Research Article

On Hermite-Hadamard Type Inequalities for s -Convex Functions on the Coordinates via Riemann-Liouville Fractional Integrals

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We obtain some Hermite-Hadamard type inequalities for s -convex functions on the coordinates via Riemann-Liouville integrals. Some integral inequalities with the right-hand side of the fractional Hermite-Hadamard type inequality are also established.

1. Introduction

If $f : I \rightarrow \mathbb{R}$ is a convex function on the interval I , then, for any $a, b \in I$ with $a \neq b$, we have the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2}. \quad (1)$$

This remarkable result is well known in the literature as the Hermite-Hadamard inequality.

Since then, some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (e.g., [1–7]).

Definition 1 (see [8]). Let $s \in (0, 1]$ be a fixed real number. A function $f : I \subset [0, \infty) \rightarrow [0, \infty)$ is said to be s -convex in the second sense, or that f belongs to the class K_s^2 , if the inequality,

$$f(\alpha x + (1-\alpha)y) \leq \alpha^s f(x) + (1-\alpha)^s f(y), \quad (2)$$

holds for all $x, y \in I$ and $\alpha \in [0, 1]$.

It can be easily seen that, for $s = 1$, s -convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

In [4], Dragomir defined convex functions on the coordinates as follows.

Let us consider the bidimensional interval $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$; a mapping $f : \Delta \rightarrow \mathbb{R}$ is said to be convex on Δ if the inequality,

$$\begin{aligned} f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \\ \leq \lambda f(x, y) + (1-\lambda)f(z, w), \end{aligned} \quad (3)$$

holds for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$.

A function $f : \Delta \rightarrow \mathbb{R}$ is said to be coordinated convex on Δ if the partial mappings $f_y : [a, b] \rightarrow \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \rightarrow \mathbb{R}$, $f_x(v) = f(x, v)$ are convex for all $y \in [c, d]$ and $x \in [a, b]$.

A formal definition for convex functions on the coordinates may be stated as follows.

Definition 2. A function $f : \Delta \rightarrow \mathbb{R}$ is said to be convex on coordinates on Δ if the inequality,

$$\begin{aligned} f(\lambda x + (1-\lambda)z, t y + (1-t)w) \\ \leq \lambda t f(x, y) + \lambda(1-t)f(x, w) + (1-\lambda)t f(z, y) \\ + (1-\lambda)(1-t)f(z, w), \end{aligned} \quad (4)$$

holds for all $(x, y), (z, y), (x, w), (z, w) \in \Delta$, and $t, \lambda \in [0, 1]$.

In [4], Dragomir established the following Hadamard-type inequalities for convex functions on the coordinates in a rectangle from the plane $\mathbb{R}^2$.

Theorem 3 (see [4]). Suppose that $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}$ is convex on the coordinates on Δ . Then one has the inequalities:

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\ &\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned} \quad (5)$$

The concept of s -convex functions on the coordinates in the second sense was introduced by Alomari and Darus in [9].

Let us consider the bidimensional interval $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$; a mapping $f : \Delta \rightarrow \mathbb{R}$ is s -convex on Δ if the inequality,

$$\begin{aligned} f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \\ \leq \lambda^s f(x, y) + (1-\lambda)^s f(z, w), \end{aligned} \quad (6)$$

holds for all $(x, y), (z, w) \in \Delta$ with $\lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$.

A function $f : \Delta \rightarrow \mathbb{R}$ is said to be s -convex on the coordinates on Δ in the second sense if the partial mappings $f_y : [a, b] \rightarrow \mathbb{R}$, $f_y(u) = f(u, y)$, and $f_x : [c, d] \rightarrow \mathbb{R}$, $f_x(v) = f(x, v)$ are s -convex in the second sense for all $y \in [c, d]$ and $x \in [a, b]$ with some fixed $s \in (0, 1]$.

A formal definition for convex functions on the coordinates in the second sense may be stated as follows.

