Research Article

The Diagonally Dominant Degree and Disc Separation for the Schur Complement of Ostrowski Matrix

Jianxing Zhao, 1 Feng Wang, 1,2 and Yaotang Li 1

1 School of Mathematics and Statistics, Yunnan University, Kunming, Yunnan 650091, China
2 Department of Mathematics, Heze University, Heze, Shandong 274015, China

Correspondence should be addressed to Yaotang Li; liyaotang@ynu.edu.cn

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By applying the properties of Schur complement and some inequality techniques, some new estimates of diagonally and doubly diagonally dominant degree of the Schur complement of Ostrowski matrix are obtained, which improve the main results of Liu and Zhang (2005) and Liu et al. (2012). As an application, we present new inclusion regions for eigenvalues of the Schur complement of Ostrowski matrix. In addition, a new upper bound for the infinity norm on the inverse of the Schur complement of Ostrowski matrix is given. Finally, we give numerical examples to illustrate the theory results.

1. Introduction

Let \( C^{n \times n} \) denote the set of all \( n \times n \) complex matrices, \( N = \{1, 2, \ldots, n\} \), and \( A = (a_{ij}) \in C^{n \times n}(n \geq 2) \). Denote \( R_i(A) = \sum_{j \neq i} |a_{ij}| \).

We know that \( A \) is called a strictly diagonally dominant matrix if

\[
|a_{ii}| > R_i(A), \quad \forall i \in N. 
\]

(2)

\( A \) is called a generalized Ostrowski matrix if

\[
|a_{ii}| |a_{j j}| \geq R_i(A) R_j(A), \quad \forall i, j \in N, \ i \neq j. 
\]

(3)

\( A \) is called Ostrowski matrix if all strict inequalities in (3) hold (see [1]).

\( SD_n \) and \( OS_n(GOS_n) \) will be used to denote the sets of all \( n \times n \) strictly diagonally dominant matrices and the sets of all \( n \times n \) (generalized) Ostrowski matrices, respectively.

As shown in [2], for all \( i \in N \), we call \( |a_{ii}| - R_i(A) \) and \( |a_{ii}| |a_{j j}| - R_i(A) R_j(A) \) the \( i \)th diagonally and doubly diagonally dominant degree of \( A \), respectively.

The infinity norm of \( A \) is defined as

\[
\|A\|_\infty = \max_{1 \leq i \leq n} \{R_i(A) + |a_{ii}|\}. 
\]

(4)

For \( \beta \subseteq N \), denote by \( |\beta| \) the cardinality of \( \beta \) and \( \overline{\beta} = N/\beta \). If \( \beta, \gamma \subseteq N \), then \( A(\beta, \gamma) \) is the submatrix of \( A \) with row indices in \( \beta \) and column indices in \( \gamma \). In particular, \( A(\beta, \beta) \) is abbreviated to \( A(\beta) \). Assuming that \( \beta = \{i_1, i_2, \ldots, i_k\} \subseteq N \), \( \overline{\beta} = N/\beta = \{j_1, j_2, \ldots, j_l\} \) and the elements of \( \beta \) and \( \overline{\beta} \) are both conventionally arranged in an increasing order. For \( 1 \leq t \leq l \), we denote

\[
A_t = A(\beta \cup \{j_t\}). 
\]

(5)

If \( A(\beta) \) is nonsingular,

\[
A(\beta) = \frac{1}{A(\beta)} A(\beta, \beta) \left[A(\beta)\right]^{-1} A(\beta, \beta) 
\]

is called the Schur complement of \( A \) with respect to \( A(\beta) \).

The comparison matrix of \( A \), \( \mu(A) = (\alpha_{ij}) \), is defined by

\[
\alpha_{ij} = \begin{cases} 
|a_{ij}|, & \text{if } i = j, \\
-|a_{ij}|, & \text{if } i \neq j.
\end{cases} 
\]

(7)

A matrix \( A = (a_{ij}) \in C^{n \times n} \) is called an \( M \)-matrix if there exist a nonnegative matrix \( B \) and a number \( s > \rho(B) \) such that \( A = sI - B \), where \( \rho(B) \) is the spectral radius of \( B \). We know that \( A \) is an \( H \)-matrix if and only if \( \mu(A) \) is an \( M \)-matrix, and if
A is an $M$-matrix, then the Schur complement of $A$ is also an $M$-matrix and det $B > 0$ (see [3]). $H_n$ and $M_n$ will denote the set of all $n \times n$ $H$-matrices and the set of all $n \times n$ $M$-matrices, respectively.

