Research Article

Strong Convergence Theorem for Bregman Strongly Nonexpansive Mappings and Equilibrium Problems in Reflexive Banach Spaces

Jinhua Zhu, 1 Shih-sen Chang, 2 and Min Liu 1

1 Department of Mathematics, Yibin University, Yibin, Sichuan 644007, China
2 College of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan 650221, China

Correspondence should be addressed to Shih-sen Chang; changss@yahoo.cn

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By using a new hybrid method, a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of Bregman strongly nonexpansive mappings in a reflexive Banach space is proved.

1. Introduction

Throughout this paper, we denote by $\mathbb{R}$ and $\mathbb{R}^+$ the set of all real numbers and all nonnegative real numbers, respectively. We also assume that $E$ is a real reflexive Banach space, $E^*$ is the dual space of $E$, $C$ is a nonempty closed convex subset of $E$, and $\langle \cdot, \cdot \rangle$ is the pairing between $E$ and $E^*$. Let $\Phi$ be a bifunction from $C \times C \to \mathbb{R}$. The equilibrium problem is to find

$$x^* \in C \text{ such that } \Phi(x^*, y) \geq 0, \quad \forall y \in C. \quad (1)$$

The set of such solutions $x^*$ is denoted by $EP(\Phi)$.

Recall that a mapping $T : C \to C$ is said to be nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (2)$$

We denote by $F(T)$ the set of fixed points of $T$.

Numerous problems in physics, optimization, and economics reduce to find a solution of the equilibrium problem. Some methods have been proposed to solve the equilibrium problem in a Hilbert space; see, for instance, Blum and Oettli [1], Combettes and Hirstoaga [2], and Moudafi [3]. Recently, Tada and Takahashi [4, 5] and S. Takahashi and W. Takahashi [6] obtained weak and strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. In particular, Tada and Takahashi [4] established a strong convergence theorem for finding a common element of two sets by using the hybrid method introduced by Nakajo and Takahashi [7]. The authors also proved such a strong convergence theorem in a uniformly convex and uniformly smooth Banach space.

In this paper, motivated by Takahashi et al. [8], we prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a Bregman strongly nonexpansive mapping in a real reflexive Banach space by using the shrinking projection method. Using this theorem, we obtain two new strong convergence results for finding a solution of an equilibrium problem and a fixed point of Bregman strongly nonexpansive mappings in a real reflexive Banach space.

2. Preliminaries and Lemmas

In the sequel, we begin by recalling some preliminaries and lemmas which will be used in the proof.

Let $E$ be a real reflexive Banach space with the norm $\| \cdot \|$ and $E^*$ the dual space of $E$. Throughout this paper, $f : E \to (-\infty, +\infty]$ is a proper, lower semicontinuous, and convex function. We denote by $\text{dom } f$ the domain of $f$, that is, the set $\{x \in E : f(x) < +\infty\}$. 
Let \( x \in \text{int dom } f \). The subdifferential of \( f \) at \( x \) is the convex set defined by

\[
\partial f (x) = \{ x^* \in E^* : f (x) + \langle x^*, y-x \rangle \leq f(y), \forall y \in E \},
\]

where the Fenchel conjugate of \( f \) is the function \( f^*: E^* \to (-\infty, +\infty] \) defined by

\[
f^*(x^*) = \sup \{ \langle x^*, x \rangle - f(x) : x \in E \}. \quad (4)
\]

We know that the Young-Fenchel inequality holds:

\[
\langle x^*, x \rangle \leq f(x) + f^*(x^*), \quad \forall x \in E, x^* \in E^*. \quad (5)
\]

A function \( f \) on \( E \) is coercive [9] if the sublevel set of \( f \) is bounded; equivalently,

\[
\lim_{\|x\| \to +\infty} f(x) = +\infty. \quad (6)
\]

A function \( f \) on \( E \) is said to be strongly coercive [10] if

\[
\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty. \quad (7)
\]

For any \( x \in \text{int dom } f \) and \( y \in E \), the right-hand derivative of \( f \) at \( x \) in the direction \( y \) is defined by

\[
f^*(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \quad (8)
\]

The function \( f \) is said to be Gâteaux differentiable at \( x \) if

\[
\lim_{y \to 0} \frac{f(x + ty) - f(x)}{t} = \partial f(x) \quad \text{exists for any } y \in E.
\]

The function \( f \) is said to be Fréchet differentiable at \( x \) if this limit is attained uniformly in \( \|y\| = 1 \). Finally, \( f \) is said to be uniformly Fréchet differentiable on a subset \( C \subset E \) if \( f \) is Gâteaux differentiable for \( x \in C \) and \( \|y\| = 1 \). It is known that if \( f \) is Gâteaux differentiable (resp., Fréchet differentiable) on int dom \( f \), then \( f \) is continuous and its Gâteaux derivative \( Vf \) is norm-to-weak* continuous (resp., continuous) on int dom \( f \) (see also [11, 12]). We will need the following result.

