Research Article

Oscillation Criteria for Nonlinear Fractional Differential Equations

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Received 5 October 2013; Accepted 3 December 2013

Academic Editor: Nazim Idrisoglu Mahmudov

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Several oscillation criteria are established for nonlinear fractional differential equations of the form

\[ \{a(t)[(r(t)D_\alpha x(t))]^\eta\}^\prime - F(t, \int_{1}^{\infty} (v - t)^{-\alpha} x(v) dv) = 0, \]

where \( D_\alpha x(t) \) is the Liouville right-side fractional derivative of order \( \alpha \in (0,1) \) of \( x \) and \( \eta \) is a quotient of two odd positive integers. We also give some examples to illustrate the main results. To the best of our knowledge, the results are initiation for the oscillatory behavior of the equations.

1. Introduction

In this paper, we are dealing with the oscillation problem of nonlinear fractional differential equations of the form

\[ \{a(t)[(r(t)D_\alpha x(t))]^\eta\}^\prime - F(t, \int_{1}^{\infty} (v - t)^{-\alpha} x(v) dv) = 0, \quad t \geq t_0 > 0, \]

where \( \alpha \in (0,1) \) is a constant, \( \eta \) is a ratio of two odd positive integers. \( D_\alpha x(t) \) is the Liouville right-side fractional derivative of order \( \alpha \) of \( x(t) \) defined by

\[ D_\alpha x(t) = - \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_{t_0}^{\infty} (v - t)^{-\alpha} x(v) dv, \quad t \in R_+ = (0, \infty). \]

Throughout this paper, we will suppose that the following conditions hold:

\( (A_1) \) \( a(t) \in C([t_0, \infty); R_+), r(t) \in C^2([t_0, \infty); R_+), \)

\[ \int_{t_0}^{\infty} (ds/a_1^\eta(s)) = \infty, \int_{t_0}^{\infty} (ds/r(s)) = \infty; \]

\( (A_2) \) \( F(t, G) \in C([t_0, \infty) \times R; R), \)

there exists a function \( q(t) \in C([t_0, \infty); R_+) \) such that \( F(t, G)/G^\gamma \geq q(t) \) for \( G \neq 0 \) and \( x \neq 0, t \geq t_0. \)

By a solution of (1), we mean a function \( x(t) \in C(R_+; R) \) such that \( \int_{t_0}^{\infty} (v - t)^{-\alpha} x(v) dv \in C^1(R_+; R), r(t)D_\alpha x(t) \in C^2(R_+; R) \) and satisfies (1) on \([T_x, \infty)\). A nontrivial solution of (1) is called oscillatory if it has arbitrary large zero. Otherwise, it is called nonoscillatory. Equation (1) is called oscillatory if all of its solutions are oscillatory.

The theory of fractional derivatives goes back to Leibniz’s note in [1], and this led to the appearance of the theory of derivatives and integrals of arbitrary order. The theory had taken a more or less finished form due primarily to Liouville, Grünwald, Letnikov, and Riemann by the end of nineteenth century. We can see some of the books such as [2, 3] on the subject of fractional derivatives and fractional integrals.

Fractional differential equations are generalizations of classical differential equations of integer order. In the last few decades, many researchers found that fractional derivatives and fractional integrals were applied in widespread fields of science and engineering, especially in mathematical modeling and simulation of systems and processes, instead of simply being applied in pure theoretical fields of mathematics. Nowadays, many articles have investigated some aspects of fractional differential equations, such as the existence, the uniqueness and stability of solutions, the methods for explicit and numerical solutions, and the stability of solutions (we refer the reader to see [4–9] and the references cited in there). In very recent days, the research on oscillation of various
fractional differential equations is being a hot topic; see [10–16].

In [10], Grace et al. discussed the oscillation of the following question:

\[
D^\alpha_t x + f_1(t, x) = v(t) + f_2(t, x), \quad \lim_{t \to \infty} f_1^{-\eta} x(t) = b_1,
\]

where \(D^\alpha_t\) denotes the Riemann-Liouville differential operator of order \(q\) with \(0 < q \leq 1\), and the functions \(f_1, f_2, v\) are continuous functions.

In [11], Chen et al. established several oscillation criteria for (3) with some additional initial conditions and \(m - 1 \leq q \leq m\), \(m \geq 1\) is an integer. They improved and extended some results of [10].