The Schur complement has been proved to be a useful tool in many fields such as control theory, statistics, and computational mathematics. A lot of work has been done on it (see [2, 4–15]). It is well known that the Schur complements of $S_D^\infty$ and $O_S^\infty$ are $S_D^\infty$ and $O_S^\infty$, respectively. These properties have been used for the derivation of matrix inequalities in matrix analysis and for the convergence of iterations in numerical analysis (see [16–19]). Meanwhile, estimating the upper bound for the infinity norm of the inverse of the Schur complement is of great significance. We know that the upper bound of $\| A^{-1} \|_\infty$ plays an important role in some iterations for large scale nonhomogeneous system of linear equation $Ax = b$ (see [20]).

The paper is organized as follows. In Section 2, we give several new estimates of diagonally and doubly diagonally dominant degree on the Schur complement of matrices. In Section 3, new inclusion regions for eigenvalues of the Schur complement are obtained. A new upper bound of $\| (A/\beta)^{-1} \|_\infty$ is given in Section 4. In Section 5, we present numerical examples to illustrate the theory results.

2. The Diagonally Dominant Degree for the Schur Complement

In this section, we give several new estimates of diagonally and doubly diagonally dominant degree on the Schur complement of $O_S\infty$.

**Lemma 1** (see [3]). If $A \in H_n$, then $[\mu(A)]^{-1} \geq |A^{-1}|$.

**Lemma 2** (see [3]). If $A \in S_D^n$ or $A \in O_S\infty$, then $A \in H_n$; that is, $\mu(A) \in M_n$.

**Lemma 3** (see [6]). If $A \in S_D^n$ or $A \in O_S\infty$ and $\beta \subseteq N$, then the Schur complement of $A$ is in $SD^\beta$ or $O_S^\beta\infty$, where $\beta = N - \beta$ is the complement of $\beta$ in $N$ and $|\beta|$ is the cardinality of $\beta$.

**Lemma 4** (see [12]). Let $A \in S_D^n$, $\beta = \{i_1, i_2, \ldots, i_k\} \subset N$, $\beta^\prime = \{j_1, j_2, \ldots, j_k\}$, and $k + l = n$. For any $j_i \in \beta^\prime$, denote

$$B_{j_i} = \begin{pmatrix} x & -a_{i_1j_i} & \cdots & -a_{i_1j_l} \\ -a_{i_1j_1} & \ddots & \vdots & \vdots \\ \vdots & \ddots & x & -a_{i_lj_i} \\ -a_{i_1j_k} & \cdots & -a_{i_lj_k} & x \end{pmatrix}, \quad x > 0.$$  \hfill (8)

Then $B_{j_i} \in GOS_{k+1}$ if and only if

$$x \geq \max_{1 \leq w \leq k} R_{i_wj_i}(A) \sum_{s=1}^{k} |a_{i_sj_i}|.$$  \hfill (9)

When the strict inequality in (9) holds, $B_{j_i} \in M_{k+1}$, and thus $\det B_{j_i} > 0$. If the equality in (9) occurs, then $\det B_{j_i} \geq 0$.

**Lemma 5.** Let $A = (a_{ij}) \in O_S\infty$ and $\beta = \{i_1, i_2, \ldots, i_k\}$ with an index $i_d$ ($1 \leq d \leq k$) satisfying $|a_{i_dj_d}| \leq R_{i_d}(A)$, $|a_{i_dj_d}| \geq \sum_{s \neq \beta \in \{i_1, i_2, \ldots, i_k\}, |a_{i_dj_s}|, j_s \in \beta} 1 \leq k < n$, and $A/\beta = (a'_{ij})$. Then, for all $1 \leq t \leq l$,

$$a_{ij} - R_t \left( A \right) \geq \left| a_{ij} \right| - R_{i_d}(A)$$

$$+ \frac{|a_{i_dj_t}| - P_{i_d}(A)}{|a_{i_dj_t}|} \sum_{s=1}^{k} |a_{i_sj_s}|$$

$$\geq |a_{ij} - R_{i_d}(A)| > 0,$$

where

$$h = \max \left\{ \frac{|a_{i_dj_s}| - R_{i_d}(A)}{|a_{i_dj_s}|}, 1 \leq s \leq k \right\}.$$  \hfill (10)

**Proof.** From Lemmas 2 and 3, we know that $A(\beta) \in H_k$ and $\mu(A(\beta)) \in M_k$. Further, by Lemma 1, we have

$$|\mu(A(\beta))|^{-1} \geq |\mu(A)\beta^{-1}|.$$  \hfill (11)

