A Concentration Phenomenon for p-Laplacian Equation

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It is proved that if the bounded function of coefficient $Q_n$ in the following equation $-\text{div}(|u|^{p-2}u) + V(x)|u|^{p-2}u = Q_n(x)|u|^{q-2}u$, $u(x) = 0$ as $x \in \partial\Omega$, $u(x) \to 0$ as $|x| \to \infty$ is positive in a region contained in $\Omega$ and negative outside the region, the sets $\{Q_n > 0\}$ shrink to a point $x_0 \in \Omega$ as $n \to \infty$, and then the sequence $u_n$ generated by the nontrivial solution of the same equation, corresponding to $Q_n$, will concentrate at $x_0$ with respect to $W^{1,p}_0(\Omega)$ and certain $L^p(\Omega)$-norms. In addition, if the sets $\{Q_n > 0\}$ shrink to finite points, the corresponding ground states $\{u_n\}$ only concentrate at one of these points. These conclusions extend the results proved in the work of Ackermann and Szulkin (2013) for case $p = 2$.

1. Introduction

We study a new concentration phenomenon for the following $p$-Laplacian equations:

$$-\text{div}(|u|^{p-2}u) + V(x)|u|^{p-2}u = Q_n(x)|u|^{q-2}u,$$

where $\Omega \subset \mathbb{R}^N$ is a smooth domain and $V(x) \geq 0 \in L^\infty(\Omega)$, and $p < q < p^*$, where $p^* := Np/(N - p)$ if $N \geq p$ and $p^* := \infty$ if $N < p$. If $\Omega$ is unbounded, we assume additionally that $\sigma(-\text{div}(|V|^{p-2}V) + V) \cdot |\cdot|^{p-2} < (0, \infty)$.

And an assumption of $Q_n$ is as follows.

(*) The set $\{x \in \Omega | Q_n(x) > 0\}$ contained in the neighborhood of zero has positive measure, and $\|Q_n\|_{L^1(\Omega)} \leq C$ with the constant $C$ is independent of $n$. Moreover, for each $\epsilon > 0$ there exist constants $\delta_\epsilon(> 0)$ and $N_\epsilon$ such that $Q_n \leq -\epsilon$ whenever $x \notin B_\epsilon(0)$ and $n \geq N_\epsilon$.

As it is known, $u = 0$ is the only solution to (1) if $Q_n(x) \leq 0$ for all $x \in \Omega$. In addition, if $Q_n(x) > 0$ is based on a bounded set of positive measures, it is clear that there exists a solution $u \neq 0$ (see Theorem 1). Hence, without loss of generality, we assume that $0 \in \Omega$ and let $Q = Q_n$ be such that $Q_n > 0$ on the ball $B_{1/n}(0)$ and $Q_n < 0$ on $\Omega \setminus B_{2/n}(0)$ and $u_n \neq 0$ are the solutions to (1) associated with $Q_n(x)$.

Accordingly, the question is what happens to $u_n$ as $n \to \infty$. Furthermore, this phenomenon can be found in physics. For instance, considering the materials separately from $Q$ positive or negative (see [1]), it corresponds to investigating the existence of bright ($Q > 0$) or dark ($Q < 0$) solitons.

Equations of these types have been studied extensively in many monographs and lectures (e.g., [2–10] for $p = 2$, [11–18] for general $p$). In [2], Byeon and Wang considered the standing wave solutions $\psi(x,t) \equiv \exp(-iEt/\hbar)v(x)$ for the nonlinear Schrödinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2} \Delta \psi - V(x) \psi + |\psi|^{p-1} \psi = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$  

Thus, they needed only to discuss the function $v$ which satisfies

$$\frac{\hbar^2}{2} \Delta v - (V(x) - E) v + |v|^{p-1} v = 0, \quad x \in \mathbb{R}^N,$$

and rewrote it in the following form:

$$\epsilon^2 \Delta v - V(x) v + \epsilon^p v = 0, \quad \epsilon > 0, \quad x \in \mathbb{R}^N,$$

$$\lim_{|x| \to 0} v(x) = 0.$$
By a rescaling, it is transformed to
\[ \Delta u - V'(\epsilon x)u + u^p = 0, \quad u > 0, \ x \in \mathbb{R}^N \]
\[ \lim_{|x| \to 0} u(x) = 0. \] (5)