Definition 4. A function $f : \Delta \rightarrow \mathbb{R}$ is said to be s -convex on coordinates in the second sense on Δ if the inequality,

$$\begin{aligned} f(\lambda x + (1-\lambda)z, t y + (1-t)w) \\ \leq \lambda^s t^s f(x, y) + \lambda^s (1-t)^s f(x, w) \\ + (1-\lambda)^s t^s f(z, y) + (1-t)^s (1-\lambda)^s f(z, w), \end{aligned} \quad (7)$$

holds for all $(x, y), (z, y), (x, w), (z, w) \in \Delta$ with $t, \lambda \in [0, 1]$ and some fixed $s \in (0, 1]$.

In [10], Alomari and Darus proved the following inequalities based on the above definition.

Theorem 5 (see [10]). Suppose that $f : \Delta = [a, b] \times [c, d] \subseteq [0, \infty)^2 \rightarrow [0, \infty)$ is s -convex on the coordinates in the second sense on Δ . Then one has the inequalities:

$$\begin{aligned} 4^{s-1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\ &\leq (f(a, c) + f(a, d) + f(b, c) \\ &\quad + f(b, d)) \left((s+1)^2\right)^{-1}. \end{aligned} \quad (8)$$

It is remarkable that Sarikaya et al. [11] proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 6 (see [11]). Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \leq \frac{f(a) + f(b)}{2}, \quad (9)$$

with $\alpha > 0$.

We remark that the symbol $J_{a^+}^\alpha$ and $J_{b^-}^\alpha f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \geq 0$ with $a \geq 0$ which are defined by

$$J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a, \quad (10)$$

$$J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$.

Definition 7 (see [12]). Let $f \in L_1[a, b] \times [c, d]$. The Riemann-Liouville fractional integrals $J_{a^+, c^+}^{\alpha, \beta}$, $J_{a^+, d^-}^{\alpha, \beta}$, $J_{b^-, c^+}^{\alpha, \beta}$ and $J_{b^-, d^-}^{\alpha, \beta}$ of order $\alpha, \beta > 0$ with $a, c \geq 0$ are defined by

$$\begin{aligned} J_{a^+, c^+}^{\alpha, \beta} f(x, y) \\ = \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^x \int_c^y (x-t)^{\alpha-1} (y-s)^{\beta-1} f(t, s) ds dt, \\ x > a, \quad y > c, \end{aligned}$$

$$\begin{aligned} J_{a^+, d^-}^{\alpha, \beta} f(x, y) \\ = \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^x \int_y^d (x-t)^{\alpha-1} (s-y)^{\beta-1} f(t, s) ds dt, \\ x > a, \quad y < d, \end{aligned}$$

$$\begin{aligned} J_{b^-, c^+}^{\alpha, \beta} f(x, y) \\ = \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_x^b \int_c^y (t-x)^{\alpha-1} (y-s)^{\beta-1} f(t, s) ds dt, \\ x < b, \quad y > c, \end{aligned}$$

$$\begin{aligned} J_{b^-, d^-}^{\alpha, \beta} f(x, y) \\ = \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_x^b \int_y^d (t-x)^{\alpha-1} (s-y)^{\beta-1} f(t, s) ds dt, \\ x < b, \quad y < d. \end{aligned} \quad (11)$$

In [12], Sarikaya proposed the following Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals by using convex functions of two variables on the coordinates.

Theorem 8 (see [12]). Let $f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be convex functions on the coordinates on $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $0 \leq a < b$, $0 \leq c < d$, and $f \in L_1(\Delta)$. Then one has the inequalities:

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \\ &\times \left[J_{a^+,c^+}^{\alpha,\beta} f(b,d) + J_{a^+,d^-}^{\alpha,\beta} f(b,c) \right. \\ &\quad \left. + J_{b^-,c^+}^{\alpha,\beta} f(a,d) + J_{b^-,d^-}^{\alpha,\beta} f(a,c) \right] \\ &\leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}. \end{aligned} \quad (12)$$

In [12], Sarikaya established some Hermite-Hadamard inequalities for convex functions on the coordinates in the second sense via fractional integrals based on the following lemma.