In [12], Chen considered the oscillation of the fractional differential equation

\[
\left[ r(t) (D^\alpha_t y)(t) \right]' - q(t) f \left( \int_0^t (v - t)^{-\alpha} v(v) dv \right) = 0, \quad t > 0
\]

with \(0 < \alpha \leq 1\).

In [13], Han et al. have established some oscillation criteria for the equation

\[
\left[ a(t) [r(t) D^\alpha_t x(t)]' \right]' + p(t) [r(t) D^\alpha_t x(t)]' - q(t) \int_0^\infty (\xi - t)^{-\alpha} x(\xi) d\xi = 0, \quad t > t_0
\]

with \(0 < \alpha < 1\) and established some new interval oscillation criteria by using a generalized Riccati transformation and inequality technique.

In [14], Qi and Cheng studied the following equation:

\[
\left( a(t) [r(t) D^\alpha_t x(t)]' \right)' + p(t) [r(t) D^\alpha_t x(t)]' - q(t) \int_0^\infty (\xi - t)^{-\alpha} x(\xi) d\xi = 0, \quad t > t_0
\]

with \(0 < \alpha < 1\) and established some new interval oscillation criteria by using a generalized Riccati transformation and inequality technique.

In [15], Feng and Meng paid attention to the oscillation of the fractional differential equation

\[
D^\alpha_t \left[ r(t) \psi(x(t)) D^\alpha_t x(t) \right] + q(t) f(x(t)) = e(t), \quad t > t_0 > 0, \quad 0 < \alpha < 1.
\]

In [16], Chen considered the oscillation of the fractional differential equation

\[
\left[ r(t) (D^\alpha_t y)(t) \right]' - q(t) f \left( \int_0^t (s - t)^{-\alpha} x(s) ds \right) = 0, \quad t > 0, \quad 0 < \alpha < 1.
\]

The purpose of this paper is to establish some oscillation criteria for (1) by generalized Riccati function and present some applications for our results.

In order to prove our theories, we use the general weighted functions from the class \(X\). We say that a function \(H = H(t, s)\) belongs to the function class \(X\), if \(H \in C(D; R_+)\), where \(D = \{(t, s) : t_0 \leq s \leq t < \infty\}\), which satisfies \(H(t, t) = 0\), \(H(t, s) > 0\) for \(t > s\), and has nonpositive continuous partial derivative \(H'_t(t, s)\) on \(D\).

2. Main Results

First, we set

\[
G(t) = \int_t^\infty (v - t)^{-\alpha} x(v) dv,
\]

then it follows that

\[
G'(t) = -\Gamma(1 - \alpha) (-D^\alpha_x x)(t) .
\]

We give the following lemmas for our results.

Lemma 1 (see [17]). Let \(X\) and \(Y\) be nonnegative; then

\[
m X^{m-1} - X^m \leq (m - 1) Y^m \quad \text{for } m > 1.
\]

Lemma 2. Assume that \(x(t)\) is an eventually positive solution of (1), and

\[
\int_{t_0}^\infty \frac{1}{r(\xi)} \int_{t_0}^\infty \left( \frac{1}{a(\tau)} \int_{t_0}^\infty q(s)ds \right)^{1/\eta} d\tau d\xi = \infty;
\]

then there exists a sufficiently large \(T\) such that \((r(t)D^\alpha x(t))' < 0\) on \([T, \infty)\), and one of the following two conditions hold:

(i) \(D^\alpha x(t) < 0\) on \([T, \infty)\),

(ii) \(D^\alpha x(t) > 0\) on \([T, \infty)\) and \(\lim_{t \to \infty} G(t) = 0\).