Let the zero set \( \mathcal{Z} \triangleq \{ x \in \mathbb{R}^N \mid V(x) = 0 \} \) and \( A \) be an isolated component of \( \mathcal{Z} \), and they distinguished three cases of \( A \) to prove the concentration as \( \epsilon \to 0 \). And then, in [3] by replacing \( v^p \) with a fairly general class nonlinearity \( f(v) \), they also obtained the concentration. Furthermore, in [4], Byeon and Jeanjean gave the almost optimal condition on \( \epsilon(x) \).

\[ \int_{|x| < \epsilon} |u|^q dx. \] (14)

Suppose that \( (v_k) \) is a minimizing sequence for \( s_n \), normalized by \( J_n(v_k) = 1 \); then \( \|v_k\| \) is bounded. Hence, \( v_n \to v \) in \( E \) and \( v_n(x) \to v(x) \) a.e. in \( \Omega \) (by choosing a subsequence). Note that \( Q_n < 0 \) on \( |x| > 1 \) for \( n \) large. The Rellich-Kondrachov Theorem and Fatou’s Lemma say that

\[ s_n = \lim_{k \to \infty} \frac{\|v_k\|}{J_n(v_k)} = \lim_{k \to \infty} \frac{\|v_k\|}{\|v_k\|^p} = \frac{\inf_{\|v\|=1} \int_{\Omega} Q_n|v|^q dx}{\inf_{\|v\|=1} \int_{\Omega} Q_n|v|^q dx} \] (10)

Thus \( v \) is a minimizer.

And then, the lagrange multiple rule implies that \( u_n = c_n v_n \) is a solution to (1) for some appropriate constant \( c_n > 0 \). Moreover, since \( v_n \) may be replaced by \( |v_n| - v_n \) and \( u_n \geq 0 \) (and hence \( u_n \geq 0 \)). To show that \( u_n \to 0 \), we note that \( u_n \) satisfies

\[ -\div (|\nabla v|^{p-2} \nabla v) + (V(x) u_n^{p-2} + Q_n(x) u_n(x)^{p-2}) v = Q_n(x) u_n(x) v \geq 0, \] (11)

where \( Q_n := \max \{ \pm Q_n(x), 0 \} \). Since \( V(x) u_n^{p-2} + Q_n(x) u_n(x)^{p-2} \geq 0 \), it follows from the strong maximum principle (see [20, 21]) that \( u_n > 0 \).

If \( u_n \neq 0 \) is a solution to (1), then, via multiplying the equation by \( u_n \), integrating by parts, and using the Sobolev inequality, one deduces that

\[ \|u_n\|^p = \int_{\Omega} Q_n u_n|^q dx \leq c_1|u_n|^p \leq c_2\|u_n\|^q; \] (12)

hence, \( \|u_n\| \geq \alpha \) for some \( \alpha > 0 \) and all large \( n \).

The next step is to consider the property of the nontrivial solution \( \{u_n\} \) to (1) and \( w_n := u_n/\|u_n\| \).

**Lemma 2.** Consider

\[ \|u_n\| \to \infty \text{ as } n \to \infty. \] (13)

**Proof.** We present an abridged version of the proof highlighting the main differences to that in [19]. It will be proved by contradiction. Assume \( u_n \to u \) in \( E \) and \( u_n \to u \) in \( L_{\text{loc}}^q(\Omega) \) after passing to a subsequence. Multiplying (1) (with \( u = u_n \)) by \( u_n \), integrating by parts, and recalling that \( Q_n < 0 \) for each \( \epsilon > 0 \) and \( n \geq N_\epsilon \), it holds that

\[ \limsup_{n \to \infty} \|u_n\|^p = \limsup_{n \to \infty} \int_{\Omega} Q_n|u_n|^q dx \]

\[ \leq \limsup_{n \to \infty} \int_{|x| \leq \epsilon} Q_n|u_n|^q dx \leq c \int_{|x| \leq \epsilon} |u|^q dx. \] (14)
If $\epsilon \to 0$, $u_n \to 0$ in $E$. It is a contradiction to $\|u_n\| \geq \alpha > 0$ given in Theorem 1.

