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Research Article

Gap Functions and Algorithms for Variational Inequality Problems

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We solve several kinds of variational inequality problems through gap functions, give algorithms for the corresponding problems, obtain global error bounds, and make the convergence analysis. By generalized gap functions and generalized D-gap functions, we give global bounds for the set-valued mixed variational inequality problems. And through gap function, we equivalently transform the generalized variational inequality problem into a constraint optimization problem, give the steepest descent method, and show the convergence of the method.

1. Introduction

Variational inequality problem (VIP) provides us with a simple, natural, unified, and general frame to study a wide class of equilibrium problems arising in transportation system analysis [1, 2], regional science [3, 4], elasticity [5], optimization [6], and economics [7]. Canonical VIP can be described as follows: find a point $x \in K \subset \mathbb{R}^n$ such that

$$\langle T(x), y - x\rangle \geq 0, \quad \forall y \in K, \tag{1}$$

where K is a nonempty closed convex subset of $\mathbb{R}^n$, T is a mapping from $\mathbb{R}^n$ into itself, and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$.

In recent years, considerable interest has been shown in developing various, useful, and important extensions and generalizations of VIP, both for its own sake and for its applications, such as general variational inequality problem (GVIP) [8] and set-valued (mixed) variational inequality problem (SMVIP) [9]. There are significant developments of these problems related to multivalued operators, nonconvex optimization, iterative methods, and structural analysis. More recently, much attention has been given to reformulate the VIP as an optimization problem. And gap functions, which can constitute an equivalent optimization problem, turn out to be very useful in designing new globally convergent algorithms and in analyzing the rate of convergence of some iterative methods. Various gap functions for VIP have been suggested and proposed by many authors in [8, 10–13] and the references therein. Error bounds are functions which provide a measure of the distance between a solution set and an arbitrary point. Therefore, error bounds play an important role in the analysis of global or local convergence analysis of algorithms for solving VIP.

For the VIP defined in (1), the authors in [10] provided an equivalent optimization problem formulation through regularized gap function $G_\alpha : H \to \mathbb{R}$ defined by

$$G_\alpha(x) = \max_{y\in K}\left\{\langle F(x), x - y\rangle - \frac{\alpha}{2}\|x - y\|^2\right\}, \tag{2}$$

where α is a parameter. The authors proved that x is the solution of problem (1) if and only if x is global minimizer of function $G_\alpha(x)$ in K and $G_\alpha(x) = 0$. In order to expand the definition of regularized gap function, the authors in [14] gave the definition of generalized regularized gap function defined by

$$G_\alpha(x) = \max_{y\in K}\left\{\langle F(x), x - y\rangle - \alpha\phi(x, y)\right\}, \tag{3}$$

where ϕ is an abstract function which satisfies conditions ranked as follows:

(C1) ϕ is continuous differentiable on $H \times H$;

(C2) ϕ is nonnegative on $H \times H$;

(C3) ϕ is uniformly strongly convex on H; that is, there exists a positive number λ such that

$$\phi(x, y_1) - \phi(x, y_2) \geq \langle \nabla_2 \phi(x, y_2), y_1 - y_2 \rangle + \lambda \|y_1 - y_2\|^2, \quad \forall x, y_1, y_2 \in H; \tag{4}$$

(C4) $\phi(x, y) = 0 \Leftrightarrow x = y$;

(C5) $\nabla_2(x, y)$ is uniformly Lipschtiz continuous on H; that is, there exists a constant $L' > 0$ such that

$$\|\nabla_2 \phi(x, y_1) - \nabla_2 \phi(x, y_2)\| \leq L' \|y_1 - y_2\|, \quad \forall x, y_1, y_2 \in H. \tag{5}$$

Note that ∇_2 is the partial of ϕ with respect to the second component and conditions (C1)–(C5) can make sense. One can refer to [10, 14] and so forth for more details.