Lemma 9 (see [12]). Let $f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $0 \leq a < b$, $0 \leq c < d$. If $\partial^2 f / \partial t \partial s \in L(\Delta)$, then the following equality holds:

$$\begin{aligned} &\frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \\ &\times \left[J_{a^+,c^+}^{\alpha,\beta} f(b,d) + J_{a^+,d^-}^{\alpha,\beta} f(b,c) + J_{b^-,c^+}^{\alpha,\beta} f(a,d) \right. \\ &\quad \left. + J_{b^-,d^-}^{\alpha,\beta} f(a,c) \right] - A \\ &= \frac{(b-a)(d-c)}{4} \\ &\times \left\{ \int_0^1 \int_0^1 t^\alpha s^\beta \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) ds dt \right. \\ &\quad - \int_0^1 \int_0^1 (1-t)^\alpha s^\beta \frac{\partial^2 f}{\partial t \partial s} \\ &\quad \times (ta + (1-t)b, sc + (1-s)d) ds dt \\ &\quad - \int_0^1 \int_0^1 t^\alpha (1-s)^\beta \frac{\partial^2 f}{\partial t \partial s} \\ &\quad \times (ta + (1-t)b, sc + (1-s)d) ds dt \\ &\quad \left. + \int_0^1 \int_0^1 (1-t)^\alpha (1-s)^\beta \frac{\partial^2 f}{\partial t \partial s} \right. \\ &\quad \left. \times (ta + (1-t)b, sc + (1-s)d) ds dt \right\}, \end{aligned} \quad (13)$$

where

$$\begin{aligned} A &= \frac{\Gamma(\beta+1)}{4(d-c)^\beta} \left[J_{c^+}^\beta f(a,d) + J_{c^+}^\beta f(b,d) + J_{d^-}^\beta f(a,c) \right. \\ &\quad \left. + J_{d^-}^\beta f(b,c) \right] \\ &\quad + \frac{\Gamma(\alpha+1)}{4(b-a)^\alpha} \left[J_{a^+}^\alpha f(b,c) + J_{a^+}^\alpha f(b,d) + J_{b^-}^\alpha f(a,c) \right. \\ &\quad \left. + J_{b^-}^\alpha f(a,d) \right]. \end{aligned} \quad (14)$$

In this paper, we establish some Hermite-Hadamard type inequalities for s -convex functions on the coordinates functions via Riemann-Liouville integrals. Some integral inequalities with the right-hand side of the fractional Hermite-Hadamard type inequality are also given.

2. Fractional Inequalities for s -Convex Functions on the Coordinates

Theorem 10. Suppose that $f : \Delta = [a, b] \times [c, d] \subseteq [0, \infty)^2 \rightarrow [0, \infty)$ is s -convex function on the coordinates in the second sense on Δ . Then one has the inequalities:

$$\begin{aligned} &4^{s-1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ &\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \\ &\times \left[J_{a^+,c^+}^{\alpha,\beta} f(b,d) + J_{a^+,d^-}^{\alpha,\beta} f(b,c) + J_{b^-,c^+}^{\alpha,\beta} f(a,d) \right. \\ &\quad \left. + J_{b^-,d^-}^{\alpha,\beta} f(a,c) \right] \\ &\leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \\ &\times \left(\frac{1}{(\alpha+s)(\beta+s)} + \frac{1}{\alpha+1} B(\beta, s+1) + \frac{1}{\beta+1} \right. \\ &\quad \left. \times B(\alpha, s+1) + B(\beta, s+1) B(\alpha, s+1) \right). \end{aligned} \quad (15)$$

Proof. From (7), with $x = t_1 a + (1-t_1)b$, $y = (1-t_1)a + t_1 b$, $z = t_2 c + (1-t_2)d$, $w = (1-t_2)c + t_2 d$, and $t = \lambda = 1/2$, we get