**Lemma 1** (see [13]). If \( f : E \to \mathbb{R} \) is uniformly Fréchet differentiable and bounded on bounded subsets of \( E \), then \( Vf \) is uniformly continuous on bounded subsets of \( E \) from the strong topology of \( E \) to the strong topology of \( E^* \).

**Definition 2** (see [14]). The function \( f \) is said to be

(i) essentially smooth, if \( \partial f \) is both locally bounded and single valued on its domain,

(ii) essentially strictly convex, if \( (\partial f)^{-1} \) is locally bounded on its domain and \( f \) is strictly convex on every convex subset of dom \( \partial f \),

(iii) Legendre if it is both essentially smooth and essentially strictly convex.

**Remark 3.** Let \( E \) be a reflexive Banach space. Then we have the following.

(i) \( f \) is essentially smooth if and only if \( f^* \) is essentially strictly convex (see [14, Theorem 5.4]).

(ii) \( (\partial f)^{-1} = \partial f^* \) (see [12]).

(iii) \( f \) is Legendre if and only if \( f^* \) is Legendre (see [14, Corollary 5.5]).

(iv) If \( f \) is Legendre, then \( Vf \) is a bijection satisfying \( Vf = (Vf^*)^{-1} \), ran \( Vf = \text{dom } Vf^* = \text{int } \text{dom } f^* \), and ran \( Vf^* = \text{dom } Vf = \text{int } \text{dom } f \) (see [14, Theorem 5.10]).

Examples of Legendre functions were given in [14, 15]. One important and interesting Legendre function is \( (1/p) \cdot \|y\|^p \) \((1 < p < \infty) \) when \( E \) is a smooth and strictly convex Banach space. In this case, the gradient \( Vf \) of \( f \) is coincident with the generalized duality mapping of \( E \); that is, \( Vf = J_p \) \((1 < p < \infty) \). In particular, \( Vf = I \) the identity mapping in Hilbert spaces. In the rest of this paper, we always assume that \( f : E \to (-\infty, +\infty] \) is Legendre.

Let \( f : E \to (-\infty, +\infty] \) be a convex and Gâteaux differentiable function. The function \( D_f : \text{dom } f \times \text{int dom } f \to [0, +\infty) \) defined as

\[
D_f(y,x) := f(y) - f(x) - \langle Vf(x), y-x \rangle \quad (9)
\]

is called the Bregman distance with respect to \( f \) [16].

Recall that the Bregman projection [17] of \( x \in \text{int dom } f \) onto the nonempty closed and convex set \( C \subset \text{dom } f \) is the necessarily unique vector \( P_C^f(x) \in C \) satisfying

\[
D_f \left(P_C^f(x), x\right) = \inf \{D_f(y,x) : y \in C\}. \quad (10)
\]

Concerning the Bregman projection, the following are well known.

**Lemma 4** (see [18]). Let \( C \) be a nonempty, closed, and convex subset of a reflexive Banach space \( E \). Let \( f : E \to \mathbb{R} \) be a Gâteaux differentiable and totally convex function and let \( x \in E \). Then

(a) \( z = P_C^f(x) \) if and only if \( \langle Vf(x)-Vf(z), y-z \rangle \leq 0 \), for all \( y \in C \).

(b) \( D_f(y,P_C^f(x)) + D_f(P_C^f(x),x) \leq D_f(y,x), \quad (11)
\]

\( \forall x \in E, y \in C \).

