Research Article

On Some Properties for the Sequence of Brualdi-Li Matrices

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Let B_{2n} denote the Brualdi-Li matrix, and let $\rho_{2n} = \rho(B_{2n})$ denote the Perron value of the Brualdi-Li matrix of order $2n$. We prove that $2n(n - 1/2 - \rho_{2n})$ is monotonically decreasing for all n and $\rho_{2n} < n - 1/2 - (e^2 - 1)/4(e^2 + 1)n$, where $e = 2.718281828459045 \dots$

1. Introduction

A tournament matrix of order n is a $(0, 1)$ matrix T satisfying the equation $T + T^t = J - I$, where J is the all ones matrix, I is the identity matrix, and T^t is the transpose of T . If n is odd, then the maximum possible Perron value of a tournament matrix T of order n is $(n-1)/2$, attained when the row sums of T are all equal [1]. However, if n is even, it is difficult to attain. Brualdi and Li [2] conjectured that the matrix

$$\mathcal{B}_{2n} = \begin{pmatrix} U_n & U_n^t \\ I + U_n^t & U_n \end{pmatrix}, \quad (1)$$

where U_n is strictly upper triangular tournament matrix (all of its entries above the main diagonal are equal to one), that is,

$$U_n = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n \times n}, \quad (2)$$

is the tournament matrix of order $2n$ which maximizes the Perron value. This conjecture has recently been confirmed in [3].

The matrix $\mathcal{B}_{2n}$ has been dubbed by the Brualdi-Li matrix [4]. Let $\rho_{2n} = \rho(B_{2n})$ denote the Perron value of the Brualdi-Li matrix of order $2n$.

It is well known that $n - 1 < \rho_{2n} < n$.

Friedland [4] has paid particular attention to the behavior of the Perron value of $\mathcal{B}_{2n}$. Friedland and Katz [5] posed

the problem of determining the behavior of the sequence $2n(n - 1/2 - \rho_{2n})$. Kirkland [6] conjectured that $2n(n - 1/2 - \rho_{2n})$ is monotonically decreasing for all n . We investigate this conjecture and obtain the following results.

Theorem 1. *Let $n \geq 2$ be an integer, and let $\rho_{2n} = \rho(\mathcal{B}_{2n})$ be the Perron value of the Brualdi-Li matrix $\mathcal{B}_{2n}$. Then $2n(n - 1/2 - \rho_{2n})$ is monotonically decreasing for all n .*

By Theorem 1 and Lemma 4, one can easily prove the following corollaries.

Corollary 2. *Let $n \geq 2$ be an integer, and let $\rho_{2n} = \rho(\mathcal{B}_{2n})$ be the Perron value of the Brualdi-Li matrix $\mathcal{B}_{2n}$. Then*

$$\rho_{2n} < n - \frac{1}{2} - \frac{e^2 - 1}{4(e^2 + 1)n}, \quad (3)$$

where $e = 2.718281828459045 \dots$

In [6], Kirkland proved that

$$\lim_{n \rightarrow +\infty} n^3 \left(n - \frac{1}{2} - \frac{\gamma}{4n} - \rho_{2n} \right) = \frac{\gamma^2}{16} + \frac{e^2(3\gamma + 1)}{12(e^2 + 1)^2}, \quad (4)$$

where $\gamma = (e^2 - 1)/(e^2 + 1)$.

We make the following conjecture and remark that it is supported by computations for $n \leq 100$.

Conjecture 3. *$n^3(n - 1/2 - \gamma/4n - \rho_{2n})$ is monotonically decreasing for all $n > 1$.*

We give some lemmas in Section 2 and prove Theorem 1 in Section 3.

2. Some Lemmas

2.1. Intermediate Value Theorem. If f is a real-valued continuous function on the closed interval $[a, b]$, and μ is a number between $f(a)$ and $f(b)$, then there is $\xi \in (a, b)$ such that $f(\xi) = \mu$, where (a, b) is open interval.