Many gap functions have been explored during the past two decades as it is shown in [10–16] and the references therein. Motivated by their work, in this paper, we solve some classes of VIP through gap functions, give algorithms for the corresponding problems, obtain global error bounds, and make the convergence analysis. We consider generalized gap functions and generalized D-gap functions for SMVIP and give global bounds for the problem through the two functions, respectively. And for GVIP, we equivalently transform it into a constraint optimization problem through gap function, introduce the steepest descent method, and show the convergence of the method.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let K be a nonempty closed convex set in H and let 2^H be the family of all nonempty compact subsets of H.

Let $F, f : H \to H$ be nonlinear operators. The GVIP can be described as follows: Find $x \in H$, $f(x) \in K$ such that

$$\langle F(x), f(y) - f(x) \rangle \geq 0, \quad \forall y \in H, \ f(y) \in K. \tag{6}$$

For single-valued operator $f : H \to \mathbb{R} \cup \{+\infty\}$, which is proper convex and lower semicontinuous, and for given multivalued operator $T : H \to 2^H$, the SMVIP can be described as follows: Find $x \in K$, $w \in T(x)$ such that

$$\langle w, y - x \rangle + f(y) - f(x) \geq 0, \quad \forall y \in K. \tag{7}$$

Note that when $f = 0$, the original problem (7) reduces to a set-valued variational inequality problem; when $f = 0$ and T is a single-valued operator, problem (7) is the right problem discussed in (1).

Recall that the multivalued operator $T : K \subset H \to 2^H$ is said to be strongly monotone with modulus $\beta > 0$ on K if

$$\langle w - w', x - x' \rangle \geq \beta \|x - x'\|^2, \quad \forall (x, w), (x', w') \in \operatorname{graph}(T). \tag{8}$$

And T is said to be Lipschtiz continuous on a nonempty bounded set $B \subset K$, if there exists a positive constant L such that

$$H(T(x), T(y)) \leq L \|x - y\|, \quad \forall x, y \in B, \tag{9}$$

where $H(\cdot, \cdot)$ is the Hausdorff metric on B defined by

$$H(T(x), T(y)) = \max \left\{ \sup_{r \in T(x)} \inf_{s \in T(y)} \|r - s\|, \sup_{s \in T(y)} \inf_{r \in T(x)} \|r - s\| \right\}, \quad \forall x, y \in B. \tag{10}$$

Let $F : \mathbb{R}^n \to \mathbb{R}^n$. Then F is a P_0-function if $\max_{1 \leq i \leq n, x_i \neq y_i} (x_i - y_i)(F_i(x) - F_i(y)) \geq 0$, for all $x, y \in \mathbb{R}^n$ and $x \neq y$. Assume $F_\mu(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ ($\mu > 0$). F_μ is called smoothing approximation function of F, if there exists a positive constant k such that

$$\|F_\mu(x) - F(x)\| \leq ku, \quad \forall u > 0, \ x \in \mathbb{R}^n. \tag{11}$$

And F_μ is a uniform approximation if k is independent of x.

A matrix $M \in \mathbb{R}^{n \times n}$ is a P_0-matrix if each of its principal minors is nonnegative.

We need the following lemmas. The parameters involved in the lemmas can be found in the following sections.

Lemma 1 (see [11]). *If abstract function ϕ satisfies condition (C1), then the following holds:*

$$\langle \nabla_2 \phi(x, y_1) - \nabla_2 \phi(x, y_2), y_1 - y_2 \rangle \geq 2\lambda \|y_1 - y_2\|^2, \quad \forall y_1, y_2 \in H; \tag{12}$$

that is, $\nabla_2(x, \cdot)$ is strong monotone in H, and by (C5), one obtains that $2\lambda \leq L'$.

Lemma 2 (see [17]). *If abstract function ϕ satisfies conditions (C1)–(C4), then*

$$\nabla_2 \phi(x, y) = 0 \Longleftrightarrow x = y. \tag{13}$$

Lemma 3 (see [18]). *If abstract function ϕ satisfies conditions (C1)–(C5) and λ and L' are the corresponding coefficients defined above, then one has*

$$\lambda \|x - y\|^2 \leq \phi(x, y) \leq (L' - \lambda) \|x - y\|^2, \quad \forall x, y \in H. \tag{14}$$