Let \( f : E \to (-\infty, +\infty] \) be a convex and Gâteaux differentiable function. The modulus of total convexity of \( f \) at \( x \in \text{int dom } f \) is the function \( \nu_f(x, \cdot) : [0, +\infty) \to [0, +\infty] \) defined by

\[
\nu_f(x,t) := \inf \{D_f(y,x) : y \in \text{dom } f, \|y-x\| = t\}. \quad (12)
\]

The function \( f \) is called totally convex at \( x \) if \( \nu_f(x,t) > 0 \) whenever \( t > 0 \). The function \( f \) is called totally convex if
it is totally convex at any point \( x \in \text{int dom } f \) and is said to be totally convex on bounded sets if \( \gamma_f(B,t) > 0 \) for any nonempty bounded subset \( B \) of \( E \) and \( t > 0 \), where the modulus of total convexity of the function \( f \) on the set \( B \) is the function \( \gamma_f : \text{int dom } f \times [0, +\infty) \to [0, +\infty) \) defined by

\[
\gamma_f(B,t) := \inf \left\{ \gamma_f(x,t) : x \in B \cap \text{dom } f \right\}.
\]

The next lemma will be useful in the proof of our main results.

**Lemma 5** (see [19]). If \( x \in \text{dom } f \), then the following statements are equivalent.

(i) The function \( f \) is totally convex at \( x \).

(ii) For any sequence \( \{y_n\} \subset \text{dom } f \),

\[
\lim_{n \to +\infty} D_f(y_n,x) = 0 \iff \lim_{n \to +\infty} \|y_n - x\| = 0.
\]

Recall that the function \( f \) is called sequentially consistent [18] if, for any two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( E \) such that the first one is bounded,

\[
\lim_{n \to +\infty} D_f(y_n,x_n) = 0 \iff \lim_{n \to +\infty} \|y_n - x_n\| = 0.
\]

**Lemma 6** (see [20]). The function \( f \) is totally convex on bounded sets if and only if the function \( f \) is sequentially consistent.

**Lemma 7** (see [21]). Let \( f : E \to \mathbb{R} \) be a Gâteaux differentiable and totally convex function. If \( x_0 \in E \) and the sequence \( \{D_f(x_n,x_0)\} \) is bounded, then the sequence \( \{x_n\} \) is bounded too.

**Lemma 8** (see [21]). Let \( f : E \to \mathbb{R} \) be a Gâteaux differentiable and totally convex function, \( x_0 \in E \), and let \( C \) be a nonempty, closed, and convex subset of \( E \). Suppose that the sequence \( \{x_n\} \) is bounded and any weak subsequential limit of \( \{x_n\} \) belongs to \( C \). If \( D_f(x_n,x_0) \leq D_f(p,C,x_0,x_0) \) for any \( n \in \mathbb{N} \), then \( \{x_n\} \) converges strongly to \( p \in \text{cl} C \cap x_0 \).

Let \( C \) be a convex subset of \( \text{int dom } f \) and let \( T \) be a self-mapping of \( C \). A point \( p \in C \) is called an asymptotic fixed point of \( T \) (see [22, 23]) if \( C \) contains a sequence \( \{x_n\} \) which converges weakly to \( p \) such that \( \lim_{n \to +\infty} \|x_n - Tx_n\| = 0 \). We denote by \( \bar{F}(T) \) the set of asymptotic fixed points of \( T \).

**Definition 9.** A mapping \( T \) with a nonempty asymptotic fixed point set \( \bar{F}(T) \) is said to be

(i) Bregman strongly nonexpansive (see [24, 25]) with respect to \( \bar{F}(T) \) if

\[
D_f(p,Tx) \leq D_f(p,x), \quad \forall x \in C, \quad p \in \bar{F}(T),
\]

and if, whenever \( \{x_n\} \subset C \) is bounded, \( p \in \bar{F}(T) \) and

\[
\lim_{n \to +\infty} (D_f(p,x_n) - D_f(p,Tx_n)) = 0,
\]

it follows that

\[
\lim_{n \to +\infty} D_f(x_n,Tx_n) = 0.
\]

(ii) Bregman firmly nonexpansive [26] if, for all \( x, y \in C \),

\[
\langle \nabla f( Tx) - \nabla f( Ty), Tx - Ty \rangle \leq \langle \nabla f( x) - \nabla f( y), Tx - Ty \rangle
\]

or, equivalently,

\[
D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x).
\]

The existence and approximation of Bregman firmly nonexpansive mappings were studied in [26]. It is also known that if \( T \) is Bregman firmly nonexpansive and \( f \) is Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of \( E \), then \( F(T) = \bar{F}(T) \) and \( F(T) \) is closed and convex (see [26]). It also follows that every Bregman firmly nonexpansive mapping is Bregman strongly nonexpansive with respect to \( F(T) = \bar{F}(T) \).

