Research Article

New Quasi-Coincidence Point Polynomial Problems

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Let $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a real-valued polynomial function of the form $F(x, y) = a_s(x)y^s + a_{s-1}(x)y^{s-1} + \cdots + a_0(x)$, where the degree s of y in $F(x, y)$ is greater than or equal to 1. For arbitrary polynomial function $f(x) \in \mathbb{R}[x]$, $x \in \mathbb{R}$, we will find a polynomial solution $y(x) \in \mathbb{R}[x]$ to satisfy the following equation: $(*) : F(x, y(x)) = af(x)$, where $a \in \mathbb{R}$ is a constant depending on the solution $y(x)$, namely, a quasi-coincidence (point) solution of $(*)$, and a is called a quasi-coincidence value. In this paper, we prove that (i) the leading coefficient $a_s(x)$ must be a factor of $f(x)$, and (ii) each solution of $(*)$ is of the form $y(x) = -a_{s-1}(x)/sa_s(x) + \lambda p(x)$, where λ is arbitrary and $p(x) = c(f(x)/a_s(x))^{1/s}$ is also a factor of $f(x)$, for some constant $c \in \mathbb{R}$, provided the equation $(*)$ has infinitely many quasi-coincidence (point) solutions.

1. Introduction and Preliminaries

Let $F : \mathbb{Q}(\alpha) \times \mathbb{Q}(\alpha) \rightarrow \mathbb{Q}(\alpha)$ (where α is an algebraic number) be a polynomial function. Lenstra [1] investigated that $F(x, y(x)) = 0$. He solved a polynomial function $y = y(x) \in \mathbb{Q}(\alpha)[x]$ and derived to find a polynomial $y = y(x)$ satisfying an $x \in \mathbb{Q}(\alpha)[x]$ as a fixed point of the polynomial equation. That is,

$$F(x, y(x)) = x \quad (1)$$

has a polynomial solution $y(x) \in \mathbb{Q}(\alpha)[x]$.

Further, Tung [2, 3] extended (1) to solve $y(x) \in \mathbb{K}[x]$ ($\mathbb{K}$ is a field) for the following equation:

$$F(x, y(x)) = ax^m, \quad (2)$$

where $a \in \mathbb{K}$ is a constant depending on the polynomial solution $y(x)$ and $m \in \mathbb{N}$ a given positive integer.

Recently, Lai and Chen [4, 5] extended (2) to solve $y(x) \in \mathbb{R}[x]$ to satisfy the polynomial equation as the form:

$$F(x, y(x)) = ap^m(x), \quad x \in \mathbb{R}, \quad (3)$$

where $a \in \mathbb{R}$, $p(\cdot)$ is an irreducible polynomial in $x \in \mathbb{R}$ and the polynomial function $F(x, y) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is written by

$$F(x, y) = \sum_{i=0}^s a_i(x)y^i \quad \text{with } s \geq 1, \quad (4)$$

where $s = \deg_y F$ denotes the degree of y in $F(x, y)$.

Definition 1 (see [4]). A polynomial function $y = y(x)$ satisfying (3) is called a quasi-fixed solution corresponding to some real number a . This number a is called a quasi-fixed value corresponding to the polynomial solutions $y = y(x)$.

In mathematics, a coincidence point (or simply coincidence) of two mappings is a point in their domain having the same image point under both mappings. Coincidence theory (the study of coincidence points) is, in most settings, a generalization of fixed point theory.

In this paper, we define a more general coincidence (point) problem in which the $f(x) \in \mathbb{R}[x]$ is replaced by the irreducible polynomial power $p^m(x) \in \mathbb{R}[x]$ throughout this

paper, where $f(x)$ is an arbitrary polynomial. Then, we restate (3) as the following equation:

$$F(x, y) = af(x). \quad (5)$$

It is a new development coincidence point-like problem. We call the polynomial solution $y = y(x)$ for (5) as a quasi-coincidence (point) solution. Precisely, we give the following definition like Definition 1.

Definition 2. A polynomial function $y = y(x)$ satisfying (5) is called a quasi-coincidence (point) solution corresponding to some real number a . This number a is called a quasi-coincidence value corresponding to the polynomial solutions $y = y(x)$.

The number of all solutions in (5) may be infinitely many, or finitely many, or not solvable.

Since there may have many solutions corresponding to the number a , for convenience, we use the following notations to represent different situations:

- (1) Qcs_F , the set of all quasi-coincidence solutions satisfying (5),
- (2) Qcv_F , the set of all quasi-coincidence values satisfying (5),
- (3) $Qcs_F(a)$, the set of all quasi-coincidence solutions $y(x)$ corresponding to a fixed quasi-coincidence value a .

