi}_{j,k_3}^0 \delta_2], \\
\vec{\Psi}_{j,k}^{\text{div},(1,1)} &= \frac{1}{\sqrt{4^{j_3} + 1}} [2^{j_3} \bar{\varphi}_{j_{\min},k_1}^0 \cdot \bar{\varphi}_{j_{\min},k_2}^D \cdot \bar{\psi}_{j_3,k_3}^0 \delta_2 \\
&\quad - \bar{\varphi}_{j_{\min},k_1}^0 \cdot \bar{\gamma}_{j_{\min},k_2} \cdot \bar{\psi}_{j_3,k_3}^D \delta_3] \\
\vec{\Psi}_{j,k}^{\text{div},(1,2)} &= \frac{1}{\sqrt{4^{j_2} + 1}} [\bar{\varphi}_{j_{\min},k_1}^0 \cdot \bar{\psi}_{j_2,k_2}^D \cdot \bar{\gamma}_{j_{\min},k_3} \delta_2 \\
&\quad - 2^{j_2} \bar{\varphi}_{j_{\min},k_1}^0 \cdot \bar{\psi}_{j_2,k_2}^0 \cdot \bar{\varphi}_{j_{\min},k_3}^D \delta_3]
\end{aligned}$$

$$\begin{aligned}
\vec{\Psi}_{j,k}^{\text{div},(1,3)} &= \frac{1}{\sqrt{2}} [\bar{\psi}_{j_1,k_1}^0 \cdot \bar{\varphi}_{j_{\min},k_2}^D \cdot \bar{\gamma}_{j_{\min},k_3} \delta_2 \\
&\quad - \bar{\psi}_{j_1,k_1}^0 \cdot \bar{\gamma}_{j_{\min},k_2} \cdot \bar{\varphi}_{j_{\min},k_3}^D \delta_3] \\
\vec{\Psi}_{j,k}^{\text{div},(1,4)} &= \frac{1}{\sqrt{4^{j_2} + 4^{j_3}}} [2^{j_3} \bar{\varphi}_{j_{\min},k_1}^0 \cdot \bar{\psi}_{j_2,k_2}^D \cdot \bar{\psi}_{j_3,k_3}^0 \delta_2 \\
&\quad - 2^{j_2} \bar{\varphi}_{j_{\min},k_1}^0 \cdot \bar{\psi}_{j_2,k_2}^0 \cdot \bar{\psi}_{j_3,k_3}^D \delta_3] \\
\vec{\Psi}_{j,k}^{\text{div},(1,5)} &= \frac{1}{\sqrt{4^{j_3} + 1}} [2^{j_3} \bar{\psi}_{j_1,k_1}^0 \cdot \bar{\varphi}_{j_{\min},k_2}^D \cdot \bar{\psi}_{j_3,k_3}^0 \delta_2 \\
&\quad - \bar{\psi}_{j_1,k_1}^0 \cdot \bar{\gamma}_{j_{\min},k_2} \cdot \bar{\psi}_{j_3,k_3}^D \delta_3] \\
\vec{\Psi}_{j,k}^{\text{div},(1,6)} &= \frac{1}{\sqrt{4^{j_2} + 1}} [\bar{\psi}_{j_1,k_1}^0 \cdot \bar{\psi}_{j_2,k_2}^D \cdot \bar{\gamma}_{j_{\min},k_3} \delta_2 \\
&\quad - 2^{j_2} \bar{\psi}_{j_1,k_1}^0 \cdot \bar{\psi}_{j_2,k_2}^0 \cdot \bar{\varphi}_{j_{\min},k_3}^D \delta_3] \\
\vec{\Psi}_{j,k}^{\text{div},(1,7)} &= \frac{1}{\sqrt{4^{j_2} + 4^{j_3}}} [2^{j_3} \bar{\psi}_{j_1,k_1}^0 \cdot \bar{\psi}_{j_2,k_2}^D \cdot \bar{\psi}_{j_3,k_3}^0 \delta_2 \\
&\quad - 2^{j_2} \bar{\psi}_{j_1,k_1}^0 \cdot \bar{\psi}_{j_2,k_2}^0 \cdot \bar{\psi}_{j_3,k_3}^D \delta_3], \\
\vec{\Psi}_{j,k}^{\text{div},(2,1)} &= \frac{1}{\sqrt{4^{j_3} + 1}} [\bar{\gamma}_{j_{\min},k_1} \cdot \bar{\varphi}_{j_{\min},k_2}^0 \cdot \bar{\psi}_{j_3,k_3}^D \delta_3 \\
&\quad - 2^{j_3} \bar{\varphi}_{j_{\min},k_1}^D \cdot \bar{\varphi}_{j_{\min},k_2}^0 \cdot \bar{\psi}_{j_3,k_3}^0 \delta_1] \\
\vec{\Psi}_{j,k}^{\text{div},(2,2)} &= \frac{1}{\sqrt{2}} [\bar{\gamma}_{j_{\min},k_1} \cdot \bar{\psi}_{j_2,k_2}^0 \cdot \bar{\varphi}_{j_{\min},k_3}^D \delta_3 \\
&\quad - \bar{\varphi}_{j_{\min},k_1}^D \cdot \bar{\psi}_{j_2,k_2}^0 \cdot \bar{\gamma}_{j_{\min},k_3} \delta_1] \\
\vec{\Psi}_{j,k}^{\text{div},(2,3)} &= \frac{1}{\sqrt{4^{j_1} + 1}} [2^{j_1} \bar{\psi}_{j_1,k_1}^0 \cdot \bar{\varphi}_{j_{\min},k_2}^0 \cdot \bar{\varphi}_{j_{\min},k_3}^D \delta_3 \\
&\quad - \bar{\psi}_{j_1,k_1}^D \cdot \bar{\varphi}_{j_{\min},k_2}^0 \cdot \bar{\gamma}_{j_{\min},k_3} \delta_1] \\
\vec{\Psi}_{j,k}^{\text{div},(2,4)} &= \frac{1}{\sqrt{4^{j_3} + 1}} [\bar{\gamma}_{j_{\min},k_1} \cdot \bar{\psi}_{j_2,k_2}^0 \cdot \bar{\psi}_{j_3,k_3}^D \delta_3 \\
&\quad - 2^{j_3} \bar{\varphi}_{j_{\min},k_1}^D \cdot \bar{\psi}_{j_2,k_2}^0 \cdot \bar{\psi}_{j_3,k_3}^0 \delta_1] \\
\vec{\Psi}_{j,k}^{\text{div},(2,5)} &= \frac{1}{\sqrt{4^{j_1} + 4^{j_3}}} [2^{j_1} \bar{\psi}_{j_1,k_1}^0 \cdot \bar{\varphi}_{j_{\min},k_2}^0 \cdot \bar{\psi}_{j_3,k_3}^D \delta_3 \\
&\quad - 2^{j_3} \bar{\psi}_{j_1,k_1}^D \cdot \bar{\varphi}_{j_{\min},k_2}^0 \cdot \bar{\psi}_{j_3,k_3}^0 \delta_1] \\
\vec{\Psi}_{j,k}^{\text{div},(2,6)} &= \frac{1}{\sqrt{4^{j_1} + 1}} [2^{j_1} \bar{\psi}_{j_1,k_1}^0 \cdot \bar{\psi}_{j_2,k_2}^0 \cdot \bar{\varphi}_{j_{\min},k_3}^D \delta_3 \\
&\quad - \bar{\psi}_{j_1,k_1}^D \cdot \bar{\psi}_{j_2,k_2}^0 \cdot \bar{\gamma}_{j_{\min},k_3} \delta_1] \\
\vec{\Psi}_{j,k}^{\text{div},(2,7)} &= \frac{1}{\sqrt{4^{j_1} + 4^{j_3}}} [2^{j_1} \bar{\psi}_{j_1,k_1}^0 \cdot \bar{\psi}_{j_2,k_2}^0 \cdot \bar{\psi}_{j_3,k_3}^D \delta_3 \\
&\quad - 2^{j_3} \bar{\psi}_{j_1,k_1}^D \cdot \bar{\psi}_{j_2,k_2}^0 \cdot \bar{\psi}_{j_3,k_3}^0 \delta_1], \\
\vec{\Psi}_{j,k}^{\text{div},(3,1)} &= \frac{1}{\sqrt{2}} [\bar{\varphi}_{j_{\min},k_1}^D \cdot \bar{\gamma}_{j_{\min},k_2} \cdot \bar{\psi}_{j_3,k_3}^0 \delta_1 \\
&\quad - \bar{\gamma}_{j_{\min},k_1} \cdot \bar{\varphi}_{j_{\min},k_2}^D \cdot \bar{\psi}_{j_3,k_3}^0 \delta_2]
\end{aligned}$$