Lemma 4 (see [19]). *If abstract function ϕ satisfies conditions (C1)–(C4), then $G_\alpha(x) \geq \alpha\lambda \|x - \pi_\alpha(x)\|^2$. Moreover, when $G_\alpha(x) = 0$, x is a solution of SMVIP.*

Lemma 5 (see [10]). *If abstract function ϕ satisfies conditions (C1)–(C4), then $g_\alpha(x)$ is differentiable and*

$$\nabla g_\alpha(x) = \nabla g(x) F(x) + \nabla F(x)(g(x) - y_\alpha(x)) - \alpha \nabla_x \phi(g(x), y_\alpha(x)). \tag{15}$$

Lemma 6 (see [10, 19]). *If abstract function ϕ satisfies conditions (C1)–(C4), then g_α is nonnegative, and $g_\alpha(x) = 0 \Leftrightarrow x$ is a solution of GVIP.*

Lemma 7 (see [10]). *Let abstract function ϕ satisfy conditions (C1)–(C4). If $\nabla g_\alpha(x) = 0$ and $\nabla F(x)$ is positive definite, then x is a solution of GVIP(F, f).*

3. Gap Functions and Error Bounds for SMVIP

In this section, by introducing appropriate gap functions, we give global error bound for SMVIP. Firstly, we need the following propositions.

Proposition 8. *Let C be a nonempty closed convex set in H and let f be strictly convex in C. Then f has only one minimum in C.*

Proof. We use proof by contradiction to show the desired result. Let $x_1, x_2 \in C$ be two minimal points of f; that is, $f(x_1) = f(x_2) = \min f(x)$. Since f is strictly convex, one obtains that

$$\begin{aligned} f(\alpha x_1 + (1-\alpha) x_2) &< \alpha f(x_1) + (1-\alpha) f(x_2) \\ &= f(x_1), \quad \forall \alpha \in (0,1). \end{aligned} \tag{16}$$

This implies that there exists a point $x_3 = \alpha x_1 + (1-\alpha)x_2 \in C$, such that $f(x_3) < f(x_1)$, which is a contradiction. This completes the proof. □

Let T, f, and ϕ be defined as above and let K be a nonempty closed convex set in H. Now, we can introduce generalized gap function G_α of SMVIP(T, K) defined as follows:

$$\begin{aligned} G_\alpha(x) &= \max_{y\in H} \Psi_\alpha(x, y) \\ &= \max_{y\in H} \{\langle w, x-y\rangle + f(x) - f(y) - \alpha\phi(x, y)\}, \\ &\qquad \forall x, y \in H, \ \alpha > 0. \end{aligned} \tag{17}$$

From uniform convex of $\phi(x, \cdot)$, one obtains that $-\Psi_\alpha(x, \cdot)$ is also uniform convex in H. By Proposition 8, there exists a minimal point $\pi_\alpha(x)$ of $\phi(x, \cdot)$ in H, such that

$$\begin{aligned} G_\alpha(x) = \langle w, x - \pi_\alpha(x)\rangle &+ f(x) - f(\pi_\alpha(x)) \\ &- \alpha\phi(x, \pi_\alpha(x)). \end{aligned} \tag{18}$$

Proposition 9. *If abstract function ϕ satisfies conditions (C1)–(C4) and $f : H \to \mathbb{R} \cup \{+\infty\}$ is proper convex and lower semicontinuous, then for all $\alpha > 0$, $x = \pi_\alpha(x) \Leftrightarrow x$ is a solution of SMVIP(T, K).*

Proof. From the definition of $\pi_\alpha(x)$, one has

$$\begin{aligned} 0 &\in \partial(-\Psi(x, \pi_\alpha(x))) \\ &= w + \partial f(\pi_\alpha(x)) + \alpha\nabla_2\phi(x, \pi_\alpha(x)). \end{aligned} \tag{19}$$

By the definition of subgradient, we have

$$f(y) \geq f(\pi_\alpha(x)) - \langle w + \alpha\nabla_2\phi(x, \pi_\alpha), y - \pi_\alpha(x)\rangle, \tag{20}$$

which is equivalent to

$$\begin{aligned} &\langle w, y - \pi_\alpha(x)\rangle + f(y) - f(\pi_\alpha(x)) \\ &\quad \geq \alpha\langle -\nabla_2\phi(x, \pi_\alpha(x)), y - \pi_\alpha(x)\rangle. \end{aligned} \tag{21}$$