Evidently,

$$Qcs_F = \bigcup_{a \in Qcv_F} Qcs_F(a), \quad (6)$$

$$Qcs_F(a) \cap Qcs_F(b) = \emptyset \quad (7)$$

for any $a \neq b$ in Qcv_F . Moreover, for each $a \in \mathbb{R}$, the cardinal number of $Qcs_F(a)$, denoted by $|Qcs_F(a)|$, satisfies the following condition:

$$|Qcs_F(a)| \leq \deg_y F(x, y). \quad (8)$$

In Section 2, we derive some properties of quasi-coincidence solutions of $F(x, y)$. If (5) has infinitely many quasi-coincidence solutions, the concerned properties are described in Section Section 3.

Throughout the paper, we denote the polynomial function by

$$\begin{aligned} F(x, y) &= a_s(x)y^s + a_{s-1}(x)y^{s-1} \\ &\quad + \cdots + a_1(x)y + a_0(x) \\ &= \sum_{i=0}^s a_i(x)y^i. \end{aligned} \quad (9)$$

2. Auxiliary Lemmas

For convenience, we explain some interesting properties of quasi-coincidence point solutions as the following lemmas. Throughout this paper, we consider (5) for polynomial function (9) and arbitrary polynomial $f(x)$ in $\mathbb{R}[x]$.

Lemma 3. Let $y_1(x) \in Qcs_F(a)$, $y_2(x) \in Qcs_F(b)$, $a \neq b$ in Qcv_F . Then,

$$y_1(x) - y_2(x) = dp(x) \quad \text{for some } d \in \mathbb{R}, \quad (10)$$

and this $p(x)$ is divisible $f(x)$; that is, $p(x) \mid f(x)$.

Proof. Since $y_1(x) \neq y_2(x)$ in Qcs_F correspond to $a \neq b$ in Qcv_F , respectively, thus

$$\begin{aligned} F(x, y_1(x)) &= af(x), \\ F(x, y_2(x)) &= bf(x). \end{aligned} \quad (11)$$

Subtracting the above two equations and using binomial formula, it yields

$$\begin{aligned} (a-b)f(x) &= F(x, y_1(x)) - F(x, y_2(x)) \\ &= a_s(x)[y_1^s(x) - y_2^s(x)] + a_{s-1}(x) \\ &\quad \times [y_1^{s-1}(x) - y_2^{s-1}(x)] \\ &\quad + \cdots + a_1(x)[y_1(x) - y_2(x)] \\ &= [y_1(x) - y_2(x)] \\ &\quad \times [a_s(x)G_s(y_1(x), y_2(x)) \\ &\quad + a_{s-1}(x)G_{s-1}(y_1(x), y_2(x)) \\ &\quad + \cdots + a_1(x)] \\ &= [y_1(x) - y_2(x)]Q(x, y_1(x), y_2(x)), \end{aligned} \quad (12)$$

where $G_j(y_1(x), y_2(x)) = y_1^{j-1}(x) + y_1^{j-2}(x)y_2(x) + \cdots + y_2^{j-1}(x)$, for $j = s, s-1, \dots, 2, 1$. Evidently, the factor $y_1(x) - y_2(x)$ is divisible to the term $(a-b)f(x)$.

Since $a \neq b$,

$$y_1(x) - y_2(x) = dp(x) \quad (13)$$

for a real number $d \in \mathbb{R}$ and some factor $p(x)$ of $f(x)$. $\square$

In Lemma 3, the difference of any two distinct quasi-coincidence solutions corresponding to distinct values is a factor of $f(x)$. Thus, we define a class of this factor as follows.

Notation. (i) Let $p(x)$ be a factor of $f(x)$, and we denote $\Phi(p(x)) = \{\alpha p(x) : \alpha \in \mathbb{R}\}$.

(ii) Let $y(x)$ be an arbitrary polynomial in $\mathbb{R}[x]$, and we denote $y(x) + \Phi(p(x)) = \{y(x) + \alpha p(x) : \alpha \in \mathbb{R}\}$.

It is obvious that for any $y(x), p(x) \in \mathbb{R}[x]$, then the cardinal number

$$|y(x) + \Phi(p(x))| = \infty. \quad (14)$$

For convenience, we explain the relations of Qcs_F and $\Phi(p(x))$ in the following lemma.

Lemma 4. Let $y(x) \in Qcs_F(a)$ for some $a \in \mathbb{R}$, then

$$Qcs_F = Qcs_F(a) \cup \left(\bigcup_{p(x)|f(x)} \{(y(x) + \Phi(p(x))) \cap Qcs_F\} \right). \quad (15)$$

Proof. For any $y_1(x) \in Qcs_F \setminus Qcs_F(a)$, then $y_1(x) \in Qcs(b)$ for some $b \in Qcv_F$. By Lemma 3, we have

$$y_1(x) - y(x) \in \Phi(p(x)) \quad (16)$$

for some factor $p(x)$ of $f(x)$. Then,

$$y_1(x) \in \bigcup_{p(x)|f(x)} \{y(x) + \Phi(p(x))\}. \quad (17)$$

That is,

$$Qcs_F \subseteq Qcs_F(a) \cup \left(\bigcup_{p(x)|f(x)} \{y(x) + \Phi(p(x))\} \right). \quad (18)$$

Moreover, by (6), $Qcs_F(a) \subseteq Qcs_F$, then it follows that

$$Qcs_F = Qcs_F(a) \cup \left(\bigcup_{p(x)|f(x)} \{(y(x) + \Phi(p(x))) \cap Qcs_F\} \right). \quad (19)$$

□

We will use the definitions of “the pigeonhole principle;” it could concert to Grimaldi [6], and the relation can be explained as the following lemma.