$$\begin{aligned}
\tilde{\Psi}_{j,k}^{\text{div},(3,2)} &= \frac{1}{\sqrt{4j_2+1}} \left[2^{j_2} \tilde{\varphi}_{j_{\min},k_1}^D \cdot \tilde{\psi}_{j_2,k_2}^0 \cdot \tilde{\varphi}_{j_{\min},k_3}^0 \delta_1 \right. \\
&\quad \left. - \tilde{\gamma}_{j_{\min},k_1} \cdot \tilde{\psi}_{j_2,k_2}^D \cdot \tilde{\varphi}_{j_{\min},k_3}^0 \delta_2 \right] \\
\tilde{\Psi}_{j,k}^{\text{div},(3,3)} &= \frac{1}{\sqrt{4j_1+1}} \left[\tilde{\psi}_{j_1,k_1}^D \cdot \tilde{\gamma}_{j_{\min},k_2} \cdot \tilde{\varphi}_{j_{\min},k_3}^0 \delta_1 \right. \\
&\quad \left. - 2^{j_1} \tilde{\psi}_{j_1,k_1}^0 \cdot \tilde{\varphi}_{j_{\min},k_2}^D \cdot \tilde{\varphi}_{j_{\min},k_3}^0 \delta_2 \right] \\
\tilde{\Psi}_{j,k}^{\text{div},(3,4)} &= \frac{1}{\sqrt{4j_2+1}} \left[2^{j_2} \tilde{\varphi}_{j_{\min},k_1}^D \cdot \tilde{\psi}_{j_2,k_2}^0 \cdot \tilde{\psi}_{j_3,k_3}^0 \delta_1 \right. \\
&\quad \left. - \tilde{\gamma}_{j_{\min},k_1} \cdot \tilde{\psi}_{j_2,k_2}^D \cdot \tilde{\psi}_{j_3,k_3}^0 \delta_2 \right] \\
\tilde{\Psi}_{j,k}^{\text{div},(3,5)} &= \frac{1}{\sqrt{4j_1+1}} \left[\tilde{\psi}_{j_1,k_1}^D \cdot \tilde{\gamma}_{j_{\min},k_2} \cdot \tilde{\psi}_{j_3,k_3}^0 \delta_1 \right. \\
&\quad \left. - 2^{j_1} \tilde{\psi}_{j_1,k_1}^0 \cdot \tilde{\varphi}_{j_{\min},k_2}^D \cdot \tilde{\psi}_{j_3,k_3}^0 \delta_2 \right] \\
\tilde{\Psi}_{j,k}^{\text{div},(3,6)} &= \frac{1}{\sqrt{4j_1+4j_2}} \left[2^{j_2} \tilde{\psi}_{j_1,k_1}^D \cdot \tilde{\psi}_{j_2,k_2}^0 \cdot \tilde{\varphi}_{j_{\min},k_3}^0 \delta_1 \right. \\
&\quad \left. - 2^{j_1} \tilde{\psi}_{j_1,k_1}^0 \cdot \tilde{\psi}_{j_2,k_2}^D \cdot \tilde{\varphi}_{j_{\min},k_3}^0 \delta_2 \right] \\
\tilde{\Psi}_{j,k}^{\text{div},(3,7)} &= \frac{1}{\sqrt{4j_1+4j_2}} \left[2^{j_2} \tilde{\psi}_{j_1,k_1}^D \cdot \tilde{\psi}_{j_2,k_2}^0 \cdot \tilde{\psi}_{j_3,k_3}^0 \delta_1 \right. \\
&\quad \left. - 2^{j_1} \tilde{\psi}_{j_1,k_1}^0 \cdot \tilde{\psi}_{j_2,k_2}^D \cdot \tilde{\psi}_{j_3,k_3}^0 \delta_2 \right]. \tag{38}
\end{aligned}$$