For two variables, the Implicit Function Theorem is as follows.

2.2. Implicit Function Theorem. If a real-valued function F is defined on an open disk containing point (x_0, y_0) , satisfying

- (i) $F(x_0, y_0) = 0$,
- (ii) $\partial F / \partial y|_{(x_0, y_0)} \neq 0$,
- (iii) $\partial F / \partial x$ and $\partial F / \partial y$ are continuous on the open disk,

then the equation $F(x, y) = 0$ defines y as a function of x near the point (x_0, y_0) and the derivative of this function is given by

$$\frac{dy}{dx} = -\frac{\partial F / \partial x}{\partial F / \partial y}, \quad (5)$$

where $\partial F / \partial x$ and $\partial F / \partial y$ indicate the partial derivatives of F with respect to x and y .

Lemma 1 (see [6]). Let $n \geq 2$ be an integer, and let $\rho_{2n} = \rho(\mathcal{B}_{2n})$ be the Perron value of the Brualdi-Li matrix $\mathcal{B}_{2n}$. Then

$$(2\rho_{2n}^2 + 2(1-n)\rho_{2n} + 1 - n)((1 + \rho_{2n}^{-1})^{2n} + 1) - 1 = 0. \quad (6)$$

Let $F = F(m, \rho) = (2\rho^2 + 2(1-m)\rho + 1 - m)((1 + \rho^{-1})^{2m} + 1) - 1$ and $D = \{(m, \rho) \mid m > 2, m - 1 < \rho < m, m \text{ and } \rho \text{ are real numbers}\}$, then the partial derivatives of F are

$$\begin{aligned} \frac{\partial F}{\partial m} &= -(2\rho + 1)((1 + \rho^{-1})^{2m} + 1) \\ &\quad + 2(2\rho^2 + 2(1-m)\rho + 1 - m) \\ &\quad \times (1 + \rho^{-1})^{2m} \ln(1 + \rho^{-1}), \\ \frac{\partial F}{\partial \rho} &= 2(2\rho - m + 1)((1 + \rho^{-1})^{2m} + 1) \\ &\quad - 2m\rho^{-2}(2\rho^2 + 2(1-m)\rho + 1 - m) \\ &\quad \times (1 + \rho^{-1})^{2m-1}. \end{aligned} \quad (7)$$

Evidently, $\partial F / \partial m$ and $\partial F / \partial \rho$ are derivable on the open disk D , and $\partial F / \partial \rho = 2(2\rho - m + 1) + 2((2\rho - m + 1)(1 + \rho^{-1}) - m\rho^{-2}(2\rho^2 + 2(1-m)\rho + 1 - m)) \cdot (1 + \rho^{-1})^{2m-1} > 0$, $(m, \rho) \in D$. By the Implicit Function Theorem, the equation $F(m, \rho) = 0$

defines ρ as a function in terms of m and the derivative of this function is given by

$$\frac{d\rho}{dm} = -\frac{\partial F / \partial m}{\partial F / \partial \rho}. \quad (8)$$

Lemma 2. Let $m \geq 3$ be a real number, and let ρ be a function in terms of m satisfying the equation $F(m, \rho) = 0$. Then

$$m - \frac{1}{2} - \frac{1}{5m} < \rho < m - \frac{1}{2} - \frac{10}{53m}, \quad (9)$$

where $F(m, \rho) = (2\rho^2 + 2(1-m)\rho + 1 - m)((1 + \rho^{-1})^{2m} + 1) - 1$, $(m, \rho) \in D$.

Proof. For fixed real number $m \geq 3$, $F(m, \rho)$ is continuous real function in terms of ρ .