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{4^s} \left[f(t_1 a + (1-t_1)b, t_2 c + (1-t_2)d) \right. \\ &\quad + f(t_1 a + (1-t_1)b, (1-t_2)c + t_2 d) \\ &\quad + f((1-t_1)a + t_1 b, t_2 c + (1-t_2)d) \\ &\quad \left. + f((1-t_1)a + t_1 b, (1-t_2)c + t_2 d) \right]. \end{aligned} \quad (16)$$

Multiplying both sides of above inequality by $t_1^{\alpha-1}t_2^{\beta-1}$ then integrating the resulting inequality with respect to t_1, t_2 over $[0, 1] \times [0, 1]$, we obtain

$$\begin{aligned} & \frac{1}{\alpha\beta} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq \frac{1}{4^s} \left[\iint_0^1 t_1^{\alpha-1} t_2^{\beta-1} f(t_1 a + (1-t_1)b, t_2 c \right. \\ & \quad \left. + (1-t_2)d) dt_2 dt_1 \right. \\ & \quad + \iint_0^1 t_1^{\alpha-1} t_2^{\beta-1} f(t_1 a + (1-t_1)b, (1-t_2)c \\ & \quad \left. + t_2 d) dt_2 dt_1 \right. \\ & \quad + \iint_0^1 t_1^{\alpha-1} t_2^{\beta-1} f((1-t_1)a + t_1 b, t_2 c \\ & \quad \left. + (1-t_2)d) dt_2 dt_1 \right. \\ & \quad \left. + \iint_0^1 t_1^{\alpha-1} t_2^{\beta-1} f((1-t_1)a + t_1 b, (1-t_2)c \right. \\ & \quad \left. + t_2 d) dt_2 dt_1 \right]. \end{aligned} \quad (17)$$

Using the change of the variable, we get

$$\begin{aligned} & \frac{1}{\alpha\beta} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq \frac{1}{4^s} \left[\int_a^b \int_c^d (b-x)^{\alpha-1} (d-y)^{\beta-1} f(x, y) dy dx \right. \\ & \quad + \int_a^b \int_c^d (b-x)^{\alpha-1} (y-c)^{\beta-1} f(x, y) dy dx \\ & \quad + \int_a^b \int_c^d (x-a)^{\alpha-1} (d-y)^{\beta-1} f(x, y) dy dx \\ & \quad \left. + \int_a^b \int_c^d (x-a)^{\alpha-1} (y-c)^{\beta-1} f(x, y) dy dx \right], \end{aligned} \quad (18)$$

by which the first inequality is proved. For the proof of the second inequality, we note that f is s -convex on coordinates, then

$$\begin{aligned} & f(t_1 a + (1-t_1)b, t_2 c + (1-t_2)d) \\ & \leq t_1^s t_2^s f(a, c) + t_1^s (1-t_2)^s f(a, d) \\ & \quad + (1-t_1)^s t_2^s f(b, c) + (1-t_1)^s (1-t_2)^s f(b, d), \\ & f(t_1 a + (1-t_1)b, (1-t_2)c + t_2 d) \\ & \leq t_1^s (1-t_2)^s f(a, c) + t_1^s t_2^s f(a, d) \\ & \quad + (1-t_1)^s (1-t_2)^s f(b, c) + (1-t_1)^s t_2^s f(b, d), \end{aligned}$$

$$\begin{aligned} & f((1-t_1)a + t_1 b, t_2 c + (1-t_2)d) \\ & \leq (1-t_1)^s t_2^s f(a, c) + (1-t_1)^s (1-t_2)^s f(a, d) \\ & \quad + t_1^s t_2^s f(b, c) + t_1^s (1-t_2)^s f(b, d), \\ & f((1-t_1)a + t_1 b, (1-t_2)c + t_2 d) \\ & \leq (1-t_1)^s (1-t_2)^s f(a, c) + (1-t_1)^s t_2^s f(a, d) \\ & \quad + t_1^s (1-t_2)^s f(b, c) + t_1^s t_2^s f(b, d). \end{aligned} \quad (19)$$