**Lemma 10** (see [27]). Let \( E \) be a real reflexive Banach space and \( f : E \to (-\infty, +\infty] \) a proper lower semicontinuous function; then \( f^* : E^* \to (-\infty, +\infty] \) is a proper weak* lower semicontinuous and convex function. Thus, for all \( z \in E \), we have

\[
D_f(z, \nabla f^* \left( \sum_{i=1}^{N} t_i \nabla f(x_i) \right)) \leq \sum_{i=1}^{N} t_i D_f(z, x_i).
\]

In order to solve the equilibrium problem, let us assume that a bifunction \( \Theta : C \times C \to \mathbb{R} \) satisfies the following conditions [28]:

(i) \( \Theta(x,x) = 0 \), for all \( x \in C \).

(ii) \( \Theta \) is monotone; that is, \( \Theta(x, y) + \Theta(y, x) \leq 0 \), for all \( x, y \in C \).

(iii) \( \lim \sup_{t \downarrow 0} \Theta(x + t(z - x), y) \leq \Theta(x, y) \) for all \( x, z, y \in C \).

(iv) \( \Theta \) is convex and lower semicontinuous.

The resolvent of a bifunction \( \Theta \) [29] is the operator \( \text{Res}_{\Theta}^f : E \to 2^C \) defined by

\[
\text{Res}_{\Theta}^f(x) = \{ z \in C : \Theta(z, y) + \langle \nabla f^*(z) - \nabla f^*(x), y - z \rangle \geq 0, \forall y \in C \}.
\]

From Lemma 1 in [24], if \( f : E \to (-\infty, +\infty] \) is a strongly coercive and Gâteaux differentiable function and \( \Theta \) satisfies conditions \((A_1 \sim A_4)\), then \( \text{dom (Res}_{\Theta}^f) = E \). We also know the following lemma which gives us some characterizations of the resolvent \( \text{Res}_{\Theta}^f \).
Lemma 11 (see [24]). Let $E$ be a real reflexive Banach space and $C$ a nonempty closed convex subset of $E$. Let $f : E \to (-\infty, +\infty]$ be a Legendre function. If the bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies the conditions $(A_1)$–$(A_4)$, then the followings hold:

(i) $\text{Res}_f^\Theta$ is single-valued;

(ii) $\text{Res}_f^\Theta$ is a Bregman firmly nonexpansive operator;

(iii) $F(\text{Res}_f^\Theta) = EP(\Theta)$;

(iv) $EP(\Theta)$ is a closed and convex subset of $C$;

(v) for all $x \in E$ and for all $q \in F(\text{Res}_f^\Theta)$, we have

$$D_f(q, \text{Res}_f^\Theta(x)) + D_f(\text{Res}_f^\Theta(x), x) \leq D_f(q, x).$$

(23)

3. Strong Convergence Theorem

In this section, we proved a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and a fixed point of Bregman strongly nonexpansive mapping in a real reflexive Banach space by using the shrinking projection method.

Theorem 12. Let $C$ be a nonempty, closed, and convex subset of a real reflexive Banach space $E$ and $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of $E$. Let $g$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A_1)$–$(A_4)$ and let $T$ be a Bregman strongly nonexpansive mapping from $C$ into itself such that $F(T) = \tilde{F}(T)$ and $G = F(T) \cap EP(g) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $C_0 = C$ and $x_{n+1} = P^{f}_{C_n} x$ for some $n \geq 1$. Since the inequality $D_f(z, u_n) \leq D_f(z, x_n)$ is equivalent to

$$\langle \nabla f(x_n), z - x_n \rangle - \langle \nabla f(u_n), z - u_n \rangle \leq f(u_n) - f(x_n).$$

Therefore, we have

$$C_{n+1} = \{ z \in C_n : \langle \nabla f(x_n), z - x_n \rangle - \langle \nabla f(u_n), z - u_n \rangle \leq f(u_n) - f(x_n) \}. \tag{26}$$

This implies that $C_{n+1}$ is closed and convex. The desired conclusions are proved. These in turn show that $P^{f}_{F(T) \cap EP(g)} x$ and $P^{f}_{C_n} x$ are well defined.