Let $\underline{\rho} = m - 1/2 - 1/5m$ and $\bar{\rho} = m - 1/2 - 10/53m$. It is easy to show that the real functions $y_1(x) = (1 + (1/(x - 1/2 - 1/5x)))^{2x}$ and $y_2(x) = (1 + (1/(x - 1/2 - 10/53x)))^{2x}$ are monotonically decreasing for real number $x > 2$. Thus

$$\begin{aligned} F(m, \underline{\rho}) &= (2\underline{\rho}^2 + 2(1-m)\underline{\rho} + 1 - m) \\ &\quad \times ((1 + \underline{\rho}^{-1})^{2m} + 1) - 1 \\ &= \left(2\left(m - \frac{1}{2} - \frac{1}{5m}\right)^2\right. \\ &\quad \left.+ 2(1-m)\left(m - \frac{1}{2} - \frac{1}{5m}\right) + 1 - m\right) \\ &\quad \cdot \left(\left(1 + \frac{1}{m - 1/2 - 1/5m}\right)^{2m} + 1\right) - 1 \\ &= \left(\frac{1}{2} - \frac{2}{5} + \frac{2}{5^2 m^2}\right) \\ &\quad \times \left(\left(1 + \frac{1}{m - 1/2 - 1/5m}\right)^{2m} + 1\right) - 1 \\ &\leq \left(\frac{1}{2} - \frac{2}{5} + \frac{2}{5^2 \cdot 3^2}\right) \\ &\quad \times \left(\left(1 + \frac{1}{3 - 1/2 - 1/15}\right)^6 + 1\right) - 1 \\ &= -0.0319 \dots < 0. \end{aligned} \quad (10)$$

Notice that

$$\begin{aligned} \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x - 1/2 - 10/53x}\right)^{2x} \\ = e^2 = 7.38905609893065 \dots, \end{aligned} \quad (11)$$

where $e = 2.718281828459045 \dots$. We have

$$\begin{aligned}
 F(m, \bar{\rho}) &= \left(2 \left(m - \frac{1}{2} - \frac{10}{53m} \right)^2 + 2(1-m) \right. \\
 &\quad \times \left(m - \frac{1}{2} - \frac{10}{53m} \right) + 1 - m \Big) \\
 &\quad \cdot \left(\left(1 + \frac{1}{m - 1/2 - 10/53m} \right)^{2m} + 1 \right) - 1 \\
 &= \left(\frac{1}{2} - \frac{20}{53} + \frac{200}{(53m)^2} \right) \\
 &\quad \times \left(\left(1 + \frac{1}{m - 1/2 - 10/53m} \right)^{2m} + 1 \right) - 1 \\
 &> \left(\frac{1}{2} - \frac{20}{53} \right) (e^2 + 1) - 1 > 0.
 \end{aligned} \tag{12}$$

Hence $F(m, \rho)F(m, \bar{\rho}) < 0$. By the Intermediate Value Theorem, there is $\rho \in (\rho, \bar{\rho})$ such that $F(m, \rho) = 0$; that is,

$$m - \frac{1}{2} - \frac{1}{5m} < \rho < m - \frac{1}{2} - \frac{10}{53m}. \tag{13}$$

We are done. $\square$

Lemma 3. Let $m \geq 3$ be a real number, and let ρ be a function in terms of m satisfying the equation $F(m, \rho) = 0$, and let $G = G(m, \rho) = 2\rho^2 + 2(1-m)\rho + 1 - m$, $(m, \rho) \in D$. Then

$$\begin{aligned}
 &\left(\frac{1}{10} + \frac{2}{25m^2} \right) \left(\frac{93}{106} - \frac{200}{2089m^2} \right) \\
 &< G(m, \rho) (1 - G(m, \rho)) \\
 &< \left(\frac{9}{10} - \frac{2}{25m^2} \right) \left(\frac{13}{106} + \frac{200}{2089m^2} \right).
 \end{aligned} \tag{14}$$