Multiplying both sides of above inequalities by $t_1^{\alpha-1}t_2^{\beta-1}$, then integrating the resulting inequality with respect to t_1, t_2 over $[0, 1] \times [0, 1]$, we obtain

$$\begin{aligned} & \iint_0^1 t_1^{\alpha-1} t_2^{\beta-1} [f(t_1 a + (1-t_1)b, t_2 c + (1-t_2)d) \\ & \quad + f(t_1 a + (1-t_1)b, (1-t_2)c + t_2 d) \\ & \quad + f((1-t_1)a + t_1 b, t_2 c + (1-t_2)d) \\ & \quad + f((1-t_1)a + t_1 b, (1-t_2)c + t_2 d)] dt_2 dt_1 \\ & \leq [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \\ & \quad \times \iint_0^1 t_1^{\alpha-1} t_2^{\beta-1} [t_1^s t_2^s + t_1^s (1-t_2)^s + (1-t_1)^s t_2^s \\ & \quad + (1-t_1)^s (1-t_2)^s] dt_2 dt_1 \\ & = [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \\ & \quad \times \left(\frac{1}{(\alpha+s)(\beta+s)} + \frac{1}{\alpha+1} B(\beta, s+1) \right. \\ & \quad \left. + \frac{1}{\beta+1} B(\alpha, s+1) + B(\beta, s+1) B(\alpha, s+1) \right). \end{aligned} \quad (20)$$

Here, using the change of the variable, we have

$$\begin{aligned} & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta} [J_{a^+, c^+}^{\alpha, \beta} f(b, d) + J_{a^+, d^-}^{\alpha, \beta} f(b, c) \\ & \quad + J_{b^-, c^+}^{\alpha, \beta} f(a, d) + J_{b^-, d^-}^{\alpha, \beta} f(a, c)] \\ & \leq [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \\ & \quad \times \left(\frac{1}{(\alpha+s)(\beta+s)} + \frac{1}{\alpha+1} B(\beta, s+1) \right. \\ & \quad \left. + \frac{1}{\beta+1} B(\alpha, s+1) + B(\beta, s+1) B(\alpha, s+1) \right). \end{aligned} \quad (21)$$

The proof is completed. $\square$

Remark 11. Applying Theorem 10 for $s = 1$, we get Theorem 8. We note that the Beta functions is defined by

$$B(x, y) = \int_0^1 (1-t)^{y-1} t^{x-1} dt, \quad x > 0, \quad y > 0. \quad (22)$$

3. Inequalities for Differentiable Functions

Theorem 12. Let $f : \Delta = [a, b] \times [c, d] \subseteq [0, \infty)^2 \rightarrow [0, \infty)$ be a partial differentiable mapping with $0 \leq a < b, 0 \leq c < d$. If $|\partial^2 f / \partial t_1 \partial t_2|$ is s -convex on the coordinates in the second sense on Δ for some fixed $s \in (0, 1]$. Then one has the inequalities:

$$\begin{aligned} & \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} + \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{4(b-a)^\alpha (d-c)^\beta} \right. \\ & \times \left[J_{a^+, c^+}^{\alpha, \beta} f(b, d) + J_{a^+, d^-}^{\alpha, \beta} f(b, c) \right. \\ & \quad \left. + J_{b^-, c^+}^{\alpha, \beta} f(a, d) + J_{b^-, d^-}^{\alpha, \beta} f(a, c) \right] - A \Bigg| \\ & = \frac{(b-a)(d-c)}{4} \left[\frac{1}{\alpha + s + 1} + B(s+1, \alpha+1) \right] \\ & \times \left[\frac{1}{\beta + s + 1} + B(s+1, \beta+1) \right] \\ & \times \left[\left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(a, c) \right| + \left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(b, c) \right| \right. \\ & \quad \left. + \left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(a, d) \right| + \left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(b, d) \right| \right], \end{aligned} \quad (23)$$

where A is as given in Lemma 9.