Proof. From the hypothesis, there exists a \(t_1\) such that \(x(t) > 0\) on \([t_1, \infty)\), so that \(G(t) > 0\) on \([t_1, \infty)\), and we have

\[
\left[ a(t) \left[ r(t) D^\alpha x(t) \right]' \right]' = F(t) \int_0^\infty (v - t)^{-\alpha} x(v) dv
\]

\[
\geq q(t) G^\alpha(t) > 0, \quad t > T_1.
\]

Then \(a(t)[r(t)D^\alpha x(t)]'^\eta\) is strictly increasing on \([t_1, \infty)\), and we can conclude that \((r(t)D^\alpha x(t))'\) is eventually of one sign. We claim that \((r(t)D^\alpha x(t))' < 0\) on \([t_2, \infty)\), where \(t_2 > t_1\) is sufficiently large. Otherwise, if there exists a \(t_3 > t_2\) such that \((r(t)D^\alpha x(t))'|_{t_3} > 0\), then we can get \((r(t)D^\alpha x(t))' > 0\) on \([t_3, \infty)\); from those conditions we get

\[
\delta = a(t) \left[ (r(t) D^\alpha x(t))' \right]'^\eta \geq a(t) \left[ (r(t) D^\alpha x(t))' \right]' lb_1 \geq \delta > 0, \quad t \geq t_5,
\]

that is,

\[
(r(t) D^\alpha x(t))' \geq \frac{\delta}{a(t)^{1/\eta}} > 0.
\]
Integrating two sides of the previous inequality from $t_3$ to $t$ leads to
\[ r(t) D^\alpha_x(t) - r(t_3) D^\alpha_x(t_3) \geq 6 \int_{t_3}^{t} \frac{ds}{(a(s))^{1/\eta}}. \] (16)

Then from (A1), we have $\lim_{t \to \infty} r(t)D^\alpha_x(t) = +\infty$, which implies that for a certain constant $t_4 > t_3$, $D^\alpha_x(t) > 0, t \in [t_4, \infty)$, then
\[
G(t) - G(t_4) = \int_{t_4}^{t} G'(s) \, ds \\
= -\Gamma(1-\alpha) \int_{t_4}^{t} \frac{r(s) D^\alpha_x(s)}{r(s)} \, ds \\
\leq -\Gamma(1-\alpha) r(t_4) D^\alpha_x(t_4) \int_{t_4}^{t} \frac{ds}{(s)^{1/\eta}};
\] (17)

by (A1) we obtain $\lim_{t \to \infty} G(t) = -\infty$, which contradicts to $G(t) > 0$ on $[t_4, \infty)$.

So we have $(r(t)D^\alpha_x(t))^\eta < 0$ on $[t_2, \infty)$, and $D^\alpha_x(t)$ is eventually of one sign. There are two possibilities: (i) $D^\alpha_x(t) < 0$ on $[T, \infty)$, (ii) $D^\alpha_x(t) > 0$ on $[T, \infty)$, where $T$ is sufficiently large.

Assume that $D^\alpha_x(t) > 0, t \in [t_5, \infty)$ for certain sufficiently large constant $t_5 > t_4$; then $G'(t) < 0, t \in [t_5, \infty)$ and we have $\lim_{t \to \infty} G(t) = \beta \geq 0$. We claim that $\beta = 0$. Otherwise, let $\beta > 0$; then $G(t) \geq \beta$ on $[t_5, \infty)$, and by (13), we have
\[
\{a(t)[(r(t)D^\alpha_x(t))]^\eta\}_t \geq q(t) G'^{\eta}(t) \geq q(t) \beta^{\eta}, \\
t \in [t_5, \infty).
\] (18)

Integrating two sides of above inequality from $t$ to $\infty$ leads to
\[
-a(t)[(r(t)D^\alpha_x(t))]^\eta \geq \int_{t}^{\infty} q(s) \beta^{\eta} ds - \lim_{t \to \infty} a(t)[(r(t)D^\alpha_x(t))]^\eta \\
\geq \beta^{\eta} \int_{t}^{\infty} q(s) ds,
\] (19)

which means
\[
(r(t)D^\alpha_x(t))^\eta \\
\leq -\beta \left[ \frac{1}{a(t)} \int_{t}^{\infty} q(s) ds \right]^{1/\eta}, \quad t \in [t_5, \infty).
\] (20)

Integrating two sides of (20) from $t$ to $\infty$ yields
\[
-(r(t)D^\alpha_x(t)) \\
\leq -\lim_{t \to \infty} (r(t)D^\alpha_x(t)) - \beta \int_{t}^{\infty} \left[ \frac{1}{a(t)} \int_{t}^{\infty} q(s) ds \right]^{1/\eta} \, d\tau \\
\leq -\beta \int_{t}^{\infty} \left[ \frac{1}{a(t)} \int_{t}^{\infty} q(s) ds \right]^{1/\eta} \, d\tau, \quad t \in [t_5, \infty).
\] (21)