Proof. For fixed real number $m \geq 3$, $G(m, \rho)$ is continuous real function in terms of ρ . Let $\underline{\rho} = m - 1/2 - 1/5m$ and $\bar{\rho} = m - 1/2 - 10/53m$; then

$$G(m, \underline{\rho}) = \frac{1}{10} + \frac{2}{25m^2}, \quad G(m, \bar{\rho}) = \frac{13}{106} + \frac{200}{2089m^2}. \tag{15}$$

By Lemma 2, we have

$$G(m, \underline{\rho}) < G(m, \rho) = 2\rho^2 + 2(1-m)\rho + 1 - m < G(m, \bar{\rho}). \tag{16}$$

Hence

$$\frac{1}{10} + \frac{2}{25m^2} < G(m, \rho) < \frac{13}{106} + \frac{200}{2089m^2}; \tag{17}$$

that is,

$$0 < \frac{93}{106} - \frac{200}{2089m^2} < 1 - G(m, \rho) < \frac{9}{10} - \frac{2}{25m^2}. \tag{18}$$

We have the desired result. $\square$

Lemma 4 (see [6]). There is an M such that the sequence $2n(n - 1/2 - \rho_{2n})$ decreases monotonically to its limit $(e^2 - 1)/2(e^2 + 1)$ for $n > M$.

3. Proof of Theorem 1

In Section 2, $F = F(m, \rho) = (2\rho^2 + 2(1-m)\rho + 1 - m)((1 + \rho^{-1})^{2m} + 1) - 1$ and $G = G(m, \rho) = 2\rho^2 + 2(1-m)\rho + 1 - m$, $(m, \rho) \in D$, and ρ is the function in terms of m satisfying the equation $F(m, \rho) = 0$. Clearly, F and G are the functions in terms of m .

Let $m \geq 3$ be real number. Since $F(m, \rho) = 0$, we have

$$(1 + \rho^{-1})^{2m} = \frac{1 - G(m, \rho)}{G(m, \rho)}, \quad \frac{dF}{dm} = 0. \tag{19}$$

Further

$$\begin{aligned}
 \frac{dF}{dm} &= \frac{d}{dm} \left(G \left((1 + \rho^{-1})^{2m} + 1 \right) \right) \\
 &= G \frac{d}{dm} \left((1 + \rho^{-1})^{2m} + 1 \right) + \left((1 + \rho^{-1})^{2m} + 1 \right) \frac{dG}{dm} \\
 &= 2G(1 + \rho^{-1})^{2m} \left(\ln(1 + \rho^{-1}) - \frac{m}{\rho(1 + \rho)} \frac{d\rho}{dm} \right) \\
 &\quad + \left((1 + \rho^{-1})^{2m} + 1 \right) \left((4\rho + 2 - 2m) \frac{d\rho}{dm} - 2\rho - 1 \right) \\
 &= 2(1 - G) \left(\ln(1 + \rho^{-1}) - \frac{m}{\rho(1 + \rho)} \frac{d\rho}{dm} \right) \\
 &\quad + \frac{1}{G} \left((4\rho + 2 - 2m) \frac{d\rho}{dm} - 2\rho - 1 \right) \\
 &= \frac{1}{G} \left(2(1 - G) G \ln(1 + \rho^{-1}) \right. \\
 &\quad \left. - 2(1 - G) G \frac{m}{\rho(1 + \rho)} \frac{d\rho}{dm} \right. \\
 &\quad \left. + (4\rho + 2 - 2m) \frac{d\rho}{dm} - 2\rho - 1 \right) \\
 &= \frac{1}{\rho(1 + \rho)G} \\
 &\quad \times \left(2\rho(1 + \rho)(1 - G) G \ln(1 + \rho^{-1}) \right. \\
 &\quad \left. - 2m(1 - G) G \frac{d\rho}{dm} \right. \\
 &\quad \left. + \rho(1 + \rho) \left((4\rho + 2 - 2m) \frac{d\rho}{dm} - 2\rho - 1 \right) \right) \\
 &= \frac{1}{\rho(1 + \rho)G} \\
 &\quad \times \left(2\rho(1 + \rho)(1 - G) G \ln(1 + \rho^{-1}) \right.
 \end{aligned}$$

$$\begin{aligned}
& + 2m(G-1)G \frac{d\rho}{dm} \\
& + \rho(1+\rho)(4\rho+2-2m) \frac{d\rho}{dm} \\
& - \rho(1+\rho)(2\rho+1) \Big) = 0.
\end{aligned} \tag{20}$$