(II) we prove that $G := F(T) \cap EP(g) \subset C_n$ for all $n \geq 0$. Indeed, it is obvious that $G = F(T) \cap EP(g) \subset C_0 = C$. Suppose that $G \subset C_n$ for some $n \in \mathbb{N}$. Let $u \in G \subset C_n$; since $u_n = \text{Res}_f^\Theta(y_n)$, by Lemma 11 and (21), we have

$$D_f(u, u_n) = D_f(u, \text{Res}_f^\Theta y_n) \leq D_f(u, y_n) = D_f(u, \nabla f^\ast (\alpha_n \nabla f(x_n) + (1 - \alpha_n) \nabla f(Tx_n))) \leq \alpha_n D_f(u, x_n) + (1 - \alpha_n) D_f(u, Tx_n) \leq \alpha_n D_f(u, x_n) + (1 - \alpha_n) D_f(u, x_n) = D_f(u, x_n). \tag{27}$$

Hence, we have $u \in C_{n+1}$. This implies that

$$F(T) \cap EP(g) \subset C_n, \; \forall n \in \mathbb{N} \cup \{0\}. \tag{28}$$

So, $\{x_n\}$ is well defined.

(III) We prove that $\{x_n\}$ is a bounded sequence in $C$.

By the definition of $C_n$, we have $x_n = P^{f}_{C_n} x$ for all $n \geq 0$. It follows from Lemma 4(b) that

$$D_f(x_n, x) = D_f(P^{f}_{C_n} x, x) \leq D_f(u, x) - D_f(u, P^{f}_{C_n} x) \leq D_f(u, x), \; \forall n \geq 0, \; u \in G. \tag{29}$$

This implies that $\{D_f(x_n, x)\}$ is bounded. By Lemma 7, $\{x_n\}$ is bounded. Since $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of $E$, by Lemma 1 $\nabla f$ is uniformly continuous and bounded on bounded subsets of $E$. This implies that $\{\nabla f(x_n)\}$ is bounded.

(IV) Now we proved that $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$. From $x_{n+1} \in C_{n+1} \subset C_n$ and $x_n = P^{f}_{C_n} x$, we have

$$D_f(x_n, x) \leq D_f(x_{n+1}, x), \; \forall n \in \mathbb{N} \cup \{0\}. \tag{30}$$