Thus, for any $1 \leq t \leq l$,

$$|a_{ij} - R_t \left( A \right) |$$

$$= |a_{ij} | - \sum_{s=1, s \neq t}^{l} |a_{i_sj_t}|$$

$$= |a_{ij} | - (a_{i_1j_1}, \ldots, a_{i_lj_l}) \left( A(\beta) \right)^{-1} \left( \begin{array}{c} a_{i_1j_1} \\ \vdots \\ a_{i_lj_l} \end{array} \right)$$

$$- \sum_{s \neq t}^{l} |a_{i_sj_t}| - (a_{i_1j_1}, \ldots, a_{i_lj_l}) \left( A(\beta) \right)^{-1} \left( \begin{array}{c} a_{i_1j_1} \\ \vdots \\ a_{i_lj_l} \end{array} \right)$$

$$\geq |a_{ij} | - R_{i_d}(A)$$

$$+ \frac{|a_{i_dj_t}| - P_{i_d}(A)}{|a_{i_dj_t}|} \sum_{s=1}^{k} |a_{i_sj_s}| + \frac{P_{i_d}(A)}{|a_{i_dj_t}|} \sum_{s=1}^{k} |a_{i_sj_s}|$$

$$\geq \left( |a_{ij} | - R_{i_d}(A) \right) > 0,$$

$$\left| a_{ij} \right| \geq \left( |a_{ij} | - R_{i_d}(A) \right) > 0.$$  \hfill (12)
Further,

\[ |a'_{ij}| - R_t \left( \frac{A}{\beta} \right) \]

\[ \geq |a_{ij}| - R_j (A) + \frac{P_i (A)}{|a_{ij}|} \sum_{v=1}^{k} |a_{ij,v}| \]

\[ + \frac{1}{\det \left[ \mu (A (\beta)) \right]} \det \left( \begin{array}{cccc}
\frac{P_i (A)}{|a_{ij,d}|} \sum_{v=1}^{k} |a_{ij,v}| & - |a_{ij},| & \cdots & - |a_{ij,k}| \\
- \sum_{s=1}^{t} |a_{ij,s}| & \mu (A (\beta)) \\
\vdots & & & \\
- \sum_{s=1}^{t} |a_{ij,s}| & & & \end{array} \right) \]

\[ \times \det \left[ \begin{array}{cccc}
\mu (A (\beta)) & - |a_{ij,1}| & \cdots & - |a_{ij,k}| \\
- \sum_{s=1}^{t} |a_{ij,s}| & \mu (A (\beta)) \\
\vdots & & & \\
- \sum_{s=1}^{t} |a_{ij,s}| & & & \end{array} \right] \]

\[ \geq |a_{ij}| - R_j (A) + \frac{P_i (A)}{|a_{ij}|} \sum_{v=1}^{k} |a_{ij,v}| \]

\[ + \frac{1}{\det \left[ \mu (A (\beta)) \right]} \det B. \]

By Lemma 4, we can prove that \( \det B \geq 0 \). Thus, inequality (10) holds. \( \square \)

**Remark 6.** Note that

\[ \frac{P_i (A)}{|a_{ij,d}|} \leq \frac{R_j (A)}{|a_{ij,d}|}. \]  \( (15) \)

This shows that Lemma 5 improves Theorem 2 of [12].

**Theorem 7.** Let \( A = (a_{ij}) \in OS_\kappa, \beta = \{i_1, i_2, \ldots, i_k\} \subset N, \beta = N/\beta = \{i_1, i_2, \ldots, i_k\}, 1 \leq k < n, \) and \( A/\beta = (a'_{ij}). \)

(a) If there exists an \( i_d \in \beta (1 \leq d \leq k) \) such that \( |a_{id}| \leq R_j (A) \), then, for all \( 1 \leq s, t \leq l, t \neq s, \)

\[ |a'_{ij}| |a'_{is}| - R_t \left( \frac{A}{\beta} \right) R_s \left( \frac{A}{\beta} \right) \]

\[ \geq \left[ |a_{ij,j}| - \max_{u \in N/\{j\}} \frac{P_u (A)}{|a_{ij,j}|} R_j (A) \right] \]

\[ \times \left[ |a_{ij,j}| - \max_{i, \in \beta \setminus \{ij\}} \frac{P_i (A)}{|a_{ij,j}|} R_j (A) \right], \]

\[ \leq \left[ |a_{ij,j}| + \max_{u \in N/\{j\}} \frac{P_u (A)}{|a_{ij,j}|} R_j (A) \right] \times \left[ |a_{ij,j}| + \max_{i, \in \beta \setminus \{ij\}} \frac{P_i (A)}{|a_{ij,j}|} R_j (A) \right], \]

where

\[ P_i (A) = h \sum_{j \in N/\{i\}} |a_{ij}| + |a_{is}| (i \neq i_d), \]

and \( P_i (A) \) and \( h \) are such as in Lemma 5.