Proof. From Lemma 9, we obtain

$$\begin{aligned} & \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right. \\ & + \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{4(b-a)^\alpha (d-c)^\beta} \left[J_{a^+, c^+}^{\alpha, \beta} f(b, d) + J_{a^+, d^-}^{\alpha, \beta} f(b, c) \right. \\ & \quad + J_{b^-, c^+}^{\alpha, \beta} f(a, d) \\ & \quad \left. + J_{b^-, d^-}^{\alpha, \beta} f(a, c) \right] - A \Bigg| \\ & \leq \frac{(b-a)(d-c)}{4} \\ & \times \left\{ \iint_0^1 t_1^\alpha t_2^\beta \left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(t_1 a + (1-t_1)b, t_2 c \right. \right. \\ & \quad \left. \left. + (1-t_2)d) \right| dt_2 dt_1 \right. \end{aligned}$$

$$\begin{aligned} & + \iint_0^1 (1-t_1)^\alpha t_2^\beta \\ & \times \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right. \\ & \quad \left. \times (t_1 a + (1-t_1)b, t_2 c + (1-t_2)d) \right| dt_2 dt_1 \\ & + \iint_0^1 t_1^\alpha (1-t_2)^\beta \\ & \times \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right. \\ & \quad \left. \times (t_1 a + (1-t_1)b, t_2 c + (1-t_2)d) \right| dt_2 dt_1 \\ & + \iint_0^1 (1-t_1)^\alpha (1-t_2)^\beta \\ & \times \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} (t_1 a + (1-t_1)b, t_2 c \right. \\ & \quad \left. + (1-t_2)d) \right| dt_2 dt_1 \Bigg\}. \end{aligned} \quad (24)$$

Because $|\partial^2 f / \partial t_1 \partial t_2|$ is s -convex function on the coordinates on Δ , we obtain

$$\begin{aligned} & \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right. \\ & + \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{4(b-a)^\alpha (d-c)^\beta} \left[J_{a^+, c^+}^{\alpha, \beta} f(b, d) + J_{a^+, d^-}^{\alpha, \beta} f(b, c) \right. \\ & \quad \left. + J_{b^-, c^+}^{\alpha, \beta} f(a, d) + J_{b^-, d^-}^{\alpha, \beta} f(a, c) \right] - A \Bigg| \\ & \leq \frac{(b-a)(d-c)}{4} \\ & \times \iint_0^1 \left[t_1^\alpha t_2^\beta + (1-t_1)^\alpha t_2^\beta + t_1^\alpha (1-t_2)^\beta \right. \\ & \quad \left. + (1-t_1)^\alpha (1-t_2)^\beta \right] \\ & \times \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} (t_1 a + (1-t_1)b, t_2 c + (1-t_2)d) \right| dt_2 dt_1 \\ & \leq \frac{(b-a)(d-c)}{4} \\ & \times \iint_0^1 \left[t_1^\alpha t_2^\beta + (1-t_1)^\alpha t_2^\beta + t_1^\alpha (1-t_2)^\beta \right. \\ & \quad \left. + (1-t_1)^\alpha (1-t_2)^\beta \right] \end{aligned}$$

$$\begin{aligned}
& \times \left\{ t_1^s t_2^s \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} (a, c) \right| + (1-t_1)^s t_2^s \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} (b, c) \right| \right. \\
& \quad \left. + t_1^s (1-t_2)^s \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} (a, d) \right| \right. \\
& \quad \left. + (1-t_1)^s (1-t_2)^s \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} (b, d) \right| \right\} dt_2 dt_1 \\
& = \frac{(b-a)(d-c)}{4} \left[\frac{1}{\alpha+s+1} + B(s+1, \alpha+1) \right] \\
& \quad \times \left[\frac{1}{\beta+s+1} + B(s+1, \beta+1) \right]
\end{aligned}$$