Lemma 5. Suppose that the cardinal number $|Qcs_F| = \infty$. For any $y(x) \in Qcs_F$, there exists a factor $p(x)$ of $f(x)$ such that the cardinal number

$$|(y(x) + \Phi(p(x))) \cap Qcs_F| = \infty. \quad (20)$$

Proof. Let $y(x) \in Qcs_F$, then $y(x) \in Qcs_F(a)$ for some $a \in \mathbb{R}$. Since $|Qcs_F| = \infty$ and $|Qcs_F(a)| \leq s$, by Lemma 4, we obtain

$$\left| \bigcup_{p(x)|f(x)} (\{y(x) + \Phi(p(x))\} \cap Qcs_F) \right| = \infty, \quad (21)$$

it yields

$$\sum_{p(x)|f(x)} |(y(x) + \Phi(p(x))) \cap Qcs_F| = \infty. \quad (22)$$

Moreover, the number of all factor $p(x)$ of $f(x)$ is at most $2^{\deg f(x)}$, by pigeonhole's principle, it leads to

$$|(y(x) + \Phi(p(x))) \cap Qcs_F| = \infty \quad (23)$$

for some factor $p(x)$ of $f(x)$. □

In order to know if the intersection of two sets still has infinite solutions, we state the following result to give an explanation.

Lemma 6. Suppose that the cardinal number $|Qcs_F| = \infty$, for any $y_1(x) \neq y_2(x) \in Qcs_F$, there exist some factors $p_1(x)$ and $p_2(x)$ of $f(x)$ such that

$$|(y_1(x) + \Phi(p_1(x))) \cap (y_2(x) + \Phi(p_2(x))) \cap Qcs_F| = \infty. \quad (24)$$

Proof. Let $y_1(x) \in Qcs_F$ and since $|Qcs_F| = \infty$, by Lemma 5, there exists a factor $p_1(x)$ of $f(x)$ such that

$$|(y_1(x) + \Phi(p_1(x))) \cap Qcs_F| = \infty. \quad (25)$$

Moreover, $y_2(x) \in Qcs_F$, by Lemma 4,

$$Qcs_F \subseteq Qcs_F(b) \cup \left(\bigcup_{p_2(x)|f(x)} \{y_2(x) + \Phi(p_2(x))\} \right) \quad (26)$$

for some constant $b \in \mathbb{R}$ and some factor $p_2(x)$ of $f(x)$. Thus,

$$\begin{aligned} & (\{y_1(x) + \Phi(p_1(x))\} \cap Qcs_F) \\ & \subseteq Qcs_F \subseteq Qcs_F(b) \cup \left(\bigcup_{p_2(x)|f(x)} \{y_2(x) + \Phi(p_2(x))\} \right). \end{aligned} \quad (27)$$

Since $|Qcs_F(b)| \leq s$ and the number of all factor to $f(x)$ is at most $2^{\deg f(x)}$, by pigeonhole's principle and (25), we have

$$\begin{aligned} & |(y_2(x) + \Phi(p_2(x))) \cap Qcs_F| = \infty, \\ & |(y_1(x) + \Phi(p_1(x))) \cap (y_2(x) + \Phi(p_2(x))) \cap Qcs_F| = \infty \end{aligned} \quad (28)$$

for some factor $p_2(x)$ of $f(x)$. □

Up to now, we have not shown that the factor $p(x)$ is uniquely existed. Eventually, if the number of all solutions is infinitely many, then the factor $p(x)$ of $f(x)$ is unique up to the choice of the solution $y(x)$.

Lemma 7. Assume that the cardinal number $|Qcs_F| = \infty$, then for any $y_1(x), y_2(x) \in Qcs_F$, one has

$$y_1(x) - y_2(x) = \lambda p(x) \quad (29)$$

for some constant $\lambda \in \mathbb{R}$ and some factor $p(x)$ of $f(x)$ (this $p(x)$ is independent to the choice of $y_1(x)$ and $y_2(x)$).

Proof. Let $y_1(x) \neq y_2(x) \in Qcs_F$, by Lemma 6, we have

$$\begin{aligned} & |(y_1(x) + \Phi(p_1(x))) \cap (y_2(x) + \Phi(p_2(x))) \cap Qcs_F| = \infty \end{aligned} \quad (30)$$

for some factors $p_1(x), p_2(x)$ of $f(x)$.