On the one hand, if $x = \pi_\alpha(x)$, from Lemma 2, one obtains $\nabla_2\phi(x, \pi_\alpha(x)) = 0$, and so does $\alpha\langle -\nabla_2\phi(x, \pi_\alpha(x)), y - \pi_\alpha(x)\rangle = 0$. So, from (21), we have

$$\langle w, y - \pi_\alpha(x)\rangle + f(y) - f(\pi_\alpha(x)) \geq 0, \tag{22}$$

which implies that x is a solution of SMVIP(T, K).

On the other hand, if x is a solution of SMVIP(T, K), take $y = \pi_\alpha(x)$ in (7), then we have

$$\langle w, \pi_\alpha(x) - x\rangle + f(\pi_\alpha(x)) - f(x) \geq 0. \tag{23}$$

From condition (C3), one has

$$\begin{aligned} &\phi(x, x) - \phi(x, \pi_\alpha(x)) \\ &\quad \geq \langle \nabla_2\phi(x, \pi\pi_\alpha(x)), x - \pi_\alpha(x)\rangle \\ &\qquad + \lambda\|x - \pi_\alpha(x)\|^2. \end{aligned} \tag{24}$$

And by conditions (C2) and (C4),

$$\phi(x, x) - \phi(x, \pi_\alpha(x)) \leq 0. \tag{25}$$

So we have

$$\langle \nabla_2\phi(x, \pi_\alpha(x)), x - \pi_\alpha(x)\rangle + \lambda\|x - \pi_\alpha(x)\|^2 \leq 0. \tag{26}$$

Combining (23) with (26), we have $x = \pi_\alpha(x)$. This completes the proof. □

Based on the above discussion, one can obtain the following global error bound.

Theorem 10. *If abstract function ϕ satisfies conditions (C1)–(C5), f is closed convex, and T is strong monotone and Lipschtiz continuous with respect to the solution $\overline{x}$ of SMVIP(T, K), then one has*

$$\|x - \overline{x}\| \leq \frac{L + \alpha L'}{\beta} \|x - \pi_\alpha(x)\|, \tag{27}$$

where L and L' can be found in (5) *and* (9), *respectively.*

Proof. Since $\overline{x}$ is a solution of SMVIP(T, K), take $\overline{w} \in T(\overline{x})$, then we obtain

$$\langle \overline{w}, y - \overline{x}\rangle + f(y) - f(x) \geq 0. \tag{28}$$

Let $y = \pi_\alpha(x)$, for all $x \in H$. Then inequality (28) reduces to

$$\langle \overline{w}, \pi_\alpha(x) - \overline{x}\rangle + f(\pi_\alpha(x)) - f(x) \geq 0. \tag{29}$$

Take $y = \overline{x}$, $\widehat{w} \in T(x)$ in (21) such that $\|\widehat{w} - \overline{w}\| \leq H(T(x), T(\overline{x}))$. Then inequality (21) changes to

$$\begin{aligned} &\langle \widehat{w}, \overline{x} - \pi_\alpha(x)\rangle + f(\overline{x}) - f(\pi_\alpha(x)) \\ &\quad \geq \alpha\langle -\nabla_2\phi(x, \pi_\alpha(x)), \overline{x} - \pi_\alpha(x)\rangle. \end{aligned} \tag{30}$$

Combining (29) and (30), we have

$$\langle \widehat{w} - \overline{w}, \pi_\alpha(x) - \overline{x}\rangle \leq \alpha\langle \nabla_2\phi(x, \pi_\alpha(x)), \overline{x} - \pi_\alpha(x)\rangle. \tag{31}$$