Here, $\tilde{\gamma}_{j,k} = -\int_0^x \tilde{\varphi}_{j,k}^D(t) dt$.

Proposition 12. The families $\{\Phi_{j,k}^{\text{div},\varepsilon}, \Psi_{j,k}^{\text{div},(\varepsilon,n)} : j_1, j_2, j_3 \geq j, \varepsilon = 1, 2, 3, n = 1, 2, \dots, 7\}$ and $\{\tilde{\Phi}_{j,k}^{\text{div},\varepsilon}, \tilde{\Psi}_{j,k}^{\text{div},(\varepsilon,n)} : j_1, j_2, j_3 \geq j, \varepsilon = 1, 2, 3, n = 1, 2, \dots, 7\}$ are biorthogonal in $(L^2(\Omega))^3$.

Proof. It is easily proved by the fact that $\tilde{\psi}_{j,k}^0 = -2^j \int_0^x \tilde{\psi}_{j,k}^D(t) dt$, which is shown in (16). $\square$

Theorem 13. The set $\{\Phi_{j_{\min},k}^{\text{div},\varepsilon}, \Psi_{j,k}^{\text{div},(\varepsilon,n)} : j_1, j_2, j_3 \geq j_{\min}, \varepsilon = 1, 2, 3, n = 1, 2, \dots, 7\}$ is a Riesz basis of $\mathcal{H}_{\text{div}}(\Omega)$.

Proof. The completeness is ensured by Theorem 7 and Proposition 10. Now, it remains to prove the L^2 -stability of the basis. By assumption of 1D scaling and wavelet functions, the divergence-free wavelets $\Psi_{j,k}^{\text{div},(\varepsilon,n)}$ are compactly supported, have zero mean value, and belong to the spaces C^ε for some $\varepsilon > 0$; then they constitute a vaguelette-family ([12]). Furthermore, the Riesz stability follows from the existence of a biorthogonal wavelet family given by Proposition 12. $\square$

4. Curl-Free Wavelets on $[0, 1]^3$

The boundary condition considered in [7] is $\vec{u} \times \vec{n} = \vec{0}$ on $\Gamma = \bigcup_{k=1}^3 \Gamma_k$ with

$$\Gamma_k = \bigcup_{m=1, m \neq k}^3 [0, 1]^{m-1} \times \{0\} \times [0, 1]^{3-m}, \quad 1 \leq k \leq 3. \tag{39}$$

It holds that $\vec{u} \times \vec{n} = \vec{0}$ on Γ if and only if $u_k = 0$ on Γ_k ($1 \leq k \leq 3$), which is shown in Figure 3.

In this section, we mainly consider the following space:

$$\begin{aligned}
\mathcal{H}_{\text{curl}}(\Omega) &= \left\{ \vec{u} \in (L^2(\Omega))^3 : \text{curl } \vec{u} = \vec{0}, \vec{u} \times \vec{n} = \vec{0} \text{ on } \partial\Omega \right\} \\
&\tag{40}
\end{aligned}$$

with free-slip boundary as Figure 4.