Lemma 3. Consider

$$w_n \to 0 \text{ in } E \text{ as } n \to \infty \quad (15)$$

Proof. We prove it by contradiction as well. We may assume that $w_n \to w(\neq 0)$ in $E$. Multiplying (1) (with $u = u_n$) by $u_n/\|u_n\|^{p}$ yields that

$$1 = \|w_n\|^p = \|u_n\|^{q-p} \int \Omega Q_n|w_n|^q dx. \quad (16)$$

Due to Lemma 2 with $q > p$, $\int \Omega |w_n|^q dx \to 0$.

On the other hand, we have for $0 < \epsilon < \epsilon_1$

$$0 = \lim_{n \to \infty} \int \Omega Q_n|w_n|^q dx$$

$$= \lim_{n \to \infty} \left( \int_{|x| < \epsilon} Q_n|w_n|^q dx + \int_{|x| > \epsilon} Q_n|w_n|^q dx \right)$$

$$\leq \lim_{n \to \infty} \left( \int_{|x| < \epsilon} Q_n|w_n|^q dx + \int_{|x| > \epsilon} Q_n|w_n|^q dx \right)$$

$$\leq \epsilon \int_{|x| < \epsilon} |w_n|^q dx - \delta_{\epsilon_1} \int_{|x| > \epsilon_1} |w_n|^q dx.$$ 

We may choose small $\epsilon_1$ such that the second integral on the right-hand side above is positive as $w \neq 0$. Then we get the contradiction as $\epsilon \to 0$.

In the sequel, we study concentration of $\{u_n\}$ as $n \to \infty$. Let $\epsilon > 0$ be given and $\chi \in C^\infty(\Omega, [0, 1])$ be such that $\chi(x) = 0$ for $x \in B_{\epsilon/2}(0)$ and $\chi(x) = 1$ for $x \notin B_{\epsilon}(0)$.

Multiplying (1) (with $u = u_n$) by $u_n$ we obtain

$$\int \Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (\chi u_n) + \chi V u_n^p \right) dx = \int \Omega \chi Q_n|u_n|^q dx,$$ 

namely,

$$\int \Omega \chi \left( |\nabla u_n|^p + V u_n^p \right) dx = \int \Omega \chi Q_n|u_n|^q dx$$

$$- \int \Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \chi \cdot u_n dx. \quad (19)$$

Given $\epsilon > 0$, we have $\Omega_n \leq -\delta_\epsilon$ on supp$\chi$, provided that $n$ is large enough. Hence for such all $n$,

$$0 \leq \int_{\Omega \setminus \Omega_{\epsilon}(0)} \left( |\nabla u_n|^p + V u_n^p \right) dx + \delta_\epsilon \int_{\Omega \setminus \Omega_{\epsilon}(0)} |u_n|^q dx$$

$$\leq \int \Omega \chi \left( |\nabla u_n|^p + V u_n^p \right) dx - \int \Omega \chi Q_n|u_n|^q dx$$

$$= \int |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \chi \cdot u_n dx$$

$$\leq d_\epsilon \int_{B_{\epsilon}(0) \setminus B_{\epsilon/2}(0)} |u_n| |\nabla u_n|^{p-1} dx,$$

where $d_\epsilon$ is a constant independent of $n$. Since $w_n = u_n/\|u_n\| \to 0$ in $L^p_{loc}(\Omega)$ according to Lemma 3, it follows from Hölder inequality that

$$\int_{B_{\epsilon}(0) \setminus B_{\epsilon/2}(0)} |w_n| |\nabla w_n|^{p-1} dx \to 0. \quad (21)$$

So (20) implies

$$\int_{\Omega \setminus \Omega_{\epsilon}(0)} \left( |\nabla w_n|^p + V w_n^p \right) dx + \delta_\epsilon \int_{\Omega \setminus \Omega_{\epsilon}(0)} |w_n|^q dx = 0. \quad (22)$$

\[\square\]

Theorem 4. Suppose that $Q_n$ satisfies the assumption $(\ast)$ and $q \in (p, p^*)$. Let $u_n$ be a nontrivial solution to (1) and put $w_n = u_n/\|u_n\|$. Then for every $\epsilon > 0$ they hold that

$$\lim_{n \to \infty} \int_{\Omega \setminus \Omega_{\epsilon}(0)} \left( |\nabla w_n|^p + V w_n^p \right) dx = 0, \quad (23)$$

$$\lim_{n \to \infty} \|u_n\|^{q-p} \int_{\Omega \setminus \Omega_{\epsilon}(0)} |w_n|^q dx = 0. \quad (24)$$