Let $g_1(x) \neq g_2(x) \in (y_1(x) + \Phi(p_1(x))) \cap (y_2(x) + \Phi(p_2(x))) \cap Qcs_F - \{y_1(x), y_2(x)\}$, then

$$\begin{aligned} g_1(x) & \in y_1(x) + \Phi(p_1(x)), \\ g_2(x) & \in y_1(x) + \Phi(p_1(x)), \\ g_1(x) & \in y_2(x) + \Phi(p_2(x)), \\ g_2(x) & \in y_2(x) + \Phi(p_2(x)). \end{aligned} \quad (31)$$

By Lemma 6, it yields

$$\begin{aligned} g_1(x) - y_1(x) &= \lambda_1 p_1(x), \\ g_2(x) - y_1(x) &= \lambda_2 p_1(x), \\ g_1(x) - y_2(x) &= \lambda_3 p_2(x), \\ g_2(x) - y_2(x) &= \lambda_4 p_2(x) \end{aligned} \quad (32)$$

for some constants $\lambda_1, \lambda_2, \lambda_3$, and $\lambda_4 \in \mathbb{R}$, and consequently

$$\begin{aligned} g_2(x) - g_1(x) &= (g_2(x) - y_1(x)) - (g_1(x) - y_1(x)) \\ &= (\lambda_2 - \lambda_1) p_1(x), \\ g_2(x) - g_1(x) &= (g_2(x) - y_2(x)) \\ &\quad - (g_1(x) - y_2(x)) = (\lambda_4 - \lambda_3) p_2(x). \end{aligned} \quad (33)$$

This implies that

$$(\lambda_2 - \lambda_1) p_1(x) = (\lambda_4 - \lambda_3) p_2(x) \quad (34)$$

and $p_1(x) = p_2(x)$. Therefore,

$$\begin{aligned} y_1(x) - y_2(x) &= (g_1(x) - y_2(x)) - (g_1(x) - y_1(x)) \\ &= (\lambda_3 - \lambda_1) p_1(x). \end{aligned} \quad (35)$$

Consequently, the factor $p(x)$ is uniquely existed. $\square$

By the above preparations, at first we consider the polynomial function $F(x, y)$ with $\deg_y F = 1$ as the form

$$F(x, y) = a_1(x) y + a_0(x). \quad (36)$$

Then, we consider the theorem of problem as

$$F(x, y) = af(x). \quad (37)$$

In the following theorem, we integrate the above type as follows.

Theorem 8. Let $F(x, y)$ be a polynomial function with $\deg_y F = 1$ as the form $F(x, y) = a_1(x)y + a_0(x) = af(x)$ for some $a \in \mathbb{R}$ (where polynomial function $f(x)$ is given). If the cardinal number $|\text{Qcv}_F| \geq 2 (= \deg_y F + 1)$, then

- (i) $a_1(x)$ is some factor of $f(x)$,
- (ii) any solution of (5) is of the form:

$$y(x) = -\frac{a_0(x)}{a_1(x)} + \lambda p(x) \quad (38)$$

for some $\lambda \in \mathbb{R}$ and some factor $p(x)$ of $f(x)$,

- (iii) the cardinal number $|\text{Qcs}_F| = \infty$.

Proof. Since $|\text{Qcv}_F| \geq 2$, we see that there are two distinct quasi-fixed values $a, b \in \text{Qcv}_F$ corresponding to two distinct solutions $y_1(x), y_2(x)$ in Qcs_F such that

$$\begin{aligned} F(x, y_1(x)) &= af(x), \\ F(x, y_2(x)) &= bf(x). \end{aligned} \quad (39)$$

(i) It follows that

$$F(x, y_1(x)) = a_1(x) y_1(x) + a_0(x) = af(x), \quad (40)$$

$$F(x, y_2(x)) = a_1(x) y_2(x) + a_0(x) = bf(x). \quad (41)$$

By (40)-(41), we get

$$a_1(x) (y_1(x) - y_2(x)) = (a - b) f(x). \quad (42)$$

It follows that $a_1(x)$ must be a factor of $f(x)$ and

$$y_1(x) - y_2(x) = \frac{(a - b) f(x)}{a_1(x)} \in \mathbb{R}[x]. \quad (43)$$

(ii) By (40), we have

$$a_0(x) = af(x) - a_1(x) y_1(x) \quad (44)$$

$$= a_1(x) \left(a \frac{f(x)}{a_1(x)} - y_1(x) \right). \quad (45)$$

Thus, (i) and (45) imply $a_1(x) \mid a_0(x)$, and by (44), $y_1(x)$ can be written as

$$y_1(x) = \frac{af(x) - a_0(x)}{a_1(x)}. \quad (46)$$

Moreover, we derive

$$\begin{aligned} F(x, y) &= a_1(x) y + a_0(x) \\ &= a_1(x) (y - y_1(x)) + (a_1(x) y_1(x) + a_0(x)) \end{aligned} \quad (47)$$