An equivalent characterization is firstly given for $\mathcal{H}_{\text{curl}}(\Omega)$; and then we will give the MRA and wavelets for it.

Proposition 14. There is the characterization $\mathcal{H}_{\text{curl}}(\Omega) = \{\vec{u} = \text{grad } \varphi : \varphi \in H_0^1(\Omega)\}$.

Proof. Suppose $\varphi \in H_0^1(\Omega)$; then $\vec{u} = \text{grad } \varphi = (\partial_1 \varphi, \partial_2 \varphi, \partial_3 \varphi)^T \in (L^2(\Omega))^3$. Moreover,

$$\begin{aligned}
\text{curl } \vec{u} &= \text{curl } \cdot \text{grad } \varphi \\
&= (\partial_2 \partial_3 \varphi - \partial_3 \partial_2 \varphi, \partial_3 \partial_1 \varphi - \partial_1 \partial_3 \varphi, \partial_1 \partial_2 \varphi - \partial_2 \partial_1 \varphi)^T \\
&= \vec{0}. \tag{41}
\end{aligned}$$

Note that

$$\begin{aligned}
\partial_1 \varphi(x, y, 0) &= \lim_{\Delta x \rightarrow 0} \frac{\varphi(x + \Delta x, y, 0) - \varphi(x, y, 0)}{\Delta x} = 0, \\
\partial_1 \varphi(x, y, 1) &= \lim_{\Delta x \rightarrow 0} \frac{\varphi(x + \Delta x, y, 1) - \varphi(x, y, 1)}{\Delta y} = 0, \\
\partial_1 \varphi(x, 0, z) &= \lim_{\Delta x \rightarrow 0} \frac{\varphi(x + \Delta x, 0, z) - \varphi(x, 0, z)}{\Delta x} = 0, \\
\partial_1 \varphi(x, 1, z) &= \lim_{\Delta x \rightarrow 0} \frac{\varphi(x + \Delta x, 1, z) - \varphi(x, 1, z)}{\Delta y} = 0; \tag{42}
\end{aligned}$$

therefore,

$$\begin{aligned}
u_1(x, y, 0) &= u_1(x, y, 1) = 0, \quad \forall 0 \leq x, y \leq 1, \\
u_1(x, 0, z) &= u_1(x, 1, z) = 0, \quad \forall 0 \leq x, z \leq 1. \tag{43}
\end{aligned}$$

In the same way, one can obtain

$$\begin{aligned}
u_2(x, y, 0) &= u_2(x, y, 1) = 0, \quad \forall 0 \leq x, y \leq 1, \\
u_2(0, y, z) &= u_2(1, y, z) = 0, \quad \forall 0 \leq y, z \leq 1, \\
u_3(0, y, z) &= u_3(1, y, z) = 0, \quad \forall 0 \leq y, z \leq 1, \\
u_3(x, 0, z) &= u_3(x, 1, z) = 0, \quad \forall 0 \leq x, z \leq 1. \tag{44}
\end{aligned}$$

This is equivalent to $\vec{u} \times \vec{n} = \vec{0}$. Therefore, $\vec{u} = \text{grad } \varphi \in \mathcal{H}_{\text{curl}}(\Omega)$.

On the other hand, suppose $\vec{u} \in \mathcal{H}_{\text{curl}}(\Omega)$; then we will prove that there exists a function $\varphi \in H_0^1(\Omega)$, such that $\vec{u} = \text{grad } \varphi$. Since $\text{curl } \vec{u} = \vec{0}$, then

$$\partial_2 u_3 = \partial_3 u_2, \quad \partial_3 u_1 = \partial_1 u_3, \quad \partial_1 u_2 = \partial_2 u_1. \tag{45}$$

FIGURE 3: Slip boundary condition (curl).

FIGURE 4: Free-slip boundary condition (curl).

By Stokes formula, there exists a primitive function $\varphi \in H^1(\Omega)$ such that

$$\begin{aligned} d\varphi(x, y, z) \\ = u_1(x, y, z) dx + u_2(x, y, z) dy + u_3(x, y, z) dz. \end{aligned} \quad (46)$$

Therefore, $\partial\varphi/\partial x = u_1$, $\partial\varphi/\partial y = u_2$, and $\partial\varphi/\partial z = u_3$; that is $\vec{u} = \text{grad } \varphi$. Furthermore, $\vec{u} \times \vec{n} = \vec{0}$ means that

$$\begin{aligned} u_1(x, y, 0) = u_1(x, y, 1) = 0, \quad \forall 0 \leq x, y \leq 1, \\ u_2(0, y, z) = u_2(1, y, z) = 0, \quad \forall 0 \leq y, z \leq 1, \\ u_3(x, 0, z) = u_3(x, 1, z) = 0, \quad \forall 0 \leq x, z \leq 1. \end{aligned} \quad (47)$$

Noting that

$$\begin{aligned} \varphi(x, y, z) &= \int_{x_0}^x u_1(r, y, z) dr = \int_{y_0}^y u_2(x, s, z) ds \\ &= \int_{z_0}^z u_3(x, y, t) dt, \end{aligned} \quad (48)$$

we obtain $\varphi(x, y, 0) = \varphi(x, y, 1) = 0$, $\varphi(0, y, z) = \varphi(1, y, z) = 0$, and $\varphi(x, 0, z) = \varphi(x, 1, z) = 0$. Therefore, $\varphi \in H_0^1(\Omega)$. $\square$

Noting that $V_j^D \otimes V_j^D \otimes V_j^D$ is an MRA of $(H_0^1(\Omega))^3$, we give the following definition.