And note that

$$\begin{aligned} &\alpha\langle \nabla_2\phi(x, \pi_\alpha(x)), \overline{x} - \pi_\alpha(x)\rangle \\ &\quad = \alpha\langle \nabla_2\phi(x, \pi_\alpha(x)), \overline{x} - x\rangle \\ &\qquad + \alpha\langle \nabla_2\phi(x, \pi_\alpha(x)), x - \pi_\alpha(x)\rangle \\ &\quad = \alpha\langle \nabla_2\phi(x, \pi_\alpha(x)) - \nabla_2\phi(x, x), \overline{x} - x\rangle \\ &\qquad - \alpha\langle \nabla_2\phi(x, x) - \nabla_2\phi(x, \pi_\alpha(x)), x - \pi_\alpha(x)\rangle \\ &\quad \leq \alpha\|\nabla_2\phi(x, \pi_\alpha(x)) - \nabla_2\phi(x, x)\|\,\|\overline{x} - x\| \\ &\qquad - 2\alpha\lambda\|x - \pi_\alpha(x)\|^2 \\ &\quad \leq \alpha L'\|x - \pi_\alpha(x)\|\,\|\overline{x} - x\| \\ &\qquad - 2\alpha\lambda\|x - \pi_\alpha(x)\|^2. \end{aligned} \tag{32}$$

From (8), one has

$$\begin{aligned} &\beta\|x - \overline{x}\|^2 \\ &\quad \leq \langle \widehat{w} - \overline{w}, x - \overline{x}\rangle \\ &\quad \leq \langle \widehat{w} - \overline{w}, x - \pi_\alpha(x)\rangle + \langle \widehat{w} - \overline{w}, \pi_\alpha(x) - \overline{x}\rangle \\ &\quad \leq L\|x - \overline{x}\|\,\|x - \pi_\alpha(x)\| \\ &\qquad + \alpha L'\|x - \pi_\alpha(x)\|\,\|x - \overline{x}\| \\ &\quad \leq (L + \alpha L')\|x - \overline{x}\|\,\|x - \pi_\alpha(x)\|, \end{aligned} \tag{33}$$

so we have

$$\|x - \overline{x}\| \leq \frac{L + \alpha L'}{\beta}\|x - \pi_\alpha(x)\|. \tag{34}$$

This completes the proof. □

Theorem 11. *If abstract function ϕ satisfies conditions (C1)–(C5) and T is strong monotone for the solution $\overline{x}$ of SMVIP and is Lipschtiz continuous with module L, then $\sqrt{G_\alpha}$ has global error bound with respect to SMVIP; that is,*

$$\|x - \overline{x}\| \leq \frac{L + L'}{\beta\sqrt{\alpha\lambda}}\sqrt{G_\alpha(x)}. \tag{35}$$

Proof. By Lemma 4 and Theorem 10, one obtains

$$\begin{aligned} &G_\alpha(x) \geq \alpha\lambda\|x - \pi_\alpha(x)\|^2, \\ &\|x - \overline{x}\| \leq \frac{L + \alpha L'}{\beta}\|x - \pi_\alpha(x)\|. \end{aligned} \tag{36}$$

So we can obtain

$$G_\alpha(x) \geq \frac{\alpha\lambda\beta^2}{(L + \alpha L')^2}\|x - \overline{x}\|^2, \tag{37}$$

which implies that

$$\|x - \overline{x}\| \leq \frac{L + \alpha L'}{\beta\sqrt{\alpha\lambda}}\sqrt{G_\alpha(x)}. \tag{38}$$

This completes the proof. □

Now, we introduce generalized D-gap function $H_{\alpha\gamma}$ for SMVIP which is defined by

$$\begin{aligned} H_{\alpha\gamma}(x) &= G_\alpha(x) - G_\gamma(x) \\ &= \max_{y\in H}\Psi_\alpha(x, y) - \max_{y\in H}\Psi_\gamma(x, y) \\ &= \langle w, \pi_\gamma(x) - \pi_\alpha(x)\rangle + f(\pi_\gamma(x)) - f(\pi_\alpha(x)) \\ &\quad + \beta\phi(x, \pi_\gamma(x)) - \alpha\phi(x, \pi_\alpha(x)), \end{aligned} \tag{39}$$

where $\pi_\alpha(x)$ and $\pi_\gamma(x)$ are minimal points for $-\Psi_\alpha(x, \cdot)$ and $-\Psi_\gamma(x, \cdot)$ in H, respectively, and $0 < \alpha < \gamma$. For $H_{\alpha\gamma}(x)$, we can conclude the following result.