Thus

$$\begin{aligned}
& (2m(G-1)G + \rho(1+\rho)(4\rho+2-2m)) \frac{d\rho}{dm} \\
& = 2\rho(1+\rho)(G-1)G \ln(1+\rho^{-1}) \\
& + \rho(1+\rho)(2\rho+1).
\end{aligned} \tag{21}$$

We have

$$\begin{aligned}
\frac{d\rho}{dm} & = (2\rho(1+\rho)(G-1)G \ln(1+\rho^{-1}) + \rho(1+\rho)(2\rho+1)) \\
& \times (2m(G-1)G + \rho(1+\rho)(4\rho+2-2m))^{-1}.
\end{aligned} \tag{22}$$

Hence

$$\begin{aligned}
\frac{d\rho}{dm} - 2 + \frac{2\rho+1}{2m} & = \frac{2\rho(1+\rho)(G-1)G \ln(1+\rho^{-1}) + \rho(1+\rho)(2\rho+1)}{2m(G-1)G + \rho(1+\rho)(4\rho+2-2m)} \\
& + \frac{2\rho+1-4m}{2m} \\
& = (2m(2\rho(1+\rho)(G-1)G \ln(1+\rho^{-1}) + \rho(1+\rho)(2\rho+1))) \\
& \times (2m(\rho(1+\rho)(4\rho+2-2m) - 2m(1-G)G))^{-1} \\
& + ((2m(G-1)G + \rho(1+\rho)(4\rho+2-2m)) \\
& \times (2\rho+1-4m)) \\
& \times (2m(\rho(1+\rho)(4\rho+2-2m) - 2m(1-G)G))^{-1}.
\end{aligned} \tag{23}$$

Let $A = \rho(1+\rho)(4\rho+2-2m) - 2m(1-G)G$ and

$$\begin{aligned}
B & = 2m(2\rho(1+\rho)(G-1)G \ln(1+\rho^{-1}) \\
& + \rho(1+\rho)(2\rho+1)) \\
& + (2m(G-1)G + \rho(1+\rho)(4\rho+2-2m)) \\
& \times (2\rho+1-4m);
\end{aligned} \tag{24}$$

then

$$\begin{aligned}
\frac{d\rho}{dm} - 2 + \frac{2\rho+1}{2m} & = \frac{B}{2mA}, \\
B & = 4m\rho(1+\rho)(G-1)G \ln(1+\rho^{-1}) \\
& + 2m\rho(1+\rho)(2\rho+1) \\
& + 2m(2\rho+1-4m)(G-1)G \\
& + \rho(1+\rho)(4\rho+2-2m)(2\rho+1-4m) \\
& = 2m\rho(1+\rho)(2\rho+1) \\
& + \rho(1+\rho)(4\rho+2-2m)(2\rho+1-4m) \\
& + 4m\rho(1+\rho)(G-1)G \ln(1+\rho^{-1}) \\
& + 2m(2\rho+1-4m)(G-1)G \\
& = 2\rho(1+\rho)(m(2\rho+1) \\
& + (2\rho+1-m)(2\rho+1-4m)) \\
& + 2mG(1-G) \\
& \times (4m-2\rho-1-2\rho(1+\rho) \ln(1+\rho^{-1})).
\end{aligned} \tag{25}$$