Moreover,

$$\lim_{n \to \infty} \int_{\Omega \setminus \Omega_{\epsilon}(0)} \left( |\nabla w_n|^p + V w_n^p \right) dx = 0,$$

$$\lim_{n \to \infty} \int_{\Omega \setminus \Omega_{\epsilon}(0)} |w_n|^q dx = 0. \quad (25)$$

Proof. (23) and (24) can be easily obtained by (22). Note that

$$\int_{\Omega} \left( |\nabla w_n|^p + V w_n^p \right) dx = \|w_n\|^p = 1. \quad (26)$$

From (23), one concludes that

$$\lim_{n \to \infty} \frac{\int_{\Omega \setminus \Omega_{\epsilon}(0)} \left( |\nabla w_n|^p + V w_n^p \right) dx}{\int_{\Omega} \left( |\nabla w_n|^p + V w_n^p \right) dx} = 0.$$ 

(27)

According to (16), we get

$$\epsilon \|u_n\|^{q-p} \int_{\Omega} |w_n|^q dx \geq \|u_n\|^{q-p} \int_{\Omega} Q_n|u_n|^q dx = \|w_n\|^p = 1. \quad (28)$$

This and (24) imply

$$\lim_{n \to \infty} \frac{\int_{\Omega \setminus \Omega_{\epsilon}(0)} |w_n|^q dx}{\int_{\Omega} |w_n|^q dx} = \lim_{n \to \infty} \frac{\|u_n\|^{q-p} \int_{\Omega \setminus \Omega_{\epsilon}(0)} |w_n|^q dx}{\|u_n\|^{q-p} \int_{\Omega} |w_n|^q dx} = 0. \quad (29)$$
3. Concentration in the $L^s$-Norm

The next is to consider the concentration in other norms.

Theorem 5. Let $u_n$ denote a nontrivial solution to (1) for each $n \in \mathbb{N}$. Suppose that the assumption (*) holds and there exists $R$, $\lambda > 0$ such that $V \geq \lambda$ whenever $x \in \Omega \setminus B_R(0)$, and there exists $\varepsilon > 0$ such that $B_\varepsilon(0) \subset \Omega$; then one can get that

(a) $\exists C$, for all $s \in [1, \infty)$, $n \in \mathbb{N}$, $|u_n|_{s,\Omega \setminus B(0)} \leq C$;

(b) if $\delta = \delta_c > 0$ in (*) can be chosen independently of $\varepsilon (> 0)$, then $\lim_{n \to \infty} |u_n|_{s,\Omega \setminus B(0)} = 0$, for every $s \in [1, \infty)$;

(c) for all $s (\geq 1) \in (N(q - p)/p, \infty)$, one has $\lim_{n \to \infty} |u_n|_s = \infty$ and

\[
\lim_{n \to \infty} \frac{|u_n|_{s,\Omega \setminus B(0)}}{|u_n|_s} = 0; \quad (30)
\]

(d) if $N(q - p)/p > 1$, then for $s = N(q - p)/p$ it holds that

\[
\lim_{n \to \infty} |u_n|_s > 0. \quad (31)
\]

If the hypotheses in (b) are satisfied, then (30) also holds for this $s$.

Proof. There is clearly a positive classical solution $w$ to the equation

\[
-\text{div}(|\nabla w|^{p-2} \nabla w) = -\delta \varepsilon^{1/2} |u|^{p-2} u, \quad x \in \mathbb{R}^n \setminus B_{\varepsilon/2}(0) \quad (32)
\]

In fact, by [22, 23], the radial solution $u_p(x) = u_p(|x|)$ satisfies the ordinary differential equation

\[
(r^{n-1} u'^{p-2} u')' = -\delta \varepsilon^{1/2} r^{n-1} u^q \\
u(r) = \infty \quad \text{as} \quad r \to \varepsilon/2, \\
u(r) \to 0 \quad \text{as} \quad r \to \infty. \quad (33)
\]