(b) If \( |a_{id}| > R_j (A) \) for any \( i_d \in \beta \) \((1 \leq d \leq k)\), then, for all \( 1 \leq s, t \leq l, t \neq s, \)

\[ |a'_{ij}| |a'_{is}| - R_t \left( \frac{A}{\beta} \right) R_s \left( \frac{A}{\beta} \right) \]

\[ \geq \left[ |a_{ij,j}| - \max_{u \in N/\{j\}} \frac{R_u (A)}{|a_{ij,j}|} R_j (A) \right] \times \left[ |a_{ij,j}| - \max_{i, \in \beta \setminus \{ij\}} \frac{Q_i (A)}{|a_{ij,j}|} R_j (A) \right], \]

\[ \leq \left[ |a_{ij,j}| + \max_{u \in N/\{j\}} \frac{R_u (A)}{|a_{ij,j}|} R_j (A) \right] \times \left[ |a_{ij,j}| + \max_{i, \in \beta \setminus \{ij\}} \frac{Q_i (A)}{|a_{ij,j}|} R_j (A) \right], \]

where

\[ \eta = \max \left\{ \max_{1 \leq \omega \leq k} \frac{\sum_{v=1}^{k} |a_{i\omega,v}|}{|a_{ij,j}| - \sum_{t \neq \omega} |a_{ij,t}|}, \right. \]

\[ \left. \max_{1 \leq \omega \leq k} \frac{|a_{ij,j}|}{\sum_{v=1}^{k} |a_{i\omega,v}|} \right\}, \]

\[ Q_{\omega} (A) = \eta \sum_{v=1}^{k} |a_{i\omega,v}| + \sum_{v=1}^{k} |a_{i\omega,v}|, \]

\[ 1 \leq \omega \leq k, \]

and if there exists some \( 1 \leq \omega \leq k \) such that \( \sum_{v=1}^{k} |a_{i\omega,v}| = 0 \), one denotes \( \eta = 1 \).
Proof. (a) If there exists an \( i_d \in \beta \) such that \( |a_{i_dj_d}| \leq R_j(A) \), then, for all \( j_s \in \bar{\beta} \),

\[
\max_{u \in N/\{j_s\}} \frac{P_u(A)}{|a_{uu}|} = \max_{i, j \in \beta} \frac{P_i(A)}{|a_{ij}|} = \frac{P_j(A)}{|a_{j_sj_s}|},
\]

(22)

By Lemma 5, for all \( 1 \leq t \leq l \),

\[
|a_{t^t} - R_t \left( \frac{A}{\beta} \right)| \geq |a_{j_sj_s}| - \frac{P_j(A)}{|a_{j_sj_s}|} R_j(A) > 0.
\]

(23)

Thus, for all \( 1 \leq t, s \leq l, t \neq s \),

\[
\left[ |a_{t^t} - R_t \left( \frac{A}{\beta} \right)| \right] \left[ |a_{s^s} - R_s \left( \frac{A}{\beta} \right)| \right] > 0.
\]

(24)

From Lemma 3, \( A/\beta \) is in \( OS[\beta] \); that is, for all \( 1 \leq t, s \leq l, t \neq s \),

\[
|a_{t^t} - (a_{j_tj_t}, \ldots, a_{j_sj_s})| > 0.
\]

(25)

Further, for all \( 1 \leq t, s \leq l, t \neq s \),

\[
\left[ |a_{t^t} - R_t \left( \frac{A}{\beta} \right)| R_s \left( \frac{A}{\beta} \right) \right]
\geq \left[ |a_{t^t} - R_t \left( \frac{A}{\beta} \right)| \right]
\times \left[ |a_{s^s} - R_s \left( \frac{A}{\beta} \right)| \right]
\geq |a_{j_tj_t} - \max_{u \in N/\{j_t\}} \frac{P_u(A)}{|a_{uu}|} R_{j_t}(A) |
\times |a_{j_sj_s} - \max_{i, j \in \beta} \frac{P_i(A)}{|a_{ij}|} R_j(A) |.
\]

(26)

Therefore, inequality (16) holds. Similarly, we can prove inequality (17).

(b) If \( |a_{ijd}| > R_i(A) \) for any \( i_d \in \beta \) (1 \( \leq d \leq k \)), then, from Lemmas 1 and 2, for all \( 1 \leq t, s \leq l, t \neq s \),

\[
\left| a_{t^t} - (a_{j_tj_t}, \ldots, a_{j_sj_s}) \right| \left| a_{s^s} - (a_{j_tj_t}, \ldots, a_{j_sj_s}) \right| \left[ A(\beta) \right]^{-1} \left( \begin{array}{c} a_{j_tj_t} \\ \vdots \\ a_{j_sj_s} \end{array} \right)
\]

\[
\times \left( \begin{array}{c} a_{i_tj_t} \\ \vdots \\ a_{i_sj_s} \end{array} \right) = \xi \left( a_{j_tj_t} - (a_{j_tj_t}, \ldots, a_{j_sj_s}) \right).
\]

\[
\text{def} \in \xi = \left( a_{j_tj_t} - (a_{j_tj_t}, \ldots, a_{j_sj_s}) \right)
\]
\[
\times \left( \mu(A(\beta)) \right)^{-1} \begin{pmatrix} a_{ij} \\ \vdots \\ a_{ij} \end{pmatrix} \]