Proposition 12. *If abstract function ϕ satisfies condition (C3), then one has*

$$(\gamma - \alpha)\phi(x, \pi_\gamma(x)) \leq H_{\alpha\gamma}(x) \leq (\gamma - \alpha)\phi(x, \pi_\alpha(x)). \tag{40}$$

Proof. From the definition of $H_{\alpha\gamma}(x)$, one obtains that

$$\begin{aligned} H_{\alpha\gamma}(x) &= \max_{y\in H}\Psi_\alpha(x, y) - \max_{y\in H}\Psi_\gamma(x, y) \\ &= \Psi_\alpha(x, \pi_\alpha(x)) - \Psi_\gamma(x, \pi_\gamma(x)) \\ &\geq \Psi_\alpha(x, \pi_\gamma(x)) - \Psi_\gamma(x, \pi_\gamma(x)) \\ &= \langle w, x - \pi_\gamma(x)\rangle - \alpha\phi(x, \pi_\gamma(x)) \\ &\quad - \langle w, x - \pi_\gamma(x)\rangle + \gamma\phi(x, \pi_\gamma(x)) \\ &= (\gamma - \alpha)\phi(x, \pi_\gamma(x)). \end{aligned} \tag{41}$$

$H_{\alpha\gamma}(x) \leq (\gamma - \alpha)\phi(x, \pi_\alpha(x))$ can be proved similarly. This completes the proof. □

From Proposition 12, one has the following.

Proposition 13. *If ϕ satisfies conditions (C1)–(C4), then $H_{\alpha\gamma}(x)$ is nonnegative, and $H_{\alpha\gamma}(x) = 0 \Leftrightarrow x$ is a solution of SMVIP(T, K).*

Proof. From Proposition 12 and nonnegative property of $\phi(\cdot, \cdot)$, we have that $H_{\alpha\gamma}(x)$ is nonnegative.

On the one hand, if $H_{\alpha\gamma}(x) = 0$, then by conditions (C2) and (C4), one has $x = \pi_\alpha(x)$. Then by Proposition 9, we conclude that x is a solution of SMVIP(T, K).

On the other hand, if x is a solution of SMVIP(T, K), by Proposition 9, one obtains that $x = \pi_\alpha(x)$. From condition (C4), one has $\phi(x, \pi_\alpha(x)) = 0$. And since $H_{\alpha\gamma}(x)$ is nonnegative, we have $H_{\alpha\gamma}(x) = 0$. This completes the proof. □

By the generalized D-gap function, we have the following error bound for SMVIP(T, K).

Theorem 14. *Let ϕ satisfy conditions (C1)–(C5). T is strong monotone for the solution $\overline{x}$ of SMVIP and is Lipschtiz continuous with module L; then $H_{\alpha\gamma}(x)$ has global error bound with respect to SMVIP; that is,*

$$\|x - \overline{x}\| \le \frac{L + L'}{\beta\sqrt{\lambda(\gamma - \alpha)}}\sqrt{H_{\alpha\gamma}(x)}. \tag{42}$$

Proof. From Lemma 3, Theorem 10, and Proposition 13, we have

$$\begin{aligned} H_{\alpha\gamma}(x) &\ge (\gamma - \alpha)\,\phi\left(x - \pi_\gamma(x)\right) \\ &\ge (\gamma - \alpha)\,\lambda\left\|x - \pi_\gamma(x)\right\|^2 \\ &\ge \lambda(\gamma - \alpha)\left(\frac{\beta}{L + \alpha L'}\right)^2\|x - \overline{x}\|^2, \end{aligned} \tag{43}$$

which implies that

$$\|x - \overline{x}\| \le \frac{L + L'}{\beta\sqrt{\lambda(\gamma - \alpha)}}\sqrt{H_{\alpha\gamma}(x)}. \tag{44}$$