Let $C = m(2\rho+1) + (2\rho+1-m)(2\rho+1-4m)$ and $D = 4m-2\rho-1-2\rho(1+\rho) \ln(1+\rho^{-1})$, then

$$B = 2\rho(1+\rho)C + 2mG(1-G)D. \tag{26}$$

Obviously,

$$\begin{aligned}
C & = m(2\rho+1) + (2\rho+1-m)(2\rho+1-4m) \\
& = (2\rho+1-2m)^2 > 0.
\end{aligned} \tag{27}$$

By Lemma 2 and Taylor's Theorem, we have

$$\begin{aligned}
D & = 4m-2\rho-1-2\rho(1+\rho) \ln(1+\rho^{-1}) \\
& > 4m-2\rho-1-2\rho(1+\rho) \left(\frac{1}{\rho} - \frac{1}{2\rho^2} + \frac{1}{3\rho^3} \right) \\
& = 4m-4\rho-2 + \frac{1}{3\rho} - \frac{2}{3\rho^2} \\
& > 4m-4 \left(m - \frac{1}{2} - \frac{10}{53m} \right) - 2 + \frac{1}{3\rho} - \frac{2}{3\rho^2} \\
& = \frac{1}{3\rho} \left(1 - \frac{2}{\rho} \right) + \frac{40}{53m}.
\end{aligned} \tag{28}$$

Due to $\rho > m-1 \geq 2$, then

$$D > 0. \tag{29}$$

By Lemmas 2 and 3,

$$\begin{aligned}
 A &= \rho(1 + \rho)(4\rho + 2 - 2m) - 2m(1 - G)G \\
 &> (m - 1)m(4(m - 1) + 2 - 2m) \\
 &\quad - 2m\left(\frac{9}{10} - \frac{2}{25m^2}\right)\left(\frac{13}{106} + \frac{200}{2089m^2}\right) \\
 &= 2m\left((m - 1)^2 - \left(\frac{9}{10} - \frac{2}{25m^2}\right)\left(\frac{13}{106} + \frac{200}{2089m^2}\right)\right) \\
 &> 0.
 \end{aligned} \tag{30}$$

By Lemma 3 and (25)–(30), we have $B/2mA > 0$ for $m \geq 3$. Hence $(d/dm)(2m(m - 1/2 - \rho)) < 0$, where $m \geq 3$ is a real number.

To sum up, we obtain the following result.

Let $m \geq 3$ be a real number, and let ρ be a function of m satisfying the equation $F(m, \rho) = 0$, $(m, \rho) \in D$; then $2m(m - 1/2 - \rho)$ is monotonically decreasing function for all m .

Note that $\rho_{2n} = \rho(\mathcal{B}_{2n})$ is the Perron value of the Brualdi-Li matrix $\mathcal{B}_{2n}$, by Lemma 1, ρ_{2n} satisfy the equation

$$\begin{aligned}
 F(n, \rho_{2n}) &= (2\rho_{2n}^2 + 2(1 - n)\rho_{2n} + 1 - n) \\
 &\quad \times \left(\left(1 + \rho_{2n}^{-1}\right)^{2n} + 1 \right) - 1 = 0,
 \end{aligned} \tag{31}$$

where $n \geq 3$ is an integer. It is easy to see that $2n(n - 1/2 - \rho_{2n})$ is monotonically decreasing for all $n \geq 3$.

Since

$$\begin{aligned}
 2 \times 2 \left(2 - \frac{1}{2} - \rho_{2 \times 2} \right) &= 4(1.5 - 1.3953) \\
 &> 2 \times 3 \left(3 - \frac{1}{2} - \rho_{2 \times 3} \right) \\
 &= 6(2.5 - 2.4340),
 \end{aligned} \tag{32}$$

evidently, it follows that $2n(n - 1/2 - \rho_{2n})$ is monotonically decreasing for all integers $n \geq 2$. We complete the proof of Kirkland's conjecture.

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