\[
\times \left[ a_{ij} - \max_{u \in N \setminus \{j\}} \frac{R_u(A)}{|a_{ui}|} R_j(A) \right]
\]

\[
+ \max_{u \in N \setminus \{j\}} \frac{R_u(A)}{|a_{ui}|} R_j(A)
\]

\[
- \left( [a_{j1}, \ldots, a_{j1}] \right) \mu(A(\beta))^{-1} \left( \begin{pmatrix} a_{i1} \\ \vdots \\ a_{i1} \end{pmatrix} \right)
\]

\[
- \left\{ \sum_{v \neq s} \left[ a_{jv} - (a_{j1}, \ldots, a_{j1}) \right] \times [\mu(A(\beta))]^{-1} \left( \begin{pmatrix} a_{i1} \\ \vdots \\ a_{i1} \end{pmatrix} \right) \right\}
\]

\[
\times \left\{ \sum_{u \neq t} \left[ a_{ju} - (a_{j1}, \ldots, a_{j1}) \right] \times [\mu(A(\beta))]^{-1} \left( \begin{pmatrix} a_{i1} \\ \vdots \\ a_{i1} \end{pmatrix} \right) \right\}.
\]

Therefore,

\[
\xi = \left[ a_{j1} - (a_{j1}, \ldots, a_{j1}) \right] \times [\mu(A(\beta))]^{-1} \left( \begin{pmatrix} a_{i1} \\ \vdots \\ a_{i1} \end{pmatrix} \right)
\]

\[
\times \left[ a_{j1} - \max_{u \in N \setminus \{j\}} \frac{R_u(A)}{|a_{ui}|} R_j(A) \right]
\]

\[
+ \left[ a_{j1} - (a_{j1}, \ldots, a_{j1}) \right] \times [\mu(A(\beta))]^{-1} \left( \begin{pmatrix} a_{i1} \\ \vdots \\ a_{i1} \end{pmatrix} \right)
\]

\[
\times \left[ \max_{u \in N \setminus \{j\}} \frac{R_u(A)}{|a_{ui}|} R_j(A) \right]
\]

\[
\times \left[ \frac{R_u(A)}{|a_{ui}|} R_j(A) \right]
\]

\[
- \left( [a_{j1}, \ldots, a_{j1}] \right) \mu(A(\beta))^{-1} \left( \begin{pmatrix} a_{i1} \\ \vdots \\ a_{i1} \end{pmatrix} \right)
\]

\[
- \left\{ \sum_{v \neq s} \left[ a_{jv} - (a_{j1}, \ldots, a_{j1}) \right] \times [\mu(A(\beta))]^{-1} \left( \begin{pmatrix} a_{i1} \\ \vdots \\ a_{i1} \end{pmatrix} \right) \right\}
\]

\[
\times \left\{ \sum_{u \neq t} \left[ a_{ju} - (a_{j1}, \ldots, a_{j1}) \right] \times [\mu(A(\beta))]^{-1} \left( \begin{pmatrix} a_{i1} \\ \vdots \\ a_{i1} \end{pmatrix} \right) \right\}.
\]

Further,

\[
|a_{j1} - (a_{j1}, \ldots, a_{j1})| \mu(A(\beta))^{-1} \left( \begin{pmatrix} a_{i1} \\ \vdots \\ a_{i1} \end{pmatrix} \right)
\]

\[
= \left[ a_{j1} - \max_{i, \in [A]} \frac{Q_i(A)}{|a_{ij}|} \sum_{i=1}^{k} |a_{ij}| + \max_{i, \in [A]} \frac{Q_i(A)}{|a_{ij}|} \sum_{i=1}^{k} |a_{ij}| \right]
\]

\[
- \left( \eta |a_{j1}, \ldots, a_{j1}| \eta \mu(A(\beta))^{-1} \left( \begin{pmatrix} a_{i1} \\ \vdots \\ a_{i1} \end{pmatrix} \right) \right)
\]

\[
= \left[ a_{j1} - \max_{i, \in [A]} \frac{Q_i(A)}{|a_{ij}|} \sum_{i=1}^{k} |a_{ij}| + \frac{1}{\det(\eta \mu(A(\beta)))} \right]
\]

\[
\times \left( \begin{pmatrix} \max_{i, \in [A]} \frac{Q_i(A)}{|a_{ij}|} \sum_{i=1}^{k} |a_{ij}| - \eta |a_{j1}| & \cdots & -\eta |a_{j1}| \\ -|a_{i1}| & \cdots & \eta \mu(A(\beta)) \end{pmatrix} \right)
\]