Thus, $\{D_f(x_n, x)\}$ is nondecreasing. So, the limit of $\{D_f(x_n, x)\}$ exists. Since $D_f(x_{n+1}, x) = D_f(x_{n+1}, P^{f}_{C_n} x) \leq D_f(x_{n+1}, x) - D_f(P^{f}_{C_n} x, x) = D_f(x_{n+1}, x) - D_f(x_n, x)$ for all $n \geq 0$, we
have \( \lim_{n \to \infty} D_f(x_{n+1}, x_n) = 0 \). From \( x_{n+1} = P_{C_{n+1}} x \in C_{n+1} \), we have
\[
D_f(x_{n+1}, u_n) \leq D_f(x_{n+1}, x_n), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{31}
\]
Therefore, we have
\[
\lim_{n \to \infty} D_f(x_{n+1}, u_n) = 0. \tag{32}
\]
From Lemma 5, we have
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} \|x_{n+1} - u_n\| = 0. \tag{33}
\]
So, we have
\[
\lim_{n \to \infty} \|x_n - u_n\| = 0. \tag{34}
\]
This means that the sequence \( \{u_n\} \) is bounded. Since \( f \) is uniformly Fréchet differentiable, it follows from Lemma 5 that \( \nabla f \) is uniformly continuous. Therefore, we have
\[
\lim_{n \to \infty} \|\nabla f(x_n) - \nabla f(u_n)\| = 0. \tag{35}
\]
Since \( f \) is uniformly Fréchet differentiable on bounded subsets of \( E \), then \( f \) is uniformly continuous on bounded subsets of \( E \) (see [30, Theorem 1.8]). It follows that
\[
\lim_{n \to \infty} |f(x_n) - f(u_n)| = 0. \tag{36}
\]
From the definition of the Bregman distance, we obtain that
\[
D_f(u, x_n) - D_f(u, u_n)
= \left[ f(u) - f(x_n) - \langle \nabla f(x_n), u - x_n \rangle \right] \\
- \left[ f(u) - f(u_n) - \langle \nabla f(u_n), u - u_n \rangle \right] \tag{37}
= (f(u_n) - f(x_n)) + \langle \nabla f(u_n), u - u_n \rangle \\
+ \langle \nabla f(x_n), x_n - u_n \rangle
\]
for any \( u \in G \).
It follows from (34)–(37) that
\[
\lim_{n \to \infty} \left( D_f(u, x_n) - D_f(u, u_n) \right) = 0. \tag{38}
\]
On the other hand, from \( u_n = \text{Res}_f^T y_n \) and Lemma II(i), for any \( u \in G \) we have that
\[
D_f(u_n, y_n) = D_f(\text{Res}_f^T y_n, y_n)
\leq D_f(u, y_n) - D_f(u, \text{Res}_f^T y_n) \tag{39}
\leq D_f(u, x_n) - D_f(u, \text{Res}_f^T y_n) \\
= D_f(u, x_n) - D_f(u, u_n).
\]
So, we have from (38) that
\[
\lim_{n \to \infty} D_f(u_n, y_n) = 0. \tag{40}
\]
From Lemma 5, we have
\[
\lim_{n \to \infty} \|u_n - y_n\| = 0. \tag{41}
\]
So, from (34) and (41), we have
\[
\lim_{n \to \infty} \|x_n - y_n\| = 0. \tag{42}
\]
This means that the sequence \( \{y_n\} \) is bounded. Since \( f \) is uniformly Fréchet differentiable, it follows from Lemma 1 that
\[
\lim_{n \to \infty} \|\nabla f(x_n) - \nabla f(y_n)\| = 0. \tag{43}
\]
Since \( f \) is uniformly Fréchet differentiable on bounded subsets of \( E \), then \( f \) is uniformly continuous on bounded subsets of \( E \) (see [30]). It follows that
\[
\lim_{n \to \infty} |f(x_n) - f(y_n)| = 0. \tag{44}
\]
From the definition of the Bregman distance, we obtain that
\[
D_f(u, y_n) - D_f(u, x_n)
= \left[ f(u) - f(y_n) - \langle \nabla f(y_n), u - y_n \rangle \right] \\
- \left[ f(u) - f(x_n) - \langle \nabla f(x_n), u - x_n \rangle \right] \tag{45}
= (f(x_n) - f(y_n)) - \langle \nabla f(y_n), u - y_n \rangle \\
+ \langle \nabla f(x_n), y_n - x_n \rangle
\]
for any \( u \in G \).
It follows from (42) to (45) that
\[
\lim_{n \to \infty} \left( D_f(u, y_n) - D_f(u, x_n) \right) = 0. \tag{46}
\]
On the other hand, for any \( u_n = \text{Res}_f^T y_n \) and Lemma II(i), for any \( u \in G \) we have that
\[
D_f(u_n, y_n) = D_f(\text{Res}_f^T y_n, y_n)
\leq D_f(u, y_n) - D_f(u, \text{Res}_f^T y_n) \tag{39}
\leq D_f(u, x_n) - D_f(u, \text{Res}_f^T y_n) \\
= D_f(u, x_n) - D_f(u, u_n).
\]
This together with (46), (16), and \( \lim_{n \to \infty} \alpha_n < 1 \) shows that
\[
\lim_{k \to \infty} \left( D_f(u, T x_n) - D_f(u, x_n) \right) = 0. \tag{48}
\]
Since \( T \) is Bregman strongly nonexpansive, it follows from (48) that
\[
\lim_{n \to \infty} \|x_n - T x_n\| = 0. \tag{49}
\]
(V) Next, we prove that every weak subsequential limit of \( \{x_n\} \) belongs to \( G = F(T) \cap \text{EP}(g) \).
Since \( \{x_n\} \) is bounded, there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( x_{n_k} \rightharpoonup x^* \). Since \( T \) is a Bregman strongly nonexpansive mapping with \( F(T) = \hat{F}(T) \), we have \( x^* \in E\hat{P}(T) \).

From \( x_{n_k} \rightharpoonup x^* \) and (34), we have \( u_{n_k} \rightharpoonup x^* \).