Definition 15. For $j \geq j_{\min}$, the curl-free scaling function spaces $\vec{V}_j^{\text{curl}}$ are defined by

$$\vec{V}_j^{\text{curl}} = \text{grad}(V_j^D \otimes V_j^D \otimes V_j^D) = \text{span}\{\Phi_{j,\mathbf{k}}^{\text{curl}}\}, \quad (49)$$

where the curl-free scaling functions are given by

$$\begin{aligned} \Phi_{j,\mathbf{k}}^{\text{curl}} &=: \frac{1}{\sqrt{3}} \text{grad}(\varphi_{j,k_1}^D \cdot \varphi_{j,k_2}^D \cdot \varphi_{j,k_3}^D) \\ &= \frac{1}{\sqrt{3}} \left((\varphi_{j,k_1}^D)' \cdot \varphi_{j,k_2}^D \cdot \varphi_{j,k_3}^D, \varphi_{j,k_1}^D \cdot (\varphi_{j,k_2}^D)' \cdot \varphi_{j,k_3}^D, \right. \\ &\quad \left. \varphi_{j,k_1}^D \cdot \varphi_{j,k_2}^D \cdot (\varphi_{j,k_3}^D)' \right)^T. \end{aligned} \quad (50)$$

For convenience, we also consider the standard MRA $\vec{V}_j$ of $(L^2(\Omega))^3$:

$$\vec{V}_j = (V_j^0 \otimes V_j^1 \otimes V_j^1) \times (V_j^1 \otimes V_j^0 \otimes V_j^1) \times (V_j^1 \otimes V_j^1 \otimes V_j^0). \quad (51)$$

Theorem 16. The curl-free scaling function spaces $\{\vec{V}_j^{\text{curl}}\}_{j \geq j_{\min}}$ are a multiresolution analysis of $\mathcal{H}_{\text{curl}}(\Omega)$.

Proof. Since $\mathcal{H}_{\text{curl}}(\Omega) \cap \vec{V}_j$ is a multiresolution analysis of $\mathcal{H}_{\text{curl}}(\Omega)$, it is reduced to prove

$$\vec{V}_j^{\text{curl}} = \mathcal{H}_{\text{curl}}(\Omega) \cap \vec{V}_j. \quad (52)$$

Noting that $(d/dx)V_j^1 = V_j^0$ and $V_j^D \subseteq V_j^1$, we know $\vec{V}_j^{\text{curl}} \subset \vec{V}_j$. Furthermore, $\vec{V}_j^{\text{curl}} \subset \mathcal{H}_{\text{curl}}(\Omega)$ by construction. Therefore, $\vec{V}_j^{\text{curl}} \subset \mathcal{H}_{\text{curl}}(\Omega) \cap \vec{V}_j$.

Conversely, let $\vec{u} \in \mathcal{H}_{\text{curl}}(\Omega) \cap \vec{V}_j$, we are going to prove $\vec{u} \in \vec{V}_j^{\text{curl}}$. Let $\vec{P}_j$ be the biorthogonal projector on $\vec{V}_j$. On the one hand, since $\vec{u} \in \vec{V}_j$, we have $\vec{u} = \vec{P}_j \vec{u}$. On the other hand, since $\vec{u} \in \mathcal{H}_{\text{curl}}(\Omega)$, there exists a $\varphi \in H_0^1(\Omega)$ such that $\vec{u} = \text{grad} \varphi$. Thus,

$$\vec{u} = \vec{P}_j [\text{grad} \varphi]. \quad (53)$$

Since $(V_j^D \otimes V_j^D \otimes V_j^D)_{j \geq j_{\min}}$ forms an MRA of $H_0^1(\Omega)$, we can decompose φ as

$$\varphi = P_j^D(\varphi) + \sum_{j_1, j_2, j_3 \geq j} \sum_{n=1}^7 Q_{n,J}^D(\varphi), \quad J = (j_1, j_2, j_3), \quad (54)$$