\[
= \left[ a_{j1} - \max_{i, \in [A]} \frac{Q_i(A)}{|a_{ij}|} \sum_{i=1}^{k} |a_{ij}| + \frac{1}{\det(\eta \mu(A(\beta)))} \det B_1 \right].
\]
In $B_1$, for all $p = 1, 2, 3, \ldots, k$,
\[
\eta \left| a_{ij} \right| \max_{i \in \beta} Q_{ij}(A) \sum_{p=1}^{k} |a_{ij}| \\
\geq \eta \left| a_{ij} \right| \max_{i \in \beta} Q_{ij}(A) \sum_{p=1}^{k} |a_{ij}| \\
= \eta Q_{ij}(A) \sum_{p=1}^{k} |a_{ij}| \\
= \eta \left( \sum_{p=1}^{k} |a_{ij}| \right) \sum_{p=1}^{k} |a_{ij}| \\
= \eta \left( \sum_{p=1}^{k} |a_{ij}| \right) \sum_{p=1}^{k} |a_{ij}| \\
\geq \left( \sum_{p=1}^{k} |a_{ij}| \right) \sum_{p=1}^{k} |a_{ij}|. \\
\text{(30)}
\]
And for all $p,q = 1, 2, 3, \ldots, k$, $p \neq q$,
\[
\eta \left| a_{ij} \right| \eta \left| a_{ij} \right| > \eta R_{ij}(A) \eta R_{ij}(A) \\
= \left( \sum_{p=1}^{k} |a_{ij}| + \sum_{q=1}^{l} |a_{ij}| \right) \left( \sum_{p=1}^{k} |a_{ij}| + \sum_{q=1}^{l} |a_{ij}| \right) \\
\geq \left( \sum_{p=1}^{k} |a_{ij}| + |a_{ij}| \right) \left( \sum_{p=1}^{k} |a_{ij}| + |a_{ij}| \right). \\
\text{(31)}
\]
Hence, by (30) and (31), we have $B_1 \in \text{GOS}_{k+1}$ and so $B_1 \geq 0$. Further, by (29), we obtain
\[
|a_{ij}| - \left( |a_{ij}|, \ldots, |a_{ij}| \right) \mu (A(\beta))^{-1} \\
\geq |a_{ij}| - \max_{i,j} \frac{Q_{ij}(A)}{a_{ij}} \sum_{p=1}^{k} |a_{ij}| \\
\geq |a_{ij}| - \max_{i,j} \frac{Q_{ij}(A)}{a_{ij}} R_{ij}(A). \\
\text{(32)}
\]
By (28) and a similar method as the proof of Theorem 2.1 in [2], we can prove $\xi > 0$. Therefore, by (29) and (32), we obtain inequality (19). Similarly, we can prove inequality (20).

Remark 8. Note that
\[
0 \leq h, \quad \eta \leq 1. \\
\text{(33)}
\]
This shows that Theorem 7 improves Theorem 2.1 of [2].

3. Eigenvalue Inclusion Regions of the Schur Complement

In this section, we present new inclusion regions for eigenvalues of the Schur complement of $\text{OS}_n$. 

**Lemma 9 (Brauer Ovals theorem).** Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$. Then the eigenvalues of $A$ are in the union of the following sets:
\[
U_{ij} = \{ z \in \mathbb{C} | z - a_{ij} \leq R_{ij}(A) R_{ij}(A) \}, \\
\forall i, j = N, \quad i \neq j. \\
\text{(34)}
\]

**Theorem 10.** Let $A = (a_{ij}) \in \text{OS}_n$, $\beta = \{ i_1, i_2, \ldots, i_k \} \subset N$, $\overline{\beta} = N/\beta = \{ j_1, j_2, \ldots, j_l \}$, $1 \leq k < n$, and $A/\beta = (a_{ij})$, and let $\lambda$ be an eigenvalue of $A/\beta$. Then
\[
(\lambda - \frac{\det A(\beta)}{\det A(\beta)}) \leq 2 \left[ a_{ij} \max_{i \in \beta(\beta)} \frac{P_u(A)}{a_{ij}} R_{ij}(A) \\
+ a_{ij} \max_{i \in \beta(\beta)} \frac{P_u(A)}{a_{ij}} R_{ij}(A) \right], \\
\lambda \leq \frac{\det A(\beta)}{\det A(\beta)} - 2 \left[ a_{ij} \max_{i \in \beta(\beta)} \frac{P_u(A)}{a_{ij}} R_{ij}(A) \\
+ a_{ij} \max_{i \in \beta(\beta)} \frac{P_u(A)}{a_{ij}} R_{ij}(A) \right]. \\
\text{(36)}
\]
where $P_u(A)$ is such as in Lemma 5 and $P_u(A)$ ($v \neq d$) is such as in Theorem 7.