Set $z_n = w - u_n$ and

\[
\varphi_n(x) := (q - 1) \int_0^1 |sw(x) + (1 - s) u_n(x)|^{q-2} ds \geq 0, \\
\phi_n(x) := (p - 1) \int_0^1 |sw(x) + (1 - s) u_n(x)|^{p-2} ds \geq 0;
\]

\[
\varphi_n(x) z_n = (q - 1) \int_0^1 |sw(x) + (1 - s) u_n(x)|^{q-2} (w - u_n) ds \\
= \int_0^1 \frac{d}{ds} \left( |sw + (1 - s) u_n|^{p-2} (sw + (1 - s) u_n) \right) ds \\
= w^{p-1} - |u_n|^{p-2} u_n, \\
\phi_n(x) z_n = (p - 1) \int_0^1 |sw(x) + (1 - s) u_n(x)|^{p-2} (w - u_n) ds \\
= \int_0^1 \frac{d}{ds} \left( |sw + (1 - s) u_n|^{p-2} (sw + (1 - s) u_n) \right) ds \\
= w^{p-1} - |u_n|^{p-2} u_n \\
\geq \varphi_n(x) z_n \geq 0. \quad (34)
\]

and hence from (*)

\[
-\text{div}(|\nabla w|^{p-2} \nabla w) - \left( -\text{div}(\nabla u_n|^{p-2} \nabla u_n) \right) \\
+ (V\varphi_n(x) - Q_n \varphi_n) z_n \\
= -\text{div}|\nabla u|^{p-2} + V|w|^{p-2} w - Q_n u^{p-1} \\
- \left[ -\text{div}|\nabla u|^{p-2} u + |\nabla u|^{p-2} u - Q_n u_n|^{q-2} u_n \right] \\
= -\text{div}|\nabla u|^{p-2} + V|w|^{p-2} w - Q_n u^{p-1} \\
\geq -\text{div}|\nabla u|^{p-2} + \delta \varepsilon^{1/2} w^{p-1} = 0. \quad (35)
\]

Note that $V\varphi_n(x) - Q_n \varphi_n \geq 0$ in $\Omega \setminus B_{\varepsilon/2}(0)$ when $n > N_{\varepsilon/2}$. Due to the continuity of $u_n$ and the fact that $w_n(x) \to \infty$ as $x \to \partial B_{\varepsilon/2}(0)$, there is $r \in (\varepsilon/2, \varepsilon)$ such that $z_n \geq 0$ on $\partial B_{\varepsilon/2}(0)$. Moreover, $z_n \geq 0$ on $\partial \Omega$. If $\Omega$ is bounded, the maximum principle says that $z_n \geq 0$ in $\Omega \setminus B_{\varepsilon/2}(0)$ (see [20, 21]). If $\Omega$ is unbounded, by virtue of $u(x)$ tending to $0$ as $|x| \to \infty$ by construction, thus for any $\gamma > 0$, we may pick $R > 0$ such that $z_n \geq -\gamma$ in $\Omega \setminus B_R(0)$. Moreover, applying regularity theory to $u_n \in W^{2,p}_0(\Omega)$, we can get $u_n(x) \to 0$ as $|x| \to \infty$. Now the same maximum principle is applied on $\Omega \cap (B_R \setminus B_{\varepsilon/2}(0))$, which implies that $z_n \geq -\gamma$ in all of $\Omega \setminus B_R(0)$. Letting $\gamma \to 0$, we obtain $z_n > 0$ again. By analogy we obtain $u_n \to -w$ (take $z_n := w + u_n$); hence

\[
|z_n| \geq \omega \quad \text{in} \quad \Omega \setminus B_{\varepsilon/2}(0), \quad \forall n \geq N_{\varepsilon/2}. \quad (36)
\]

Hence (a) follows from above arguments with the fact that $w$ is continuous in $\Omega \setminus B_{\varepsilon/2}(0)$.