That is,
\[
G'(t) \leq -\beta \Gamma(1-\alpha) \int_{t}^{\infty} \left[ \frac{1}{a(t)} \int_{t}^{\infty} q(s) ds \right]^{1/\eta} \, d\tau.
\] (22)

Integrating two sides of (22) from $t_5$ to $t$, we have
\[
G(t) - G(t_5) \\
\leq -\beta \Gamma(1-\alpha) \int_{t_5}^{t} \left[ \frac{1}{a(t)} \int_{t}^{\infty} q(s) ds \right]^{1/\eta} \, d\tau \times \left\{ \int_{t}^{\infty} \frac{1}{a(t)} \int_{t}^{\infty} q(s) ds \right\}^{1/\eta} \, d\xi.
\] (23)

By (12), we have $\lim_{t \to \infty} G(t) = -\infty$, which contradicts to the fact that $G(t) > 0$. Then we get that $\beta = 0$, which is $\lim_{t \to \infty} G(t) = 0$. The proof is complete.

**Lemma 3.** Assume that $x(t)$ is an eventually positive solution of (1) such that $(r(t)D^\alpha_x(t))^\eta < 0, D^\alpha_x(t) < 0$ on $[t_1, \infty)$, where $t_1$ is sufficiently large and $t_1 > t_0$. Then one has
\[
G'(t) \geq -\Gamma(1-\alpha) \int_{t_1}^{t} \left[ \frac{1}{a(t)} \int_{t}^{\infty} q(s) ds \right]^{1/\eta} \, d\tau.
\] (24)

where $A(t_1, t) = \int_{t_1}^{t} a^{-1/\eta}(s) ds$.

**Proof.** From (13), we get that $a(t)g(r(t)D^\alpha_x(t))$ is strictly increasing on $[t_1, \infty)$, so we get
\[
-(r(t)D^\alpha_x(t)) \leq -r(t) D^\alpha_x(t) - r(t_1) D^\alpha_x(t_1) \\
= \int_{t_1}^{t} \left[ a(t) \int_{t}^{\infty} q(s) ds \right]^{1/\eta} \, d\tau \\
\leq \int_{t_1}^{t} a^{-1/\eta}(s) \left( D^\alpha_x(s) \right)^{1/\eta} \, ds \\
\leq a^{1/\eta}(t) \left( r(t) D^\alpha_x(t) \right)^{1/\eta} \int_{t_1}^{t} \frac{ds}{a^{1/\eta}(s)} \\
= A(t_1) a^{1/\eta}(t) \left( r(t) D^\alpha_x(t) \right)^{1/\eta}.
\] (25)

Then we get
\[
G'(t) \geq -\Gamma(1-\alpha) \int_{t_1}^{t} A(t_1, t) a^{1/\eta}(t) \left( r(t) D^\alpha_x(t) \right)^{1/\eta}. \] (26)

**Theorem 4.** Assume that $(A_1), (A_2),$ and (12) hold; if there exists a function $\rho \in C^1([t_0, \infty), R_+)$ such that
\[
\int_{t_0}^{\infty} \rho(s) q(s) ds - \frac{1}{(\eta + 1)^{\eta+1}} \left( \rho(t) \right)^{\eta+1} \left( \rho(s) \right)^{\eta+1} \times \frac{\rho(s) r(s) ds}{\Gamma(1-\alpha) A(t_2, s)^{1/\eta}} = \infty
\] (27)
for all sufficiently large constants \( t_2 \) and \( T \), where \( \rho'_+(s) = \max\{\rho'(s), 0\} \), \( A(t_2, t) \) is defined in Lemma 3, \( t \geq T \geq t_2 \geq t_0 > 0 \), then every solution of (1) is oscillatory or satisfies \( \lim_{t \to \infty} G(t) = 0 \).

**Proof.** Suppose to the contrary that (1) has a nonoscillatory solution \( x(t) \) on \([t_0, \infty)\); without loss of generality, we assume that \( x(t) > 0 \) for all sufficiently large constants \( t \geq t_1, \infty \), where \( t_1 \) is sufficiently large. By Lemma 2 we have \( (r(t)D^\alpha x(t))'' < 0 \) on \([t_0, \infty)\), where \( t_0 \) is sufficiently large and \( t_2 > t_1 \); then \( D^\alpha x(t) < 0 \) on \([t_2, \infty)\) or \( \lim_{t \to \infty} G(t) = 0 \). (In the case \( x(t) \) is eventually negative, it can be proved similarly.)

