Research Article

Refinements on the Hermite-Hadamard Inequalities for r -Convex Functions

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We give some new generalizations of the well-known Hermite-Hadamard inequality for r -convex functions.

1. Introduction

Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on the interval I ; then for any $a, b \in I$ with $a \neq b$, we have the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2}. \quad (1)$$

This remarkable result is well known in the literature as the Hermite-Hadamard inequality. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f . Both inequalities hold in the reversed direction if f is concave. Some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (e.g., [1–9]). The Hermite-Hadamard inequality was generalized in [10] to an r -convex positive function which is defined on an interval $[a, b]$. A positive function f is called r -convex on $[a, b]$, if for each $x, y \in [a, b]$ and $t \in [0, 1]$

$$f(tx + (1-t)y) \leq \begin{cases} [tf^r(x) + (1-t)f^r(y)]^{1/r} \\ [f(x)]^t [f(y)]^{1-t}. \end{cases} \quad (2)$$

It is obvious that 0-convex functions are simply log-convex functions and 1-convex functions are ordinary convex functions. If f is a positive r -concave function, then inequality (2) is reversed. We note that if f and g are convex and g is

increasing, then $g \circ f$ is convex; moreover, since $f = \exp(\log f)$, it follows that a log-convex function is convex. This follows directly from (2) because, by the arithmetic-geometric mean inequality, we have

$$[f(x)]^t [f(y)]^{1-t} \leq tf(x) + (1-t)f(y). \quad (3)$$

In [6], Gill et al. proved the following two theorems.

Theorem 1. Suppose that f is a positive r -convex function on $[a, b]$. Then

$$\frac{1}{b-a} \int_a^b f(t) dt \leq \frac{r}{r+1} \left(\frac{f^{r+1}(b) - f^{r+1}(a)}{f^r(b) - f^r(a)} \right). \quad (4)$$

If f is a positive r -concave function, then the inequality is reversed.

Theorem 2. Suppose that f is a positive log-convex function on $[a, b]$. Then

$$\frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(b) - f(a)}{\ln f(b) - \ln f(a)}. \quad (5)$$

If f is a positive log-concave function, then the inequality is reversed.

In [11], Sulaiman obtained the following result for r -convex functions.

Theorem 3. Let f be a positive r -convex function on $[a, b]$, $0 < r \leq 1$. Set

$$\begin{aligned} w(t) &= \left[t f^r \left(\frac{(2-t)a + tb}{2} \right) \right. \\ &\quad \left. + (1-t) f^r \left(\frac{(1-t)a + (1+t)b}{2} \right) \right]^{1/r}, \\ W(t) &= \frac{r}{r+1} \left[t \frac{f^{r+1}((1-t)a + tb) - f^{r+1}(a)}{f^r((1-t)a + tb) - f^r(a)} \right. \\ &\quad \left. + (1-t) \frac{f^{r+1}(b) - f^{r+1}((1-t)a + tb)}{f^r(b) - f^r((1-t)a + tb)} \right]. \end{aligned} \quad (6)$$

Then, the following inequality holds:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq w(t) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq W(t) \\ &\leq \frac{r}{r+1} \left(\frac{f^{r+1}(b) - f^{r+1}(a)}{f^r(b) - f^r(a)} \right) \\ &\leq \frac{f(a) + f(b)}{2}. \end{aligned} \quad (7)$$

In [12], Sulaiman obtained the following result for log-convex functions.

Theorem 4. Assume that $f : I \rightarrow \mathbb{R}$ is an increasing log-convex function. Then for all $t \in [0, 1]$, one has

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq w(a, b) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq W(t) \\ &\leq \frac{f(b) - f(a)}{\ln f(b) - \ln f(a)} \leq \frac{f(a) + f(b)}{2}, \end{aligned} \quad (8)$$

where

$$w(a, b) = \sqrt{f\left(\frac{3a+b}{4}\right) f\left(\frac{a+3b}{4}\right)} \quad (9)$$

$$\begin{aligned} W(t) &= (1-t) \frac{f(ta + (1-t)b) - f(a)}{\ln f(ta + (1-t)b) - \ln f(a)} \\ &\quad + t \frac{f(b) - f(ta + (1-t)b)}{\ln f(b) - \ln f(ta + (1-t)b)}. \end{aligned} \quad (10)$$

For recent results and generalizations concerning the Hermite-Hadamard inequality, see [7, 8] and the references given therein.

In this note, we establish some generalizations of the above results for the class of r -convex functions.