Next, the hypotheses in (b) imply that there is $\delta > 0$ such that $Q_n \leq -\delta$ on $\Omega \setminus B_{1/n}(0)$ for each $n$ large enough. Let $w_n$ be a positive solution to

\[
-\text{div}(\nabla u_n|^{p-2} \nabla u_n) = -\delta |u_n|^{p-2} u, \quad x \in \mathbb{R}^n \setminus B_{1/n}(0) \\
\lim_{|x| \to 1/n} w_n(x) = +\infty, \quad \lim_{|x| \to \infty} w_n(x) = 0. \quad (37)
\]
Then the sequence $w_n$ is monotone decreasing, by using the maximum principle to $w_n \geq w_{n+1}$ on $\partial B(0)$ for every $n \in \mathbb{N}$. Therefore, $w_n$ converges locally and uniformly to a nonnegative solution $w$ to (37) on $\mathbb{R}^N \setminus \{0\}$. It follows from our hypotheses on $N$ and $p$ that $w$ is an entire solution to (37) by applying the argument as in [24]. And then, due to [25], $w \equiv 0$. For another, the function $w_n$ dominates the solution $u_n$ on $\Omega \setminus B(0)$ for some $r \in (1/2, \epsilon)$, as seen in the proof of (a). Thus, $u_n$ also converges to 0 locally and uniformly in $\Omega \setminus B(0)$; that is, $\lim_{n \to \infty} [u_n, \Omega \setminus B(0)] = 0$.

For (c), we consider the case $s(1) \geq 1 \in (N(p-1)/p, q]$. By interpolation inequality, we have the following estimate for solution $u_n$:

$$
\|u_n\|^p = \int_{\Omega} Q_n u_n|^q dx \leq c_1 \|u_n\|^{\theta q} \|u_n\|^{(1-\theta)q} \leq c_2 \|u_n\|^{\theta q} \|u_n\|^{(1-\theta)q}.
$$

(38)

Here $c_1, c_2$ are independent of $n$ and $\theta$ satisfies

$$
\frac{1}{q} = \frac{\theta}{s} + 1 - \frac{\theta}{p^*}.
$$

(39)

According to Lemma 2, it suffices to impose that $q(1-\theta) < p$ or equivalent $s > N(q-p)/p, q$. And (a) prove the case $s \in (N(q-p)/p, q]$. And then, (38) and (a) yield $\|u_n\|_{L^q(B(0))} \to \infty$; hence $\|u_n\|_{L^q(B(0))} \to \infty$ for every $s \in \{q, \infty\}$ as $n \to \infty$. Using (a) again we get (30).

Note that (38) implies (30) for $s = N(q-p)/p$, so case (d) is easily followed.

4. Concentration at Several Points

Now we assume that the function $Q_n$ is positive in a neighbourhood of two distinct points $x_1, x_2 \in \Omega$ (indeed, the following argument is also valid for any finite number of points in $\Omega$). More precisely, we assume:

\[(*) \quad Q_n > 0 \text{ in a neighbourhood of }\{x_1\} \cup \{x_2\}, \text{ and there exists a constant } C \text{ such that } |Q_n(x)| \leq C \text{ for all } x.\]

Moreover, for each $\epsilon > 0$ there exist constants $\delta_\epsilon > 0$ and $N_\epsilon$ such that $Q_n \leq -\delta_\epsilon$ whenever $x \notin B(\frac{\epsilon}{2}) (x_1) \cup B(\frac{\epsilon}{2}) (x_2)$ and $n \geq N_\epsilon$.

As in Section 2, we put $J_n(u) = \int_{\Omega} Q_n u|^q dx$:

$$
s_n := \inf_{I_{(0)}(u)} \frac{\int_{\Omega} |V u|^p + |V u|^q dx}{\int_{\Omega} Q_n u|^q dx} \equiv \inf_{I_{(0)}(u)} \left(\frac{\int_{\Omega} |V u|^p + |V u|^q dx}{\int_{\Omega} Q_n u|^q dx}\right)^{p/q}.
$$

(40)

Theorem 6. Suppose $Q_n$ satisfies $(*)$ and $q \in (p, p^*)$, and $u_n$ is a ground state solution to (1). Then, for $n$ large, $u_n$ concentrates at $x_1$ or $x_2$. More precisely, for each $\epsilon > 0$ we have by passing to a subsequence

$$
\lim_{n \to \infty} \frac{\int_{\Omega \setminus B(\frac{\epsilon}{2}) (x_1)} |V u|^p + |V u|^q dx}{\int_{\Omega \setminus B(\frac{\epsilon}{2}) (x_1)} Q_n u|^q dx} = 0, \quad \lim_{n \to \infty} \frac{\int_{\Omega \setminus B(\frac{\epsilon}{2}) (x_2)} Q_n u|^q dx}{\int_{\Omega \setminus B(\frac{\epsilon}{2}) (x_2)} Q_n u|^q dx} = 0
$$

for $j = 1$ or $2$ (but not for $j = 1$ and 2).