If \( D^\alpha x(t) < 0 \) on \([t_2, \infty)\), then the conclusion of Lemma 3 holds. We define

\[
\omega(t) = -\rho(t) \left( a(t) \left[ (r(t)D^\alpha x(t))' \right]^\eta G^\eta(t) \right),
\]

then \( \omega(t) > 0 \) on \([t_2, \infty)\), and from (1), \((A_2)\), and Lemma 3, we have

\[
\omega'(t) = \frac{\rho'_+(t)}{\rho(t)} \omega(t) = \frac{\rho'_+(t)}{\rho(t)} w(t) - \rho(t) \frac{F(t, G(t))}{G^\eta(t)}
\]

\[
+ \eta \rho(t) \left( a(t) \left[ (r(t)D^\alpha x(t))' \right]^\eta \right) \times \left( \frac{A(t_2, t)}{r(t)} \right) \alpha^{1/\eta}(t)
\]

\[
\times \left( (r(t)D^\alpha x(t))' \right)^{-1} \right) \times (G^{\eta+1}(t)^{-1})
\]

\[
\leq \frac{\rho'_+(t)}{\rho(t)} w(t) - \rho(t) q(t) \left( \frac{1}{\eta} (1 - \alpha) \right)
\]

\[
\times \frac{A(t_2, t) a^{(\eta+1)/\eta}(t)}{r(t)} \left[ (r(t)D^\alpha x(t))' \right]^{\eta+1} G^{\eta+1}(t)
\]

\[
= \frac{\rho'_+(t)}{\rho(t)} w(t) - \rho(t) q(t) \left( \frac{1}{\eta} (1 - \alpha) \right)
\]

\[
\times \frac{A(t_2, t)}{r(t)} \frac{1}{\rho^{1/\eta}(t)} w^{(\eta+1)/\eta}(t).
\]

Let \( m = 1 + (1/\eta) \), \( X = [\eta\Gamma(1 - \alpha)A(t_2, t)/\rho^{1/\eta}(t)r(t)]^{1/m} \), \( w(t), Y_{m-1} = \eta/(\eta + 1) \), \( (\rho'(t)/\rho(t))^{1/\eta}(t)r(t)/\eta\Gamma(1 - \alpha) \).

Integrating two sides of (30) from \( t_2 \) to \( t \), we have

\[
\int_{t_2}^{t} \left( \frac{\rho'(s)}{\rho(s)} \right)^{\eta+1} \frac{\rho(s) r^{\eta}(s)}{\Gamma(1 - \alpha) A(t_2, s)^\eta} ds \leq w(t) - w(t_2) \leq w(t_2),
\]

which contradicts to (27).

If \( D^\alpha x(t) > 0 \) on \([t_2, \infty)\), then from Lemma 2, we get that \( \lim_{t \to \infty} G(t) = 0 \). This completes the proof.

**Corollary 5.** Assume that \((A_1), (A_2), \) and \((12)\) hold; if there exists a function \( \rho \in C^1([t_0, \infty); R_+) \) such that

\[
\int_{t_0}^{\infty} \left( \frac{\rho(s) q(s) ds}{\rho(s) r^{\eta}(s) A^\eta(t_2, s)} \right) ds < \infty
\]

for a sufficiently large constant \( T \), where \( \rho'_+(s) = \max\{\rho'(s), 0\} \), \( A(t_2, t) \) is defined in Lemma 3, then every solution of (1) is oscillatory or satisfies \( \lim_{t \to \infty} G(t) = 0 \).