This completes the proof. □

4. Steepest Descent Method for GVIP

In this section, by introducing appropriate generalized gap function, the original GVIP(F, f) in (6) can be changed into an optimization problem with restrictions. When one designs algorithms to solve the optimization problem, the gradient of objective function is unavoidable. We try to design a new algorithm, constructing a class of descent direction, to solve the optimization problem. In the following, we set H to be $\mathbb{R}^n$. And we introduce the following generalized gap function for GVIP(F, f):

$$\begin{aligned} g_\alpha(x) &= \max_{g(y)\in K}\Psi_\alpha(x, y) \\ &= \max_{g(y)\in K}\left\{\langle F(x), f(x) - f(y)\rangle - \alpha\phi\left(f(x), f(y)\right)\right\} \\ &= \left\{\langle F(x), f(x) - y_\alpha(x)\rangle - \alpha\phi\left(f(x), y_\alpha(x)\right)\right\}, \end{aligned} \tag{45}$$

where $y_\alpha(x)$ is a minimal point for $-\Psi_\alpha(x, \cdot)$, α is a positive parameter, and ϕ satisfies conditions (C1)–(C5) stated above. For g_α, we have the following useful results [14]:

(A1) $g_\alpha(x)$ is nonnegative in K;

(A2) $g_\alpha(x) = 0$ for some $x \in K \Leftrightarrow x$ is a solution of VIP;

(A3) $y_\alpha(x)$ is the only minimizer of $\Psi_\alpha(x, \cdot)$ in K.

And similar to the discussion in [10, 11], we also give the following two assumptions:

(a) $\nabla F(x)$ is positive definite for all $x \in K$;

(b) $\nabla_x\phi(x, y) = -\nabla_y\phi(x, y)$.

From Lemmas 5–7, we obtain that the original GVIP (6) is equivalent to the following optimization problem:

$$\min_{s.t. x\in K} g_\alpha(x). \tag{46}$$

For problem (46), we give the following algorithm.

Algorithm 15.

Step 0. Choose an initial value $x^0 \in K$, $\varepsilon, t \in (0, 1)$, and put $k = 0$.

Step 1. If $g_\alpha(x) \le \varepsilon$, then we can end the circulation.

Step 2. Compute $y_\alpha(x^k)$, and let

$$d^k = y_\alpha\left(x^k\right) - f\left(x^k\right). \tag{47}$$

Step 3. Let m_k be the minimal nonnegative integer m, such that

$$g_\alpha\left(x^k + t^m d^k\right) \le g_\alpha\left(x^k\right) - t^{2m}\left\|d^k\right\|^2. \tag{48}$$

Step 4. Let $f(x^{k+1}) = f(x^k) + t^{m_k}d^k$, $k = k + 1$; go to Step 1.

Proposition 16. *Let $\{x^k\}$ be a sequence generated by Algorithm 15. If $\{x^k\}$ are not the solutions of GVIP(F, f), then*

$$\nabla g_\alpha\left(x^k\right)^T d^k < 0; \tag{49}$$

that is, d^k is the descent direction of g_α at x^k, where d^k is defined in (47).

Proof. To begin, we show that $f(x^k) \in K$, for all positive integer k. From Algorithm 15, one obtains that $f(x^0) \in K$. We prove this result by induction. Assume $f(x^k) \in K$; we only need to show that $f(x^{k+1}) \in K$. Since $x^k, y_\alpha(x^k) \in K$, $t_k \in (0, 1)$, and K is convex, we have

$$\begin{aligned} f\left(x^{k+1}\right) &= f\left(x^k\right) + t_k d^k \\ &= (1 - t_k)\,f\left(x^k\right) + t_k y_\alpha\left(x^k\right) \in K. \end{aligned} \tag{50}$$

For simplicity, $y_\alpha(x^k)$, x^k are replaced by y_α, x, respectively. From Lemma 5, one has

$$\begin{aligned} &\nabla g_\alpha(x)^T d \\ &\quad = \left\{y_\alpha F(x) + \nabla F(x)\left(g(x) - y_\alpha\right) - \alpha\nabla_x\phi\left(g(x), y_\alpha\right)\right\}^T d \\ &\quad = \left(g(x) - y_\alpha\right)^T\nabla F(x)\left(y_\alpha - g(x)\right) \\ &\qquad + \left\{\nabla g(x)\,F(x) - \alpha\nabla_x\phi\left(g(x), y_\alpha\right)\right\}^T\left(y_\alpha - g(x)\right). \end{aligned} \tag{51}$$