2. Lemmas

Lemma 5. If $a, b > 0$, then

$$\lim_{r \rightarrow 0} (ta^r + (1-t)b^r)^{1/r} = a^t b^{1-t}. \quad (11)$$

Remark 6. Applying Lemma 5 and (2), for an r -convex f , let $r \rightarrow 0$; then f is log-convex. Now we let $r \rightarrow 0$ in Theorem 1; then we get Theorem 2 from

$$\lim_{r \rightarrow 0} \frac{r}{r+1} \left(\frac{f^{r+1}(b) - f^{r+1}(a)}{f^r(b) - f^r(a)} \right) = \frac{f(b) - f(a)}{\ln f(b) - \ln f(a)}. \quad (12)$$

Lemma 7. If $0 < a < b < c$, then

$$\begin{aligned} \frac{b-a}{\ln b - \ln a} &\leq \frac{c-a}{\ln c - \ln a}, \\ \frac{c-b}{\ln c - \ln b} &\leq \frac{c-a}{\ln c - \ln a}. \end{aligned} \quad (13)$$

Lemma 8. If $0 < a < b < c$ and $r > 0$, then

$$\begin{aligned} \frac{c^{r+1} - b^{r+1}}{c^r - b^r} &\leq \frac{c^{r+1} - a^{r+1}}{c^r - a^r}, \\ \frac{b^{r+1} - a^{r+1}}{b^r - a^r} &\leq \frac{c^{r+1} - a^{r+1}}{c^r - a^r}. \end{aligned} \quad (14)$$

Lemma 9. If $a, b > 0$, then the following inequality holds:

$$\frac{b-a}{\ln b - \ln a} \leq \frac{a+b}{2}. \quad (15)$$

Lemma 10 (see [11, Theorem 3.1]). Let $f : [a, b] \rightarrow (0, \infty)$ be r -convex on $[a, b]$ and $0 < r \leq 1$; then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \left[\frac{1}{(b-a)} \int_a^b f^r(x) dx \right]^{1/r} \\ &\leq \frac{1}{b-a} \int_a^b f(x) dx \\ &\leq \frac{r}{r+1} \left(\frac{f^{r+1}(b) - f^{r+1}(a)}{f^r(b) - f^r(a)} \right) \\ &\leq \frac{f(a) + f(b)}{2}. \end{aligned} \quad (16)$$

3. Main Results

Theorem 11. Let $f : [a, b] \rightarrow (0, \infty)$ be r -convex and non-decreasing on $[a, b]$ and $0 < r \leq 1$; for $n \in \mathbb{N}$, $\lambda_0 = 0$, $\lambda_{n+1} = 1$, and arbitrary $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq 1$, the following inequality holds:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq w(\lambda_1, \dots, \lambda_n) \\ &\leq \frac{1}{b-a} \int_a^b f(x) dx \\ &\leq W(\lambda_1, \dots, \lambda_n) \\ &\leq \frac{r}{r+1} \left(\frac{f^{r+1}(b) - f^{r+1}(a)}{f^r(b) - f^r(a)} \right) \\ &\leq \frac{f(a) + f(b)}{2}, \end{aligned} \quad (17)$$

where

$$\begin{aligned}
 w(\lambda_1, \dots, \lambda_n) &= \left[\sum_{k=0}^n (\lambda_{k+1} - \lambda_k) f^r \right. \\
 &\quad \times (2 - \lambda_{k+1} - \lambda_k) a + (\lambda_{k+1} + \lambda_k) b \\
 &\quad \left. \times (2)^{-1} \right]^{1/r}, \\
 W(\lambda_1, \dots, \lambda_n) &= \frac{r}{r+1} \\
 &\times \left[\sum_{k=0}^n (\lambda_{k+1} - \lambda_k) \right. \\
 &\quad \times (f^{r+1}((1 - \lambda_{k+1})a + \lambda_{k+1}b) \\
 &\quad - f^{r+1}((1 - \lambda_k)a + \lambda_k b)) \\
 &\quad \times (f^r((1 - \lambda_{k+1})a + \lambda_{k+1}b) \\
 &\quad - f^r((1 - \lambda_k)a + \lambda_k b))^{-1} \left. \right]. \quad (18)
 \end{aligned}$$