(b) If $|a_{ij}| > R_{ij}(A)$ for any $i_d \in \beta (1 \leq d \leq k)$, then there exist $1 \leq t, s \leq l, t \neq s$, such that
\[
\lambda \leq \frac{\det A(\beta)}{\det A(\beta)} - 2 \left[ a_{ij} \max_{i \in \beta(\beta)} \frac{P_u(A)}{a_{ij}} R_{ij}(A) \\
+ a_{ij} \max_{i \in \beta(\beta)} \frac{P_u(A)}{a_{ij}} R_{ij}(A) \right]. \\
\text{(37)}
\]
where $Q_i(A)$ is such as in Theorem 7.
Proof. By Lemma 9, we know that there exist \( 1 \leq t, s \leq l, t \neq s \), such that
\[
\left| \lambda - a_t' \right| - \left| \lambda - a_s' \right| \leq R_t \left( \frac{A}{\beta} \right) R_t \left( \frac{A}{\beta} \right).
\]  

On the other hand, for all \( 1 \leq t \leq l \),
\[
\left| \lambda - a_t' \right| = \left| \lambda - a_{j,i} + (a_{j,i}, \ldots, a_{j,i}) [A(\beta)^{-1}] \right|
\]
\[
= \left| \lambda - \det \left( \frac{A + \beta}{\beta} \right) \right| = \left| \lambda - \frac{\det(A)}{\det(A(\beta))} \right|
\]  

Therefore, by (39), (40), and (41), we obtain inequality (35). With a similar method, we can prove inequality (36).

(b) If \( |a_{j,i}| > R_i(A) \) for any \( i_d \in \beta \) (1 ≤ d ≤ k), then by (19), (32), and a similar method as the part (a), we obtain inequality (37). Similarly, we can prove inequality (38).

4. Upper Bound for the Infinity Norm on the Inverse of the Schur Complement

In this section, we present a new upper bound of \( \|(A/\beta)^{-1}\|_{\infty} \).

Lemma 11 (see [2]). Let \( A = (a_{ij}) \in \text{OS}_{n} \) and \( M = (m_{ij}) \in \mathbb{C}^{n \times n} \). Then,
\[
\left\| A^{-1} M \right\|_{\infty} \leq \max_{1 \leq i, j \leq n} \left| a_{ij} \right| \sum_{v=1}^{n} \left[ m_{v,i} + R_i(A) \sum_{v=1}^{n} \left| m_{v,j} \right| \right].
\]  

\[
\left\| \left( \frac{A}{\beta} \right)^{-1} M \right\|_{\infty} \leq \max_{1 \leq i, j \leq n} \left| a_{ij} \right| \sum_{v=1}^{n} \left[ m_{v,i} + R_i(A) \sum_{v=1}^{n} \left| m_{v,j} \right| \right].
\]  

Theorem 12. Let \( A = (a_{ij}) \in \text{SD}_{n}, M = (m_{ij}) \in \mathbb{C}^{n \times n}, \beta = \{i_1, i_2, \ldots, i_k\} \subset N, \overline{\beta} = N/\beta = \{j_1, j_2, \ldots, j_l\}, 1 \leq k < n \), and \( A/\beta = (a_{ij}) \). Then,
\[
\left\| \left( \frac{A}{\beta} \right)^{-1} M \right\|_{\infty} \leq \max_{1 \leq i, j \leq n} \left( \Delta_{j,i} \right)
\]
\[
\times \left( \left( \left| a_{j,i} \right| - \max_{v \in N/\beta} \left| a_{j,i} \right| \right) \right)^{-1},
\]  

\[
(43)
\]
By Lemma 11, we have

\[
\Delta_{j,k} = \left( |a_{j,k}| + \max_{i \in \beta} \frac{Q_i(A)}{|a_{i,j}|} R_j(A) \right) \sum_{s=1}^{l} |m_{i,s}| + R_j(A) \sum_{s=1}^{l} |m_{i,s}|,
\]

and \(Q_i(A)\) is such as in Theorem 7.

Proof. By Theorem 7, we have

\[
\left\| \left( A^{-1} \right)_{\beta} \right\|_\infty \leq \max_{1 \leq i,j \leq l} \left( |a_{j,i}| + R_j(A) + \max_{i \in \beta} \frac{Q_i(A)}{|a_{i,j}|} R_j(A) \right) \sum_{s=1}^{l} |m_{i,s}| \leq \max_{1 \leq i,j \leq l} \left( |a_{j,i}| + \max_{i \in \beta} \frac{Q_i(A)}{|a_{i,j}|} R_j(A) \right) \sum_{s=1}^{l} |m_{i,s}|.
\]

Further, by (46), (47), (48), and (49), we obtain inequality (43).

Let \( M = I = \text{diag}(1,1,1,1) \); we can prove inequality (44).

\[\square\]

5. Numerical Examples

In this section, we present several numerical examples to illustrate the theory results.

Example 1 (see Example 2 in [2]). Let

\[
A = \begin{pmatrix}
1.3 & 0.2 & 0.3 & 0.4 & 0.5 \\
0.2 & 0.4 & 0.5 & 0.1 & 0.3 \\
0.4 & 0.5 & 0.1 & 3 & 0.3 \\
0.5 & 0.1 & 0.2 & 3 & 0.3
\end{pmatrix},
\]

\(\beta = \{1, 2\}\).