**Theorem 6.** Assume that \((A_1), (A_2), \) and \((12)\) hold; if there exists functions \( \rho \in C^1([t_0, \infty); R_+) \) and \( H(t, s) \in X \) such that

\[
\lim_{t \to \infty} \frac{1}{H(t, t_0)}
\]

\[
\times \int_{t_2}^{t} \left( \frac{H(t, s) q(s)}{\eta + 1)^{\eta+1} [\Gamma(1 - \alpha) A(t_2, s) H(t, s)]^\eta} \right) ds
\]

\[
= \infty,
\]

where \( t_2 \) is sufficiently large, \( t \geq t_2 \geq t_0 \). \( A(t_2, t) \) is defined in Lemma 3, \( h(t, s) = H(t, s)(\rho'_+(s)/\rho(s)) + H'_t(t, s) \), \( h(t, s) = \max\{0, h(t, s)\} \), then every solution of (1) is oscillatory or satisfies \( \lim_{t \to \infty} G(t) = 0 \).
Proof. Suppose that \( x(t) \) is a nonoscillatory solution of (1); without loss of generality, we let \( x(t) \) be an eventually positive solution of (1). According to the proof of Theorem 4, if \( D^\alpha x(t) < 0 \) on \([t_2, \infty)\), then (29) holds. Multiplying two sides of (29) by \( H(t, s) \) and integrating it from \( t_2 \) to \( t \), we get

\[
\int_{t_2}^{t} \rho(s) q(s) H(t, s) \, ds
\]

\[
\leq - \int_{t_2}^{t} H(t, s) w'(s) \, ds + \int_{t_2}^{t} H(t, s) \frac{\rho'(s)}{\rho(s)} w(s) \, ds
\]

\[
- \eta \Gamma(1 - \alpha) \int_{t_2}^{t} H(t, s) \frac{A(t_2, s)}{\rho^\eta(s) r(s)} \omega^{(n+1)/\eta}(s) \, ds
\]

\[
\leq H(t, t_2) w(t_2)
\]

\[
+ \int_{t_2}^{t} \left[ \left( H(t, s) \frac{\rho'(s)}{\rho(s)} + H'(t, s) \right) w(s)
\right.
\]

\[
\left. - \eta \Gamma(1 - \alpha) \frac{A(t_2, s) H(t, s)}{\rho^{1/\eta}(s) r(s)} \omega^{(n+1)/\eta}(s) \right] \, ds
\]

\[
\leq H(t, t_2) w(t_2)
\]

\[
+ \int_{t_2}^{t} \left[ h_+(t, s) w(s)
\right.
\]

\[
\left. - \eta \Gamma(1 - \alpha) \frac{A(t_2, s) H(t, s)}{\rho^{1/\eta}(s) r(s)} \omega^{(n+1)/\eta}(s) \right] \, ds
\]

Then we have

\[
\int_{t_2}^{t} \left[ \rho(s) q(s) H(t, s)
\right.
\]

\[
- \frac{\rho(s) r^\eta(s) H^{n+1}_n(t, s)}{(\eta + 1)^{n+1} \Gamma(1 - \alpha) A(t, s) H(t, s)^{\eta}} \right] \, ds
\]

\[
\leq H(t, t_2) w(t_2)
\]

From above inequality we have

\[
\frac{1}{H(t, t_0)} \leq \limsup_{t \to \infty} H(t, t_0)
\]

\[
\times \int_{t_2}^{t} \left[ \rho(s) q(s) H(t, s)
\right.
\]

\[
- \frac{r^\eta(s) H^{n+1}_n(t, s)}{(\eta + 1)^{n+1} \Gamma(1 - \alpha) A(t, s) H(t, s)^{\eta}} \right] \, ds
\]

\[
\leq \omega(t_2)
\]

Letting \( t \to \infty \) we have

\[
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \leq \limsup_{t \to \infty} \int_{t_2}^{t} \left[ \rho(s) q(s) H(t, s)
\right.
\]

\[
- \frac{r^\eta(s) H^{n+1}_n(t, s)}{(\eta + 1)^{n+1} \Gamma(1 - \alpha) A(t, s) H(t, s)^{\eta}} \right] \, ds
\]

\[
< \infty,
\]

which contradicts to (33). If \( D^\alpha x(t) > 0 \) on \([t_2, \infty)\), then from Lemma 2, we have \( \lim_{t \to \infty} G(t) = 0 \). So the proof is complete. \( \square \)

**Corollary 7.** Assume that \((A_1), (A_2),\) and (12) hold; if there exist functions \( \rho \in C^1([t_0, \infty); \mathbb{R}_+) \) and \( H(t, s) \in X \) such that

\[
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_2}^{t} \rho(s) q(s) H(t, s) \, ds = \infty,
\]

\[
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_2}^{t} \rho(s) q(s) H(t, s) \, ds < \infty,
\]

where \( t_2 \) is sufficiently large, \( t \geq t_2 \geq t_0 \), \( A(t_2, t) \) is defined in Lemma 3, then every solution of (1) is oscillatory or satisfies \( \lim_{t \to \infty} G(t) = 0 \).