$$\text{by (40)} = a_1(x) (y - y_1(x)) + af(x).$$

For any $y(x) \in \text{Qcs}_F$, we have

$$F(x, y(x)) = \tilde{a} f(x) \quad (48)$$

for some $\tilde{a} \in \mathbb{R}$. By (47) and (48), it follows that

$$a_1(x) (y(x) - y_1(x)) + af(x) = \tilde{a} f(x). \quad (49)$$

Hence,

$$y(x) - y_1(x) = \frac{(\tilde{a} - a) f(x)}{a_1(x)}. \quad (50)$$

Then,

$$\begin{aligned} y(x) &= y_1(x) + \frac{(\tilde{a} - a)f(x)}{a_1(x)} \\ \text{by (46)} &= \frac{af(x) - a_0(x)}{a_1(x)} + \frac{(\tilde{a} - a)f(x)}{a_1(x)} \\ &= \frac{-a_0(x)}{a_1(x)} + \frac{\tilde{a}f(x)}{a_1(x)}. \end{aligned} \quad (51)$$

Therefore,

$$y(x) = \frac{-a_0(x)}{a_1(x)} + \lambda p(x) \quad \text{for some factor } p(x) \text{ of } f(x). \quad (52)$$

(Note that this $p(x) = f(x)/a_1(x)$ is only dependent on the choice of $F(x, y)$ and $f(x)$.)

(iii) Actually in (ii), for any $\lambda \in \mathbb{R}$, $y(x) = -a_0(x)/a_1(x) + \lambda f(x)/a_1(x)$ is also a quasi-coincidence solution for $F(x, y)$. The reason is

$$\begin{aligned} F(x, y(x)) &= a_1(x)y(x) + a_0(x) \\ &= a_1(x) \left(\frac{-a_0(x)}{a_1(x)} + \lambda \frac{f(x)}{a_1(x)} \right) + a_0(x) \\ &= \lambda f(x). \end{aligned} \quad (53)$$

This shows that $(*)$ has infinitely many solutions (i.e., $|\text{Qcs}_F| = \infty$). $\square$

Remark 9. Notice that in the case of $\deg_y F = 1$ and $|\text{Qcs}_F| < \infty$, the number of all quasi-coincidence values cannot be larger than 1. Otherwise, it will contract the result of Theorem 8; the case (iii) means that “ $|\text{Qcv}_F| \geq 2$, and then $|\text{Qcs}_F| = \infty$ ”.

3. Main Theorems

In this section, we consider (5) for polynomial function $F(x, y)$ in (9); that is,

$$F(x, y) = \sum_{i=0}^s a_i(x) y^i \quad \text{with } s \geq 2. \quad (54)$$

A given polynomial function $f(x)$ in $\mathbb{R}[x]$ and $F(x, y)$ has at least $s + 1$ distinct quasi-coincidence solutions satisfying some conditions, that is, $y_1(x), y_2(x), y_3(x), \dots, y_{s+1}(x), \dots$. According to the above assumptions, we could derive the following theorem.

Theorem 10. Suppose that the cardinal number $|\text{Qcs}_F| = \infty$ and for each $y(x) \in \text{Qcs}_F$ can be represented as the form

$$y(x) = y_1(x) + \lambda p(x), \quad \lambda \in \mathbb{R} \quad (55)$$

for some $y_1(x), p(x) \in \mathbb{R}[x]$. Then, $p^s(x) \mid f(x)$, and so the polynomial $F(x, y)$ can be represented as

$$F(x, y) = \sum_{i=0}^s c_i \frac{f(x)}{p^i(x)} (y - y_1(x))^i \quad (56)$$

for constants $c_i \in \mathbb{R}$.

Proof. Let $y_i(x)$ be distinct quasi-coincidence solutions of $F(x, y)$ corresponding to quasi-coincidence values a_i , $1 \leq i \leq s + 1$ such that

$$F(x, y_i(x)) = a_i f(x). \quad (57)$$

Choose $i = 1$, $F(x, y_1(x)) = a_1 f(x)$. When $y - y_1(x)$ divides the function $F(x, y)$, we get

$$F(x, y) = (y - y_1(x)) F_1(x, y) + a_1 f(x), \quad (58)$$

where $F_1(x, y)$ is the quotient and $a_1 f(x)$ is the remainder. From the above identity, taking $y = y_2(x)$, it becomes

$$\begin{aligned} F(x, y_2(x)) &= (y_2(x) - y_1(x)) F_1(x, y_2(x)) + a_1 f(x) \\ &= a_2 f(x). \end{aligned} \quad (59)$$

Then,

$$(y_2(x) - y_1(x)) F_1(x, y_2(x)) = (a_2 - a_1) f(x). \quad (60)$$

By (55), $y_2(x) - y_1(x) = \lambda_2 p(x)$, it yields

$$\begin{aligned} F_1(x, y_2(x)) &= \left(\frac{a_2 - a_1}{\lambda_2} \right) \frac{f(x)}{p(x)} \\ &= d_2 \frac{f(x)}{p(x)} \in \mathbb{R}[x] \quad \text{for } d_2 = \frac{a_2 - a_1}{\lambda_2}. \end{aligned} \quad (61)$$