Since $(g(x)-y_\alpha)^T\nabla F(x)(y_\alpha-g(x))<0$, we only need to show that $\{\nabla g(x)F(x)-\alpha\nabla_x\phi(g(x),y_\alpha)\}^T(y_\alpha-g(x))\le 0$. Since y_α is the unique minimizer of $-\Psi(x,\cdot)$ in K, we have

$$\begin{aligned}&\langle -\nabla_y\Psi(x,y_\alpha),u-y_\alpha\rangle\\&\quad=\left\langle\nabla g(x)F(x)+\alpha\nabla_y\phi(x,y_\alpha),u-y_\alpha\right\rangle\\&\quad\ge 0,\quad\forall u\in K.\end{aligned}\tag{52}$$

Let $u=x\in K$ in (52). One has

$$\left\{\nabla g(x)F(x)+\alpha\nabla_y\phi(x,y_\alpha)\right\}^T(y_\alpha-x)\le 0.\tag{53}$$

From assumption (b), we have

$$\left\{\nabla g(x)F(x)-\alpha\nabla_x\phi(x,y_\alpha)\right\}^T(y_\alpha-x)\le 0.\tag{54}$$

This completes the proof. □

Now, we are in a position to show the global convergence result for Algorithm 15.

Theorem 17. *Let $\{x^k\}$ be a sequence generated by Algorithm 15, and let $x^\star$ be the cluster point of $\{x^k\}$. Then $x^\star$ is a solution of GVIP(F,f).*

Proof. Let $\{x^k\}_K$ be a subsequence which converges to $x^\star$. If $g_\alpha(x^\star)=0$, then from Lemma 6, $x^\star$ is a solution of GVIP. If $g_\alpha(x^\star)\ne 0$, from the continuous property, one obtains that $\{y_\alpha(x^k)\}_K\to y_\alpha(x^\star)$ which implies that

$$\left\{d^k\right\}\longrightarrow y_\alpha(x^\star)-f(x^\star).\tag{55}$$

Now, we begin to show that the cluster point $d^{x^\star}$ of $\{d^k\}_K$ is zero. We use proof by contradiction. Assume $d^\star=y_\alpha(x^\star)-x^\star\ne 0$. On the one hand, from Proposition 16, one has that

$$\nabla g_\alpha(x^\star)^T<0.\tag{56}$$

On the other hand, from Proposition 16, we obtain that $\{g_\alpha(x^k)\}$ is monotonically decreasing and bounded; that is, the sequence $\{g_\alpha(x^k)\}$ is convergent. From step 3 of Algorithm 15, one has

$$0\le t^{2m_k}\left\|d^k\right\|^2\le g_\alpha\left(x^k\right)-g_\alpha\left(x^{k+1}\right)\longrightarrow 0,\qquad\text{as } k\longrightarrow\infty.\tag{57}$$

Hence, we have $\lim_{k\to\infty}t^{2m_k}\|d^k\|^2=0$; that is,

$$\lim_{k\to\infty}t^{2m_k}=0\quad\left(d^\star\ne 0\right).\tag{58}$$

Without loss of generality, we assume $t_k\in(0,1)$, for all k. Then one cannot find the minimal nonnegative integer m_k; that is,

$$g_\alpha\left(x^k+t^{m_k-1}d^k\right)>g_\alpha\left(x^k\right)-\left(t^{m_k}\right)^2\left\|d^k\right\|^2,\quad\forall k,\tag{59}$$

Or, equally,

$$\frac{g_\alpha\left(x^k+t^{m_k-1}d^k\right)-g_\alpha\left(x^k\right)}{t^{m_k}}>-t^{m_k}\left\|d^k\right\|^2,\quad\forall k.\tag{60}$$

Let $k\to\infty$, from (58), and g_α be continuous and differentiable; we can obtain

$$\nabla g_\alpha(x^\star)^T\ge 0.\tag{61}$$

Inequalities (56) and (61) are at odds. This completes the proof. □

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