Proof. Observing that $\sum_{k=0}^n (\lambda_{k+1} - \lambda_k) = 1$, $\sum_{k=0}^n (\lambda_{k+1}^2 - \lambda_k^2) = 1$ and Jensen's inequality for $f^r(x)$, we have

$$\begin{aligned}
 f^r\left(\frac{a+b}{2}\right) &= f^r\left(\sum_{k=0}^n (\lambda_{k+1} - \lambda_k) \right. \\
 &\quad \times \left. \frac{(2 - \lambda_{k+1} - \lambda_k)a + (\lambda_{k+1} + \lambda_k)b}{2} \right) \\
 &\leq \sum_{k=0}^n (\lambda_{k+1} - \lambda_k) \\
 &\quad \times f^r\left[\frac{(2 - \lambda_{k+1} - \lambda_k)a + (\lambda_{k+1} + \lambda_k)b}{2}\right] \\
 &= w^r(\lambda_1, \dots, \lambda_n) \\
 &\leq \sum_{k=0}^n (\lambda_{k+1} - \lambda_k) \\
 &\quad \times \left[\frac{1}{(\lambda_{k+1} - \lambda_k)(b-a)} \right. \\
 &\quad \times \left. \int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} f^r(x) dx \right] \\
 &= \sum_{k=0}^n \left[\frac{1}{(b-a)} \int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} f^r(x) dx \right] \\
 &= \frac{1}{(b-a)} \int_a^b f^r(x) dx, \quad (19)
 \end{aligned}$$

where the second inequality follows from replacing a and b by $(1 - \lambda_k)a + \lambda_k b$ and $(1 - \lambda_{k+1})a + \lambda_{k+1}b$, respectively, and (1) for $f^r(x)$. Therefore

$$\begin{aligned}
 f\left(\frac{a+b}{2}\right) &\leq \left[\frac{1}{(b-a)} \int_a^b f^r(x) dx \right]^{1/r} \\
 &\leq \frac{1}{b-a} \int_a^b f(x) dx \\
 &= \frac{1}{b-a} \sum_{k=0}^n \int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} f(x) dx \\
 &= \sum_{k=0}^n (\lambda_{k+1} - \lambda_k) \frac{1}{(\lambda_{k+1} - \lambda_k)(b-a)} \\
 &\quad \times \int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} f(x) dx \\
 &\leq \frac{r}{r+1} \\
 &\quad \times \left[\sum_{k=0}^n (\lambda_{k+1} - \lambda_k) \right. \\
 &\quad \times (f^{r+1}((1 - \lambda_{k+1})a + \lambda_{k+1}b) \\
 &\quad - f^{r+1}((1 - \lambda_k)a + \lambda_k b)) \\
 &\quad \times (f^r((1 - \lambda_{k+1})a + \lambda_{k+1}b) \\
 &\quad - f^r((1 - \lambda_k)a + \lambda_k b))^{-1} \left. \right] \\
 &= W(\lambda_1, \dots, \lambda_n) \\
 &\leq \frac{r}{r+1} \left[\sum_{k=0}^n (\lambda_{k+1} - \lambda_k) \right. \\
 &\quad \times \left. \frac{f^{r+1}(b) - f^{r+1}(a)}{f^r(b) - f^r(a)} \right] \\
 &= \frac{r}{r+1} \frac{f^{r+1}(b) - f^{r+1}(a)}{f^r(b) - f^r(a)} \\
 &\leq \frac{f(a) + f(b)}{2},
 \end{aligned} \quad (20)$$

where the third inequality follows from replacing a and b by $(1 - \lambda_k)a + \lambda_k b$ and $(1 - \lambda_{k+1})a + \lambda_{k+1}b$, respectively, and (4). The proof is completed. $\square$

Remark 12. Applying Theorem 11 for $n = 1$, we get Theorem 3.

Corollary 13. With the above notations, if $f : [a, b] \rightarrow (0, \infty)$ is r -convex and nondecreasing on $[a, b]$ and $0 < r \leq 1$, one has the following inequality:

$$\begin{aligned} \sup_{0 \leq \lambda_1 \leq \dots \leq \lambda_n \leq 1} w(\lambda_1, \dots, \lambda_n) &\leq \frac{1}{b-a} \int_a^b f(x) dx \\ &\leq \inf_{0 \leq \lambda_1 \leq \dots \leq \lambda_n \leq 1} W(\lambda_1, \dots, \lambda_n), \end{aligned} \quad (21)$$

where $W(\lambda_1, \dots, \lambda_n)$ and $w(\lambda_1, \dots, \lambda_n)$ are defined in Theorem 11.