By Theorem 10, the eigenvalues of \(A/\beta\) are in the set

\[
\Gamma_1 = \{ \lambda : |\lambda - 1.87| \lambda - 2.78 | \leq 11.20 \} \cup \{ \lambda : |\lambda - 1.87| \lambda - 2.81 | \leq 10.40 \} \cup \{ \lambda : |\lambda - 2.78| \lambda - 2.81 | \leq 14.40 \}.
\]

From Theorem 3.1 of [2], the eigenvalues of \(A/\beta\) are in the set

\[
\Gamma_1' = \{ \lambda : |\lambda - 1.87| \lambda - 2.78 | \leq 12.06 \} \cup \{ \lambda : |\lambda - 1.87| \lambda - 2.81 | \leq 11.20 \} \cup \{ \lambda : |\lambda - 2.78| \lambda - 2.81 | \leq 15.51 \}.
\]

Evidently, \(\Gamma_1 \subset \Gamma_1'\), and we use Figure 1 to show this fact. And the eigenvalues of \(A/\beta\) are denoted by “+” in Figure 1.

Example 2. Let

\[
A = \begin{pmatrix}
1.6 & 0.1 & 0.5 & 0.2 & 0.2 \\
0.3 & 1.5 & 0.2 & 0.2 & 0.1 \\
0.2 & 0.2 & 1.8 & 0.3 & 0.4 \\
0.5 & 0.3 & 0.5 & 1.0 & 0.2 \\
0.5 & 0.2 & 0.3 & 1.9 \\
\end{pmatrix},
\]

\(\beta = \{2, 4\}\).
By Theorem 10, the eigenvalues of $A/\beta$ are in the set
\[
\Gamma_2 = \{\lambda \| \lambda - 1.49\| \lambda - 1.64 \leq 7.12\}
\cup \{\lambda \| \lambda - 1.49\| \lambda - 1.84 \leq 7.64\}
\cup \{\lambda \| \lambda - 1.64\| \lambda - 1.84 \leq 8.50\}.
\] (54)

From Theorem 3.1 of [2], the eigenvalues of $A/\beta$ are in the set
\[
\Gamma'_2 = \{\lambda \| \lambda - 1.49\| \lambda - 1.64 \leq 10.68\}
\cup \{\lambda \| \lambda - 1.49\| \lambda - 1.84 \leq 11.46\}
\cup \{\lambda \| \lambda - 1.64\| \lambda - 1.84 \leq 12.75\}.
\] (55)

Evidently, $\Gamma_2 \subset \Gamma'_2$, and we use Figure 2 to show this fact. And the eigenvalues of $A/\beta$ are denoted by “+” in Figure 2.

Example 3. Let
\[
A = \begin{pmatrix}
1.5 & 0.2 & 0.3 & 0.2 & 0.1 & 0.2 \\
0.2 & 1.2 & 0.1 & 0.3 & 0.1 & 0.4 \\
0.6 & 0.2 & 1.6 & 0.1 & 0.2 & 0.1 \\
0.5 & 0.2 & 0.1 & 1.8 & 0.3 & 0.2 \\
0.2 & 0.1 & 0.3 & 1.3 & 0.1 & 0.4 \\
0.1 & 0.2 & 0.1 & 0.3 & 1.2 & 2.5
\end{pmatrix},
\] (56)

$\beta = \{1, 3, 5\}$.

By Theorem 10, the eigenvalues of $A/\beta$ are in the set
\[
\Gamma_3 = \{\lambda \| \lambda - 1.16\| \lambda - 1.68 \leq 4.79\}
\cup \{\lambda \| \lambda - 1.16\| \lambda - 2.42 \leq 6.78\}
\cup \{\lambda \| \lambda - 1.68\| \lambda - 2.42 \leq 9.85\}.
\] (57)

From Theorem 3.1 of [2], the eigenvalues of $A/\beta$ are in the set
\[
\Gamma'_3 = \{\lambda \| \lambda - 1.16\| \lambda - 1.68 \leq 5.35\}
\cup \{\lambda \| \lambda - 1.16\| \lambda - 2.42 \leq 7.60\}
\cup \{\lambda \| \lambda - 1.68\| \lambda - 2.42 \leq 11.09\}.
\] (58)

Evidently, $\Gamma_3 \subset \Gamma'_3$, and we use Figure 3 to show this fact. And the eigenvalues of $A/\beta$ are denoted by “+” in Figure 3.

Remark 13. Numerical examples show that the new eigenvalue inclusion set is tighter than that in Theorem 3.1 of [2] and the new upper bound of $\| (A/\beta)^{-1} \|\infty$ is sharper than that in Theorem 4.2 of [2].
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