where

$$\begin{aligned} P_j^D(\varphi) &= \sum_{\mathbf{k}} c_{\mathbf{k}} \varphi_{j,k_1}^D \varphi_{j,k_2}^D \varphi_{j,k_3}^D, \\ Q_{1,J}^D(\varphi) &= \sum_{j_3 \geq j} \sum_{\mathbf{k}} d_{j_3, \mathbf{k}}^1 \varphi_{j_3, k_1}^D \varphi_{j_3, k_2}^D \varphi_{j_3, k_3}^D, \\ Q_{2,J}^D(\varphi) &= \sum_{j_2 \geq j} \sum_{\mathbf{k}} d_{j_2, \mathbf{k}}^2 \varphi_{j_2, k_1}^D \varphi_{j_2, k_2}^D \varphi_{j_3, k_3}^D, \\ Q_{3,J}^D(\varphi) &= \sum_{j_1 \geq j} \sum_{\mathbf{k}} d_{j_1, \mathbf{k}}^3 \varphi_{j_1, k_1}^D \varphi_{j_2, k_2}^D \varphi_{j_3, k_3}^D, \\ Q_{4,J}^D(\varphi) &= \sum_{j_2, j_3 \geq j} \sum_{\mathbf{k}} d_{j_2, j_3, \mathbf{k}}^4 \varphi_{j_2, k_1}^D \varphi_{j_2, k_2}^D \varphi_{j_3, k_3}^D, \\ Q_{5,J}^D(\varphi) &= \sum_{j_1, j_3 \geq j} \sum_{\mathbf{k}} d_{j_1, j_3, \mathbf{k}}^5 \varphi_{j_1, k_1}^D \varphi_{j_2, k_2}^D \varphi_{j_3, k_3}^D, \\ Q_{6,J}^D(\varphi) &= \sum_{j_1, j_2 \geq j} \sum_{\mathbf{k}} d_{j_1, j_2, \mathbf{k}}^6 \varphi_{j_1, k_1}^D \varphi_{j_2, k_2}^D \varphi_{j_3, k_3}^D, \\ Q_{7,J}^D(\varphi) &= \sum_{j_1, j_2, j_3 \geq j} \sum_{\mathbf{k}} d_{j_1, j_2, j_3, \mathbf{k}}^7 \varphi_{j_1, k_1}^D \varphi_{j_2, k_2}^D \varphi_{j_3, k_3}^D \end{aligned} \quad (55)$$

are the biorthogonal projectors on, respectively, $V_j^D \otimes V_j^D \otimes V_j^D$, $V_j^D \otimes V_j^D \otimes W_{j_3}^D$, $V_j^D \otimes W_{j_2}^D \otimes V_j^D$, $W_{j_1}^D \otimes V_j^D \otimes V_j^D$, $V_j^D \otimes W_{j_2}^D \otimes W_{j_3}^D$, $W_{j_1}^D \otimes V_j^D \otimes W_{j_3}^D$, $W_{j_1}^D \otimes W_{j_2}^D \otimes V_j^D$, and $W_{j_1}^D \otimes W_{j_2}^D \otimes W_{j_3}^D$.

Noting that

$$\begin{aligned} \text{grad}(\varphi_{j,k_1}^D \varphi_{j,k_2}^D \varphi_{j,k_3}^D) &\in (V_j^0 \otimes V_j^D \otimes W_{j_3}^D) \\ &\times (V_j^D \otimes V_j^0 \otimes W_{j_3}^D) \times (V_j^D \otimes V_j^D \otimes W_{j_3}^0), \end{aligned} \quad (56)$$

then $\vec{P}_j[\text{grad} Q_{1,J}^D(\varphi)] = \vec{0}$. Similarly, $\vec{P}_j[\text{grad} Q_{n,J}^D(\varphi)] = \vec{0}$ for $2 \leq n \leq 7$. Therefore,

$$\vec{u} = \vec{P}_j [\text{grad} \varphi] = \vec{P}_j [\text{grad} P_j^D(\varphi)]. \quad (57)$$

Since $\text{grad} P_j^D(\varphi) \in \vec{V}_j^{\text{curl}} \subset \vec{V}_j$, then we obtain

$$\vec{u} = \text{grad} P_j^D(\varphi) \in \vec{V}_j^{\text{curl}}. \quad (58)$$

□

Definition 17. For j_1, j_2 , and $j_3 \geq j_{\min}$, the anisotropic curl-free wavelets and wavelet spaces are defined by

$$\begin{aligned} \Psi_{j,\mathbf{k}}^{\text{curl},1} &= \frac{1}{\sqrt{4^{j_3} + 2}} \text{grad} [\varphi_{j_{\min}, k_1}^D \cdot \varphi_{j_{\min}, k_2}^D \cdot \psi_{j_3, k_3}^D] \\ \Psi_{j,\mathbf{k}}^{\text{curl},2} &= \frac{1}{\sqrt{4^{j_2} + 2}} \text{grad} [\varphi_{j_{\min}, k_1}^D \cdot \psi_{j_2, k_2}^D \cdot \varphi_{j_{\min}, k_3}^D] \\ \Psi_{j,\mathbf{k}}^{\text{curl},3} &= \frac{1}{\sqrt{4^{j_1} + 2}} \text{grad} [\psi_{j_1, k_1}^D \cdot \varphi_{j_{\min}, k_2}^D \cdot \varphi_{j_{\min}, k_3}^D] \\ \Psi_{j,\mathbf{k}}^{\text{curl},4} &= \frac{1}{\sqrt{4^{j_2} + 4^{j_3} + 1}} \text{grad} [\varphi_{j_{\min}, k_1}^D \cdot \psi_{j_2, k_2}^D \cdot \psi_{j_3, k_3}^D] \\ \Psi_{j,\mathbf{k}}^{\text{curl},5} &= \frac{1}{\sqrt{4^{j_1} + 4^{j_3} + 1}} \text{grad} [\psi_{j_1, k_1}^D \cdot \varphi_{j_{\min}, k_2}^D \cdot \psi_{j_3, k_3}^D] \\ \Psi_{j,\mathbf{k}}^{\text{curl},6} &= \frac{1}{\sqrt{4^{j_1} + 4^{j_2} + 1}} \text{grad} [\psi_{j_1, k_1}^D \cdot \psi_{j_2, k_2}^D \cdot \varphi_{j_{\min}, k_3}^D] \\ \Psi_{j,\mathbf{k}}^{\text{curl},7} &= \frac{1}{\sqrt{4^{j_1} + 4^{j_2} + 4^{j_3}}} \text{grad} [\psi_{j_1, k_1}^D \cdot \psi_{j_2, k_2}^D \cdot \psi_{j_3, k_3}^D]. \end{aligned} \quad (59)$$