Theorem 14. Let $f : [a, b] \rightarrow (0, \infty)$ be log-convex and non-decreasing on $[a, b]$; for $n \in \mathbb{N}$, $\lambda_0 = 0$, $\lambda_{n+1} = 1$, and arbitrary $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq 1$, the following inequality holds:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq m(t) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq M(\lambda_1, \dots, \lambda_n) \\ &\leq \frac{f(b) - f(a)}{\ln f(b) - \ln f(a)} \leq \frac{f(a) + f(b)}{2}, \end{aligned} \quad (22)$$

where

$$m(t) = \left[f\left(\frac{(2-t)a + tb}{2}\right) \right]^t \left[f\left(\frac{(1-t)a + (1+t)b}{2}\right) \right]^{1-t}, \quad (23)$$

$$\begin{aligned} M(\lambda_1, \dots, \lambda_n) &= \sum_{k=0}^n (\lambda_{k+1} - \lambda_k) \\ &\quad \times (f((1-\lambda_{k+1})a + \lambda_{k+1}b) \\ &\quad - f((1-\lambda_k)a + \lambda_k b)) \\ &\quad \times (\ln f((1-\lambda_{k+1})a + \lambda_{k+1}b) \\ &\quad - \ln f((1-\lambda_k)a + \lambda_k b))^{-1}. \end{aligned} \quad (24)$$

Proof. Let f be a positive r -convex function on $[a, b]$, $0 < r \leq 1$, by Theorem 3: then

$$\begin{aligned} w(t) &= \left[t f^r\left(\frac{(2-t)a + tb}{2}\right) \right. \\ &\quad \left. + (1-t) f^r\left(\frac{(1-t)a + (1+t)b}{2}\right) \right]^{1/r} \\ &\leq \frac{1}{b-a} \int_a^b f(x) dx. \end{aligned} \quad (25)$$

Applying Lemma 5, let $r \rightarrow 0$; then f is log-convex and

$$\begin{aligned} \lim_{r \rightarrow 0} w(t) &= m(t) \leq \frac{1}{b-a} \int_a^b f(x) dx. \\ f\left(\frac{a+b}{2}\right) &= f\left(t \frac{(2-t)a + tb}{2} + (1-t) \right. \\ &\quad \left. \times \frac{(1-t)a + (1+t)b}{2}\right) \\ &\leq \left[f\left(\frac{(2-t)a + tb}{2}\right) \right]^t \\ &\quad \times \left[f\left(\frac{(1-t)a + (1+t)b}{2}\right) \right]^{1-t} \\ &= m(t). \end{aligned} \quad (26)$$

Observing that $\sum_{k=0}^n (\lambda_{k+1} - \lambda_k) = 1$,

$$\begin{aligned} \frac{1}{(b-a)} \int_a^b f(x) dx &= \frac{1}{b-a} \left[\sum_{k=0}^n \int_{(1-\lambda_k)a + \lambda_k b}^{(1-\lambda_{k+1})a + \lambda_{k+1}b} f(x) dx \right] \\ &= \sum_{k=0}^n (\lambda_{k+1} - \lambda_k) \left[\frac{1}{(\lambda_{k+1} - \lambda_k)(b-a)} \right. \\ &\quad \left. \times \int_{(1-\lambda_k)a + \lambda_k b}^{(1-\lambda_{k+1})a + \lambda_{k+1}b} f(x) dx \right] \\ &\leq \sum_{k=0}^n (\lambda_{k+1} - \lambda_k) \\ &\quad \times (f((1-\lambda_{k+1})a + \lambda_{k+1}b) \\ &\quad - f((1-\lambda_k)a + \lambda_k b)) \\ &\quad \times (\ln f((1-\lambda_{k+1})a + \lambda_{k+1}b) \\ &\quad - \ln f((1-\lambda_k)a + \lambda_k b))^{-1} \\ &= M(\lambda_1, \dots, \lambda_n) \\ &\leq \sum_{k=0}^n (\lambda_{k+1} - \lambda_k) \frac{f(b) - f(a)}{\ln f(b) - \ln f(a)} \\ &= \frac{f(b) - f(a)}{\ln f(b) - \ln f(a)} \\ &\leq \frac{f(a) + f(b)}{2}, \end{aligned} \quad (27)$$

where the first inequality follows from replacing a and b by $(1-\lambda_k)a + \lambda_k b$ and $(1-\lambda_{k+1})a + \lambda_{k+1}b$, respectively, and (5). The proof is completed. $\square$

Remark 15. Applying Theorem 14 for $t = 1/2$ and $n = 1$, $m(1/2) = w(a, b) = \sqrt{f((3a+b)/4)f((a+3b)/4)}$, we get Theorem 4.

Corollary 16. *With the above notations, suppose that $f : [a, b] \rightarrow (0, \infty)$ is log-convex and nondecreasing on $[a, b]$; one has the following inequality:*

$$\sup_{0 \leq t \leq 1} m(t) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \inf_{0 \leq \lambda_1 \leq \dots \leq \lambda_n \leq 1} M(\lambda_1, \dots, \lambda_n), \quad (28)$$

where $m(t)$ and $M(\lambda_1, \dots, \lambda_n)$ are defined in Theorem 14.

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