Hence,

$$F_1(x, y) = (y - y_2(x)) F_2(x, y) + d_2 \frac{f(x)}{p(x)}. \quad (62)$$

Continuing this process from $i = 2$ to $s - 1$, we obtain

$$F_i(x, y) = (y - y_{i+1}(x)) F_{i+1}(x, y) + d_{i+1} \frac{f(x)}{p^i(x)} \quad (63)$$

for some $d_{i+1} \in \mathbb{R}$, $i = 1, 2, \dots, s - 1$. Finally, we could get

$$F_{s-1}(x, y) = (y - y_s(x)) F_s(x) + d_s \frac{f(x)}{p^{s-1}(x)}. \quad (64)$$

$F_s(x)$ does not contain the variable y since $\deg_y F = s$. By the assumption (57), $F(x, y_{s+1}(x)) = a_{s+1} f(x)$. It follows that

$$F_s(x) = \lambda \frac{f(x)}{p^s(x)} \in \mathbb{R}[x] \quad \text{for some constant } \lambda \in \mathbb{R}. \quad (65)$$

Consequently,

$$\begin{aligned}
 F(x, y) &= (y - y_1(x)) F_1(x, y) + a_1 f(x) \\
 &= (y - y_1(x)) \\
 &\quad \times \left((y - y_2(x)) F_2(x, y) + d_2 \frac{f(x)}{p(x)} \right) + a_1 f(x) \\
 &= \dots \\
 &= (y - y_1(x)) \left((y - y_2(x)) \right. \\
 &\quad \times \left(\dots \left((y - y_s(x)) F_s(x) \right. \right. \\
 &\quad \left. \left. + d_s \frac{f(x)}{p^{s-1}(x)} \right) \dots \right) \\
 &\quad \left. + d_2 \frac{f(x)}{p(x)} \right) + a_1 f(x) \\
 &= (y - y_1(x)) \left((y - y_2(x)) \right. \\
 &\quad \times \left(\dots \left((y - y_s(x)) \lambda \frac{f(x)}{p^s(x)} \right. \right. \\
 &\quad \left. \left. + d_s \frac{f(x)}{p^{s-1}(x)} \right) \dots \right) \\
 &\quad \left. + d_2 \frac{f(x)}{p(x)} \right) + a_1 f(x). \tag{66}
 \end{aligned}$$

By (55), we have $y_i(x) = y_1(x) + \lambda_i p(x)$, $i = 2, 3, \dots, s + 1$. Then, $F(x, y)$ can be expanded to a power series in the expression:

$$\begin{aligned}
 F(x, y) &= (y - y_1(x)) \\
 &\quad \times \left((y - y_1(x) - \lambda_2 p(x)) \right. \\
 &\quad \times \left(\dots \left((y - y_1(x) - \lambda_s p(x)) \lambda \frac{f(x)}{p^s(x)} \right. \right. \\
 &\quad \left. \left. + d_s \frac{f(x)}{p^{s-1}(x)} \right) \dots \right) \\
 &\quad \left. + d_2 \frac{f(x)}{p(x)} \right) + a_1 f(x) \\
 &= \sum_{i=0}^s c_i \frac{f(x)}{p^i(x)} (y - y_1(x))^i
 \end{aligned} \tag{67}$$

for some real numbers c_j , $j = 0, 1, \dots, s$. Moreover, the leading coefficient of $F(x, y)$, $c_s(f(x)/p^s(x))$ is contained to $\mathbb{R}[x]$, and it follows $p^s(x) \mid f(x)$. $\square$

Conversely, if $F(x, y)$ is expressed as in Theorem 10, then the cardinal number $|\text{Qcs}_F| = \infty$, this is the same as the sufficient conditions.

Theorem 11. *The following two conditions are equivalent:*

- (i) $F(x, y) = \sum_{i=0}^s c_i (f(x)/p^i(x)) (y - y_1(x))^i$ for some $y_1(x) \in \mathbb{R}[x]$, $p(x)$ is a factor of $f(x)$, and $c_i \in \mathbb{R}$ for $i = 0, 1, \dots, s$,
- (ii) $|\text{Qcs}_F| = \infty$.

(In fact, if $|\text{Qcs}_F| = \infty$, then $|\text{Qcs}_F|$ = the cardinal number of $\mathbb{R}$.)

Proof. (i) $\Rightarrow$ (ii) Suppose that (i) holds. Then,

$$\begin{aligned}
 F(x, y_1(x) + \lambda p(x)) &= \sum_{i=0}^s c_i \frac{f(x)}{p^i(x)} (\lambda p(x))^i \\
 &= \left(\sum_{i=0}^s c_i \lambda^i \right) f(x) \\
 &= a p^m(x) \quad \text{for } a = \sum_{i=0}^s c_i \lambda^i \in \mathbb{R}.
 \end{aligned} \tag{68}$$

This means that $y_1(x) + \lambda p(x) \in \text{Qcs}_F$ for each $\lambda \in \mathbb{R}$. It follows that the cardinal $|\text{Qcs}_F| = \infty$.