Proposition 18. Defining the wavelet spaces $\vec{W}_j^{\text{curl},n} = \text{span}\{\Psi_{j,\mathbf{k}}^{\text{curl},n}\}$ for $n = 1, 2, \dots, 7$, then

$$\vec{V}_j^{\text{curl}} = \vec{V}_{j_{\min}}^{\text{curl}} \oplus_{j_{\min} \leq j_1, j_2, j_3 \leq j-1} (\oplus_{\varepsilon=1,2,\dots,7} \vec{W}_j^{\text{curl},\varepsilon}). \quad (60)$$

Proof. The result follows from the following fact:

$$\begin{aligned} V_j^D \otimes V_j^D \otimes V_j^D &= (V_{j_{\min}}^D \oplus_{j_1=j_{\min}}^{j-1} W_{j_1}^D) \\ &\otimes (V_{j_{\min}}^D \oplus_{j_2=j_{\min}}^{j-1} W_{j_2}^D) \\ &\otimes (V_{j_{\min}}^D \oplus_{j_3=j_{\min}}^{j-1} W_{j_3}^D) \\ &= (V_{j_{\min}}^D \otimes V_{j_{\min}}^D \otimes V_{j_{\min}}^D) \\ &\oplus_{j_3=j_{\min}}^{j-1} (V_{j_{\min}}^D \otimes V_{j_{\min}}^D \otimes W_{j_3}^D) \\ &\oplus_{j_2=j_{\min}}^{j-1} (V_{j_{\min}}^D \otimes W_{j_2}^D \otimes V_{j_{\min}}^D) \\ &\oplus_{j_1=j_{\min}}^{j-1} (W_{j_1}^D \otimes V_{j_{\min}}^D \otimes V_{j_{\min}}^D) \\ &\oplus_{j_2, j_3=j_{\min}}^{j-1} (V_{j_{\min}}^D \otimes W_{j_2}^D \otimes W_{j_3}^D) \\ &\oplus_{j_1, j_3=j_{\min}}^{j-1} (W_{j_1}^D \otimes V_{j_{\min}}^D \otimes W_{j_3}^D) \\ &\oplus_{j_1, j_2=j_{\min}}^{j-1} (W_{j_1}^D \otimes W_{j_2}^D \otimes V_{j_{\min}}^D) \\ &\oplus_{j_1, j_2, j_3=j_{\min}}^{j-1} (W_{j_1}^D \otimes W_{j_2}^D \otimes W_{j_3}^D). \end{aligned} \quad (61)$$

□

Definition 19. Biorthogonal curl-free scaling functions and wavelets are defined by

$$\begin{aligned} \vec{\Phi}_{j,\mathbf{k}}^{\text{curl}} &= \frac{1}{\sqrt{3}} [\vec{\gamma}_{j,k_1}^D \cdot \vec{\varphi}_{j,k_2}^D \cdot \vec{\varphi}_{j,k_3}^D \delta_1 + \vec{\varphi}_{j,k_1}^D \cdot \vec{\gamma}_{j,k_2}^D \cdot \vec{\varphi}_{j,k_3}^D \delta_2 \\ &\quad + \vec{\varphi}_{j,k_1}^D \cdot \vec{\varphi}_{j,k_2}^D \cdot \vec{\gamma}_{j,k_3}^D \delta_3], \end{aligned}$$