With an appropriate choice of the functions \( H \) and \( \rho \), one can derive from Theorem 6 a number of oscillation criteria for (1). For example, let \( H(t, s) = \ln(t/s), (t, s) \in D, \) and \( \rho(t) = t^\eta; \) then \( h(t, s) = (1/s)(\mu \ln(t/s) - 1), h_+(t, s) = \max[h(t, s), 0]. \) Based on the above results we obtain the following corollary.
Corollary 8. Assume that \((A_1), (A_2), \) and \((12)\) hold if

\[
\limsup_{t \to \infty} \frac{1}{\ln t} \left[ \int_{t_2}^{t} s^\mu q(s) \ln \frac{t}{s} \, ds \right] = \infty,
\]

where \(t_2\) is sufficiently large, \(t \geq t_2 \geq t_0\), \(A(t_2, t)\) is defined in Lemma 3, then every solution of \((1)\) is oscillatory or satisfies \(\lim_{t \to \infty} G(t) = 0\).

Corollary 9. Assume that \((A_1), (A_2), \) and \((12)\) hold if

\[
\limsup_{t \to \infty} \frac{1}{\ln t} \left[ \int_{t_2}^{t} s^\mu (\ln \frac{t}{s})^{-\eta} \, ds \right] = \infty,
\]

where \(t_2\) is sufficiently large, \(t \geq t_2 \geq t_0\), \(A(t_2, t)\) is defined in Lemma 3, then every solution of \((1)\) is oscillatory or satisfies \(\lim_{t \to \infty} G(t) = 0\).

3. Examples

Example 1. Consider the fractional differential equation

\[
\left\{ t^\eta \left[ \int_0^{t} D^{\alpha}_x x(t) \right] \right\}'
\]

\[- F \left( t, \int_0^{t} (v - t)^{-\alpha} x(v) \, dv \right) = 0, \quad t > t_0,
\]

where \(\alpha \in (0, 1)\) and \(\eta\) is a quotient of two odd positive integers.

Here, \(a(t) = t^{\eta/2}, r(t) = 1, F(t, G/G^\eta) \geq q(t)\) for \(G \neq 0\) and \(x \neq 0, t \geq t_0, q(t) \geq t^{(\eta+1)}\).

We set \(\rho(t) = t^\eta; \) if \(t\) is sufficiently large, then \(A(t_1, t) = \int_{t_1}^{t} (ds/s^{1/2}) = 2(\sqrt{t} - \sqrt{t_1}) \geq \sqrt{t} , \) and \(\int_{t_1}^{\infty} (ds/s^{1/2}) = \infty, \int_{t_1}^{\infty} (ds/(s/r(s))) = \infty.\) Consider

\[
\int_{t_2}^{\infty} \frac{1}{r(\xi)} \int_{\xi}^{\infty} \frac{\left( 1/\alpha(\tau) \int_{\tau}^{\infty} q(s) \, ds \right)^{1/\eta} \, d\tau \, d\xi
\]

\[
\geq \int_{t_2}^{\infty} \int_{t}^{\infty} \int_{t}^{\infty} s^{-(\eta+1)} \, ds \right]^{1/\eta} \, d\tau \, d\xi = \infty,
\]

so \((A_1), (A_2), \) and \((12)\) hold.

Furthermore, for a sufficiently large constant \(T\) and \(\eta < \mu < (3/2)\eta\), we have

\[
\int_{T}^{\infty} \rho(s) q(s) \, ds = \infty,
\]

\[
\int_{T}^{\infty} \left[ \rho_1^1 (s) \, q(s) \right]^{\eta+1} \, ds
\]

\[
\leq \int_{T}^{\infty} \left( \frac{\mu}{s} s^{\eta/2} \right)^{\eta+1} \, ds = \mu^{\eta+1} \left( \frac{s^{\eta-\eta/2}}{(s^{\eta/2} - \mu)} \right)^{\eta} < \infty.
\]

So the conditions of Corollary 5 hold, and we deduce that every solution of \((43)\) is oscillatory or satisfies \(\lim_{t \to \infty} G(t) = 0\).