Remark 7. Note that, in view of the obvious modification of Theorem 4, the limits in (41) are 0 if $\Omega \setminus B(x_j)$ is replaced by $\Omega \setminus B(x_1) \cup B(x_2)$. So if $j = 1$ in (41), then concentration occurs at $x_1$ and if $j = 2$, it occurs at $x_2$.

Proof. As in [19], we may assume that $J_n(u_n) = \int_{\Omega} Q_n u_n|^q dx = 1$ by renormalizing ($u_n$ may not be a solution to (1), but we still have $s_n := \|u_n\|^p / J_n(u_n)^{p/q}$). Let $\xi_j \in C_0^\infty (\Omega, [0, 1])$ be a function such that $\xi_j = 1$ on $B(\frac{\epsilon}{2}) (x_j)$ and $\xi_j = 0$ on $\Omega \setminus B(x_j)$, $j = 1, 2$, where $\epsilon$ is so small that $B(x_j) \subset \Omega$ and $B(x_1) \cup B(x_2) = \emptyset$. Set $v_n := \xi_1 u_n$, $w_n := \xi_2 u_n$, and $z_n := u_n - v_n - w_n$. Since $supp_\epsilon z_n \subset \Omega \setminus (B(x_1) \cup B(x_2))$ and the conclusion of Theorem 4 remains valid after a modification, we have

$$
\|u_n\|^p = \int_{\Omega} |V v_n|^p + |V w_n|^p dx
$$

$$
= \left(\int_{\Omega} |V v_n|^p + |V w_n|^p dx\right) (1 + o(1))
$$

$$
\leq \left(\|v_n\|^p + \|w_n\|^p \right) (1 + o(1)),
$$

(42)

$$
1 = I_n(u_n) = \int_{\Omega} Q_n u_n|^q dx
$$

$$
\leq \int_{\Omega} Q_n v_n|^q dx + \int_{\Omega} Q_n w_n|^q dx + o(1)
$$

$$
= J_n(v_n) + J_n(w_n) + o(1).
$$

First, we assume that $\limsup_{n \to \infty} J_n(v_n) \geq 0$ and $\limsup_{n \to \infty} J_n(w_n) \geq 0$. By passing to a subsequence, we may assume that $J_n(v_n) \to c_1 \in [0, 1]$ and $J_n(w_n) \to 1 - c_1 \in [0, 1]$. If $c_1 \in (0, 1)$, recalling that $q > p$, we get a contradiction from the following inequality:

$$
s_n = \frac{\|v_n\|^p}{J_n(v_n)} + \frac{\|w_n\|^p}{J_n(w_n)} \geq\left(\frac{\|v_n\|^p + \|w_n\|^p}{J_n(v_n) + J_n(w_n)} + o(1)\right)^{p/q} \geq s_n.
$$

So $c_1 = 0$ or 1. If $c_1 = 1$ (say), then the second limit in (41) is 0 for $j = 1$ because $supp v_n \subset B(x_1)$. The first limit is 0 as well, since $\|w_n\|^p / \|v_n\|^p$ is otherwise bounded away from 0 for large $n$, and we obtain a contradiction again from

$$
s_n = \left(\frac{\|v_n\|^p + \|w_n\|^p}{J_n(v_n) + J_n(w_n)} + o(1)\right)^{p/q} \geq \frac{\|v_n\|^p}{J_n(v_n)^{p/q}} \geq s_n.
$$

(44)

Finally, suppose $\limsup_{n \to \infty} J_n(w_n) < 0$ (the case $\limsup_{n \to \infty} J_n(v_n) < 0$ is of course analogous); it passes to
a subsequence \( J_n(w_n) \leq -\eta \) for some \( \eta > 0 \) when \( n \) is large enough. Then a contradiction (44) holds for such \( n \) because
\[
J_n(v_n) > J_n(w_n) + o(1).
\]

Conflict of Interests
The author declares that there is no conflict of interests regarding the publication of this paper.

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