(ii) $\Rightarrow$ (i) For any $y(x), y_1(x) \in \text{Qcs}_F$, since $|\text{Qcs}_F| = \infty$ and by Lemma 7, we obtain

$$y(x) - y_1(x) = d p(x) \tag{69}$$

for some factor $p(x)$ of $f(x)$. By Theorem 10, we have

$$F(x, y) = \sum_{i=0}^s c_i \frac{f(x)}{p^i(x)} (y - y_1(x))^i \tag{70}$$

for some $y_1(x), p(x) \in \mathbb{R}[x]$, and $c_i \in \mathbb{R}$ for $i = 0, 1, \dots, s$. $\square$

If the $F(x, y)$ can be represented as the form of (71) in the following lemma, then any quasi-coincidence solution can be determined.

Lemma 12. *Suppose that*

$$F(x, y) = \sum_{i=0}^s c_i \frac{f(x)}{p^i(x)} (y - y(x))^i, \tag{71}$$

where $y(x) \in \mathbb{R}[x]$, $c_i \in \mathbb{R}$, $i = 0, 1, \dots, s$ and $p(x)$ is a factor of $f(x)$. Then, $h(x) \in \mathbb{R}[x]$ is a quasi-coincidence solution of $F(x, y)$, if and only if

$$h(x) = y(x) + d p(x) \quad \text{for some } d \in \mathbb{R}. \tag{72}$$

Proof. At first, we assume that $h(x) \in \mathbb{R}[x]$ is a quasi-coincidence solution of $F(x, y)$, and we consider

$$\begin{aligned}
 F(x, y(x)) &= \sum_{i=0}^s c_i \frac{f(x)}{p^i(x)} (y(x) - y(x))^i \\
 &= c_0 f(x).
 \end{aligned} \tag{73}$$

This means $y(x) \in \text{Qcs}_F$.

By Theorem 11,

$$F(x, y) = \sum_{i=0}^s c_i \frac{f(x)}{p^i(x)} (y - y(x))^i, \quad \text{then } |\text{Qcs}_F| = \infty. \quad (74)$$

It follows from Lemma 7 that for any quasi-coincidence solution $h(x)$, we obtain

$$h(x) = y(x) + dp(x) \quad \text{for some } d \in \mathbb{R}. \quad (75)$$

Conversely, suppose $h(x) = y(x) + dp(x)$ for some factor $p(x)$ of $f(x)$ and some constant $d \in \mathbb{R}$. Substituting this $h(x)$ as y in (71), we have

$$\begin{aligned} F(x, h(x)) &= F(x, y(x) + dp(x)) \\ &= \sum_{i=0}^s c_i \frac{f(x)}{p^i(x)} (dp(x))^i \\ &= \left(\sum_{i=0}^s c_i d^i \right) f(x). \end{aligned} \quad (76)$$

Therefore, $h(x) \in \text{Qcs}_F$.

Note that not all polynomial functions $F(x, y)$ can be written as (71). Actually, almost all $F(x, y)$ are expressed as the form of the next theorem. In that situation, any solution can be written as the next form (*) if the cardinal number $|\text{Qcs}_F|$ is infinitely many in this theorem. $\square$

Theorem 13. Let $F(x, y)$ be a polynomial function with

$$F(x, y) = a_s(x) y^s + a_{s-1}(x) y^{s-1} + \cdots + a_0(x), \quad (77)$$

and $f(x)$ a polynomial. If the cardinal number $|\text{Qcs}_F|$ is infinitely many, then for each quasi-coincidence point solution of (5) must be of the form

$$-\frac{a_{s-1}(x)}{sa_s(x)} + \lambda p(x) \quad (*)$$

for arbitrary $\lambda \in \mathbb{R}$, where $p(x) = c(f(x)/a_s(x))^{1/s}$ is a factor of $f(x)$ and c is a constant.

Proof. Assume $|\text{Qcs}_F| = \infty$. By Theorem 11, we have

$$\begin{aligned} F(x, y) &= a_s(x) y^s + a_{s-1}(x) y^{s-1} + \cdots + a_0(x) \\ &= \sum_{i=0}^s c_i \frac{f(x)}{p^i(x)} (y - y(x))^i \end{aligned} \quad (78)$$

for some $c_i \in \mathbb{R}$, and $y(x) \in \text{Qcs}_F$. Comparing the coefficients of y^s and y^{s-1} in both sides, we get

$$a_s(x) = c_s \frac{f(x)}{p^s(x)}, \quad (79)$$

$$a_{s-1}(x) = -sa_s(x) y(x) + c_{s-1} \frac{f(x)}{p^{s-1}(x)}. \quad (80)$$

Consequently, by (79) and (80), we get

$$\begin{aligned} p^s(x) &= c_s \frac{f(x)}{a_s(x)}, \\ y(x) &= \frac{c_{s-1}}{sc_s} p(x) - \frac{a_{s-1}(x)}{sa_s(x)} \in \mathbb{R}[x]. \end{aligned} \quad (81)$$