Example 2. Consider the fractional differential equation

\[
\left\{ t^\eta \left[ \int_{t}^{\infty} \frac{\ln t}{D^{\alpha}_x x(t)} \right] \right\}'
\]

\[- F \left( t, \int_{t}^{\infty} (v - t)^{-\alpha} x(v) \, dv \right) = 0, \quad t > t_0,
\]

where \(\alpha \in (0, 1)\) and \(\eta\) is a quotient of two odd positive integers.

Here, \(a(t) = t^{\eta}, r(t) = \ln t/t, F(t, G/G^\eta) \geq q(t)\) for \(G \neq 0\) and \(x \neq 0, t \geq t_0, q(t) \geq t^{(\eta+1)}\).

We set \(\rho(t) = 1, H(t,s) = \ln(\ln t); \) then \(h(t,s) = (1/s) (\mu \ln(t/s) - 1), \) and \(\int_{t_0}^{\infty} (ds/a^{1/2}(s)) = \int_{t_0}^{\infty} (ds/s) = \infty, \int_{t_0}^{\infty} (ds/ r(s)) = \int_{t_0}^{\infty} (s ds/ln s) = \infty.

May as well we assume that \(t_2 < \sqrt{t}; \) then we have \(A(t_2, t) = \ln(t/t_2) > (1/2) \ln t.\) By calculation we have

\[
\int_{t_2}^{\infty} \frac{1}{r(\xi)} \int_{\xi}^{\infty} \left[ \frac{1}{\alpha(\tau)} \int_{\tau}^{\infty} q(s) \, ds \right]^{1/\eta} \, d\tau \, d\xi
\]

\[
\geq \int_{t_2}^{\infty} \int_{t}^{\infty} \int_{t}^{\infty} s^{-(\eta+1)} \, ds \right]^{1/\eta} \, d\tau \, d\xi = \infty,
\]

so \((A_1), (A_2), \) and \((12)\) hold.

Furthermore,

\[
\limsup_{t \to \infty} \frac{1}{\ln t} \left[ \int_{t_2}^{t} s^\mu q(s) \, ds \right] = \infty,
\]


and for \(\eta < \mu < 2\eta,

\[
\limsup_{t \to \infty} \frac{1}{\ln t} \left[ \int_{t_2}^{t} s^\mu q(s) \, ds \right] \leq \int_{t_2}^{t} \frac{q(s)}{\alpha(\ln t/s)} \, ds
\]

\[
= \int_{t_2}^{t} \frac{\ln t/s}{\alpha(\ln t/s)} \, ds = \infty.
\]
\[
\leq \limsup_{t \to \infty} \frac{1}{\ln t} \times \int_{t_2}^t \frac{s^{\mu} \ln \eta s \left( \left(1/s\right) \left( \mu \ln \left(t/s\right) - 1\right) \right)^{\eta+1}}{s^{\eta \left( \ln \left(s/2\right) \ln \left(t/s\right) \right)^{\eta}}} ds 
\leq 2^{(1/\eta)} \limsup_{t \to \infty} \frac{1}{\ln t} \int_{t_0}^t \frac{s^{\mu-2\eta-1} \left[ \mu \ln \left(t/s\right) \right]^{\eta+1}}{\left( \ln \left(t/s\right) \right)^{\eta}} ds 
= 2^{(1/\eta)} \limsup_{t \to \infty} \frac{1}{\ln t} \int_{t_0}^t s^{\mu-2\eta-1} \ln \frac{t}{s} \frac{ds}{s} 
= 2^{(1/\eta)} \limsup_{t \to \infty} \frac{1}{\ln t} \int_{t_0}^t s^{\mu-2\eta-1} \left[ - \ln \frac{t}{t_0} + \frac{1}{\mu - 2\eta} \left( t^{\mu-2\eta} - t_0^{\mu-2\eta} \right) \right] < \infty,
\]

the conditions of Corollary 9 hold, and we get that every solution of (46) is oscillatory or satisfies \( \lim_{t \to \infty} G(t) = 0 \).

Acknowledgment

This research is supported by National Science Foundation of China (11171178 and 11271225). The authors sincerely thank the reviewers for their valuable suggestions and useful comments.

References