By \( u_n = \text{Res}_{\hat{F}(T)}^{\hat{g}} y_n \), we have

\[
g(u_n, y) + \langle \nabla f(u_n) - \nabla f(y_n), y - u_n \rangle \geq 0, \quad \forall y \in C. \tag{50}
\]

Replacing \( n \) by \( n_k \), we have from (A_2) that

\[
\langle \nabla f(u_{n_k}) - \nabla f(y_{n_k}), y - u_{n_k} \rangle \geq -g(u_{n_k}, y) \geq g(y, u_{n_k}), \quad \forall y \in C. \tag{51}
\]

Since \( g(x, \cdot) \) is convex and lower semicontinuous, it is also weakly lower semicontinuous. So, letting \( k \to \infty \), we have from (35), (43), and (A_4) that

\[
g(y, x^*) \leq 0, \quad \forall y \in C. \tag{52}
\]

For \( t \in (0, 1] \) and \( y \in C \), letting \( y_t = ty + (1 - t)x^* \), there are \( y_t \in C \) and \( g(y_t, x^*) \leq 0 \). By condition (A_1) and (A_4), we have

\[
0 = g(y_t, y_t) \leq tg(y_t, y) + (1 - t)g(y_t, x^*) \leq tg(y_t, y). \tag{53}
\]

Dividing both sides of the above equation by \( t \), we have \( g(y_t, y) \geq 0 \), for all \( y \in C \). Letting \( t \downarrow 0 \), from condition (A_3), we have

\[
g(x^*, y) \geq 0, \quad \forall y \in C. \tag{54}
\]

Therefore, \( x^* \in E\hat{P}(g) \).

(VI) Now, we prove \( x_n \to P_{F(T)\cap E\hat{P}(g)}^f x \).

Let \( w = P_{F(T)\cap E\hat{P}(g)}^f x \). From \( w \in F(T) \cap E\hat{P}(g) \subset C_{n+1} \), we have \( D_f(x_{n+1}, x) \leq D_f(w, x) \). Therefore, Lemma 8 implies that \( \{x_n\} \) converges strongly to \( w = P_{F(T)\cap E\hat{P}(g)}^f x \), as claimed. This completes the proof of Theorem 12. \( \square \)

**Corollary 13.** Let \( C \) be a nonempty, closed, and convex subset of a real reflexive Banach space \( E \) and \( f : E \to \mathbb{R} \) a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of \( E \). Let \( g \) be a bifunction from \( C \times C \) to \( \mathbb{R} \), \( (A_1) \)–\( (A_4) \). Let \( \{x_n\} \) be a sequence generated by \( x_0 = x \in C \), \( C_0 = C \), and

\[
u_n \in C \text{ such that } g(u_n, y) + \langle \nabla f(u_n) - \nabla f(x_n), y - u_n \rangle \geq 0, \quad \forall y \in C, \tag{55}
\]

\[
C_{n+1} = \left\{ z \in C_n : D_f(z, u_n) \leq D_f(z, x_n) \right\},
\]

\[
x_{n+1} = P_{C_{n+1}}^f x
\]

for every \( n \in \mathbb{N} \cup \{0\} \). Then, \( \{x_n\} \) converges strongly to \( P_{E\hat{P}(g)}^f x \), where \( P_{E\hat{P}(g)}^f \) is the Bregman projection of \( E \) onto \( E\hat{P}(g) \).

**Proof.** Putting \( T = I \) in Theorem 12, we obtain Corollary 13. \( \square \)

**Corollary 14.** Let \( C \) be a nonempty, closed, and convex subset of a real reflexive Banach space \( E \) and \( f : E \to \mathbb{R} \) a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of \( E \). Let \( T \) be a Bregman strongly nonexpansive mapping from \( C \) into itself such that \( F(T) = \hat{F}(T) \) and \( G = F(T) \cap E\hat{P}(g) \neq \emptyset \). Let \( \{x_n\} \) be a sequence generated by \( x_0 = x \in C \), \( C_0 = C \), and

\[
u_n = P_{C_{n+1}}^f \left( \alpha_n \nabla f(x_n) + (1 - \alpha_n) \nabla f(Tx_n) \right),
\]

\[
C_{n+1} = \left\{ z \in C_n : D_f(z, u_n) \leq D_f(z, x_n) \right\}, \tag{56}
\]

\[
x_{n+1} = P_{C_{n+1}}^f x
\]

for every \( n \in \mathbb{N} \cup \{0\} \), where \( \{\alpha_n\} \subset [0, 1] \) satisfies

\[
\liminf_{n \to \infty} (1 - \alpha_n) > 0.
\]

Then, \( \{x_n\} \) converges strongly to \( P_{F(T)\cap E\hat{P}(g)}^f x \), where \( P_{F(T)\cap E\hat{P}(g)}^f \) is the Bregman projection of \( E \) onto \( F(T) \).

**Proof.** Putting \( g(x, y) = 0 \) for all \( x, y \in C \) in Theorem 12, we obtain Corollary 14. \( \square \)

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**References**