By Lemma 12 and (81), for any $d \in \mathbb{R}$, we have that any quasi-coincidence solution is represented by

$$\begin{aligned} y(x) + dp(x) &= \frac{c_{s-1}}{sc_s} p(x) - \frac{a_{s-1}(x)}{sa_s(x)} + dp(x) \\ &= -\frac{a_{s-1}(x)}{sa_s(x)} + \left(d - \frac{c_{s-1}}{sc_s} \right) p(x) \\ &= -\frac{a_{s-1}(x)}{sa_s(x)} + \lambda p(x), \end{aligned} \quad (82)$$

where $p(x) = (c_s)^{1/s} (f(x)/a_s(x))^{1/s}$ (note that since d is arbitrary, then λ is arbitrary).

This completes the proof. $\square$

Finally, we provide two examples. Example 1 explains the case of all cardinal number $|\text{Qcs}_F| = 4$.

Example 1. Let

$$\begin{aligned} F(x, y) &= (x^2 + x + 1) y^2 - x^6 \\ &\quad - 3x^5 - 6x^4 - 7x^3 - 10x^2 - 7x - 5, \\ f(x) &= (x^2 + x + 1)^2. \end{aligned} \quad (83)$$

Then,

$$\begin{aligned} F(x, y) &= (x^2 + x + 1) \\ &\quad \times [y^2 - (x^2 + x + 1)^2 - 4], \quad \deg_y F = s = 2. \end{aligned} \quad (84)$$

This polynomial The polynomial equation $F(x, y) = a(x^2 + x + 1)^2$ for some $a \in \mathbb{R}$ has exactly 4 = (s + 2) quasi-coincidence solutions as follows:

$$\begin{aligned} F(x, x^2 + x + 3) &= 4(x^2 + x + 1)^2, \\ F(x, -x^2 - x - 3) &= 4(x^2 + x + 1)^2, \\ F(x, x^2 + x - 1) &= -4(x^2 + x + 1)^2, \\ F(x, -x^2 - x + 1) &= -4(x^2 + x + 1)^2. \end{aligned} \quad (85)$$

The next example explains that the number of all quasi-coincidence solutions of (5) is infinitely many.

Example 2. Let $x \in \mathbb{R}$, $f(x) = x^4(x-1)^4$, and

$$\begin{aligned} F(x, y) &= a_3(x)y^3 + a_2(x)y^2 + a_1(x)y + a_0(x) \\ &= x(x-1)y^3 + 0y^2 + x^3(x-1)^3y + 0. \end{aligned} \quad (86)$$

We will solve all quasi-coincidence solutions of $F(x, y) = ax^4(x-1)^4$ for some $a \in \mathbb{R}$. This polynomial function has at least $6(\geq s+3, \text{ since } s=3)$ quasi-coincidence solutions as follows:

$$\begin{aligned} F(x_1, x_2, x^2 - x) &= 2x^4(x-1)^4, \\ F(x_1, x_2, 2x^2 - 2x) &= 10x^4(x-1)^4, \\ F(x_1, x_2, -x^2 + x) &= -2x^4(x-1)^4, \\ F(x_1, x_2, -2x^2 + 2x) &= -10x^4(x-1)^4, \\ F\left(x_1, x_2, \frac{x^2}{2} - \frac{x}{2}\right) &= \frac{5}{8x^4(x-1)^4}, \\ F\left(x_1, x_2, -\frac{x^2}{2} + \frac{x}{2}\right) &= -\frac{5}{8x^4(x-1)^4}. \end{aligned} \quad (87)$$

In fact, we have $|\text{Qcs}_F| = \infty$, and by (79), we obtain

$$\begin{aligned} p(x) &= c\left(\frac{f(x)}{a_s(x)}\right)^{1/s} \\ &= c\left(\frac{x^4(x-1)^4}{x(x-1)}\right)^{1/3} \\ &= cx(x-1), \end{aligned} \quad (88)$$

for some real number c .

By Theorem 13, any quasi-coincidence solution is written as

$$\begin{aligned} -\frac{a_2(x)}{sa_3(x)} + \lambda p(x) &= \frac{0}{3x(x-1)} + \lambda cx(x-1) \\ &= \mu x(x-1), \end{aligned} \quad (89)$$

where $\mu = \lambda c \in \mathbb{R}$ is arbitrary. This shows that the quasi-coincidence (point) solutions have cardinal $|\text{Qcs}_F| = \infty$.

We would like to provide one open problem as follows.

Further Development. For a real-valued polynomial function $F: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. Can we find all rational quasi-coincidence solutions $y = b(x)/a(x)$ with coprime polynomials $a(x), b(x) \in \mathbb{R}[x]$ to satisfy

$$F(x, y) = af(x) \quad (90)$$

for some polynomials $f(x) \in \mathbb{R}[x]$?

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