$$\begin{aligned}
\tilde{\Psi}_{j,k}^{\text{curl},1} &= \frac{1}{\sqrt{4j_3+2}} \left[\tilde{\gamma}_{j_{\min},k_1} \tilde{\varphi}_{j_{\min},k_2}^D \tilde{\psi}_{j_3,k_3}^D \delta_1 \right. \\
&\quad + \tilde{\varphi}_{j_{\min},k_1}^D \tilde{\gamma}_{j_{\min},k_2} \tilde{\psi}_{j_3,k_3}^D \delta_2 \\
&\quad \left. + 2^{j_3} \tilde{\varphi}_{j_{\min},k_1}^D \tilde{\varphi}_{j_{\min},k_2}^D \tilde{\psi}_{j_3,k_3}^0 \delta_3 \right], \\
\tilde{\Psi}_{j,k}^{\text{curl},2} &= \frac{1}{\sqrt{4j_2+2}} \left[\tilde{\gamma}_{j_{\min},k_1} \tilde{\psi}_{j_2,k_2}^D \tilde{\varphi}_{j_{\min},k_3}^D \delta_1 \right. \\
&\quad + 2^{j_2} \tilde{\varphi}_{j_{\min},k_1}^D \tilde{\psi}_{j_2,k_2}^0 \tilde{\varphi}_{j_{\min},k_3}^D \delta_2 \\
&\quad \left. + \tilde{\varphi}_{j_{\min},k_1}^D \tilde{\psi}_{j_2,k_2}^D \tilde{\gamma}_{j_{\min},k_3} \delta_3 \right], \\
\tilde{\Psi}_{j,k}^{\text{curl},3} &= \frac{1}{\sqrt{4j_1+2}} \left[2^{j_1} \tilde{\psi}_{j_1,k_1}^0 \tilde{\varphi}_{j_{\min},k_2}^D \tilde{\varphi}_{j_{\min},k_3}^D \delta_1 \right. \\
&\quad + \tilde{\psi}_{j_1,k_1}^D \tilde{\gamma}_{j_{\min},k_2} \tilde{\varphi}_{j_{\min},k_3}^D \delta_2 \\
&\quad \left. + \tilde{\psi}_{j_1,k_1}^D \tilde{\varphi}_{j_{\min},k_2}^D \tilde{\gamma}_{j_{\min},k_3} \delta_3 \right], \\
\tilde{\Psi}_{j,k}^{\text{curl},4} &= \frac{1}{\sqrt{4j_2+4j_3+1}} \left[\tilde{\gamma}_{j_{\min},k_1} \tilde{\psi}_{j_2,k_2}^D \tilde{\psi}_{j_3,k_3}^D \delta_1 \right. \\
&\quad + 2^{j_2} \tilde{\varphi}_{j_{\min},k_1}^D \tilde{\psi}_{j_2,k_2}^0 \tilde{\psi}_{j_3,k_3}^D \delta_2 \\
&\quad \left. + 2^{j_3} \tilde{\varphi}_{j_{\min},k_1}^D \tilde{\psi}_{j_2,k_2}^D \tilde{\psi}_{j_3,k_3}^0 \delta_3 \right], \\
\tilde{\Psi}_{j,k}^{\text{curl},5} &= \frac{1}{\sqrt{4j_1+4j_3+1}} \left[2^{j_1} \tilde{\psi}_{j_1,k_1}^0 \tilde{\varphi}_{j_{\min},k_2}^D \tilde{\psi}_{j_3,k_3}^D \delta_1 \right. \\
&\quad + \tilde{\psi}_{j_1,k_1}^D \tilde{\gamma}_{j_{\min},k_2} \tilde{\psi}_{j_3,k_3}^D \delta_2 \\
&\quad \left. + 2^{j_3} \tilde{\psi}_{j_1,k_1}^D \tilde{\varphi}_{j_{\min},k_2}^D \tilde{\psi}_{j_3,k_3}^0 \delta_3 \right], \\
\tilde{\Psi}_{j,k}^{\text{curl},6} &= \frac{1}{\sqrt{4j_1+4j_2+1}} \left[2^{j_1} \tilde{\psi}_{j_1,k_1}^0 \tilde{\psi}_{j_2,k_2}^D \tilde{\varphi}_{j_{\min},k_3}^D \delta_1 \right. \\
&\quad + 2^{j_2} \tilde{\psi}_{j_1,k_1}^D \tilde{\psi}_{j_2,k_2}^0 \tilde{\varphi}_{j_{\min},k_3}^D \delta_2 \\
&\quad \left. + \tilde{\psi}_{j_1,k_1}^D \tilde{\psi}_{j_2,k_2}^D \tilde{\gamma}_{j_{\min},k_3} \delta_3 \right], \\
\tilde{\Psi}_{j,k}^{\text{curl},7} &= \frac{1}{\sqrt{4j_1+4j_2+4j_3}} \left[2^{j_1} \tilde{\psi}_{j_1,k_1}^0 \tilde{\psi}_{j_2,k_2}^D \tilde{\psi}_{j_3,k_3}^D \delta_1 \right. \\
&\quad + 2^{j_2} \tilde{\psi}_{j_1,k_1}^D \tilde{\psi}_{j_2,k_2}^0 \tilde{\psi}_{j_3,k_3}^D \delta_2 \\
&\quad \left. + 2^{j_3} \tilde{\psi}_{j_1,k_1}^D \tilde{\psi}_{j_2,k_2}^D \tilde{\psi}_{j_3,k_3}^0 \delta_3 \right]. \tag{62}
\end{aligned}$$

Here, $\tilde{\gamma}_{j,k}$ is defined as in Definition 11.

Proposition 20. The families $\{\Phi_{j,k}^{\text{curl}}, \Psi_{j,k}^{\text{curl},\varepsilon} : j_1, j_2, j_3 \geq j, \varepsilon = 1, 2, \dots, 7\}$ and $\{\tilde{\Phi}_{j,k}^{\text{curl}}, \tilde{\Psi}_{j,k}^{\text{curl},\varepsilon} : j_1, j_2, j_3 \geq j, \varepsilon = 1, 2, \dots, 7\}$ are biorthogonal in $(L^2(\Omega))^3$.

Theorem 21. The set $\{\Phi_{j_{\min},k}^{\text{curl}}, \Psi_{j,k}^{\text{curl},\varepsilon} : j_1, j_2, j_3 \geq j_{\min}, \varepsilon = 1, 2, \dots, 7\}$ is a Riesz basis of $\mathcal{H}_{\text{curl}}(\Omega)$.

Proof. It can be proved by the same method as Theorem 13. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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