(25)

which completes the proof. $\square$

Theorem 13. Let $f : \Delta = [a, b] \times [c, d] \subseteq [0, \infty)^2 \rightarrow [0, \infty)$ be a partial differentiable mapping with $0 \leq a < b, 0 \leq c < d$. If $|\partial^2 f / \partial u \partial v|^q$ is s -convex on the coordinates in the second sense on Δ for some fixed $s \in (0, 1]$ and $q > 1$. Then, one has the inequalities:

$$\begin{aligned}
& \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \right. \\
& \quad \times \left[J_{a^+, c^+}^{\alpha, \beta} f(b, d) + J_{a^+, d^-}^{\alpha, \beta} f(b, c) + J_{b^-, c^+}^{\alpha, \beta} f(a, d) \right. \\
& \quad \left. \left. + J_{b^-, d^-}^{\alpha, \beta} f(a, c) \right] - A \right| \\
& \leq \frac{(b-a)(d-c)}{[(\alpha p + 1)(\beta p + 1)]^{1/p}} \\
& \quad \times \left(\frac{|\partial^2 f / \partial t_1 \partial t_2 (a, c)|^q + |\partial^2 f / \partial t_1 \partial t_2 (b, c)|^q + |\partial^2 f / \partial t_1 \partial t_2 (a, d)|^q + |\partial^2 f / \partial t_1 \partial t_2 (b, d)|^q}{(s+1)^2} \right)^{1/q},
\end{aligned}$$

(26)

where $(1/p) + (1/q) = 1$ and A are as given in Lemma 9.

Proof. From Lemma 9, we obtain

$$\begin{aligned}
& \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \right. \\
& \quad \times \left[J_{a^+, c^+}^{\alpha, \beta} f(b, d) + J_{a^+, d^-}^{\alpha, \beta} f(b, c) + J_{b^-, c^+}^{\alpha, \beta} f(a, d) \right. \\
& \quad \left. \left. + J_{b^-, d^-}^{\alpha, \beta} f(a, c) \right] - A \right| \\
& \leq \frac{(b-a)(d-c)}{4} \\
& \quad \times \left\{ \int_0^1 \int_0^1 t_1^\alpha t_2^\beta \right. \\
& \quad \times \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} (t_1 a + (1-t_1)b, t_2 c + (1-t_2)d) \right| dt_2 dt_1 \\
& \quad + \int_0^1 \int_0^1 (1-t_1)^\alpha t_2^\beta \\
& \quad \times \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} (t_1 a + (1-t_1)b, t_2 c + (1-t_2)d) \right| dt_2 dt_1
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 t_1^\alpha (1-t_2)^\beta \\
& \quad \times \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} (t_1 a + (1-t_1)b, t_2 c + (1-t_2)d) \right| dt_2 dt_1 \\
& + \int_0^1 (1-t_1)^\alpha (1-t_2)^\beta \\
& \quad \times \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right. \\
& \quad \times \left. (t_1 a + (1-t_1)b, t_2 c + (1-t_2)d) \right| dt_2 dt_1 \Big\}.
\end{aligned}$$

(27)

By using the well-known Hölder's inequality for double integrals, then one has

$$\begin{aligned}
& \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right. \\
& \quad \left. + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \right|
\end{aligned}$$

$$\begin{aligned}
& \times \left[J_{a^+, c^+}^{\alpha, \beta} f(b, d) + J_{a^+, d^-}^{\alpha, \beta} f(b, c) \right. \\
& \quad \left. + J_{b^-, c^+}^{\alpha, \beta} f(a, d) + J_{b^-, d^-}^{\alpha, \beta} f(a, c) \right] - A \Bigg| \\
& \leq \frac{(b-a)(d-c)}{4} \left\{ \left(\int_0^1 \int_0^1 t_1^{p\alpha} t_2^{p\beta} dt_2 dt_1 \right)^{1/p} \right. \\
& \quad \left. + \left(\int_0^1 \int_0^1 (1-t_1)^{p\alpha} t_2^{p\beta} dt_2 dt_1 \right)^{1/p} \right. \\
& \quad \left. + \left(\int_0^1 \int_0^1 t_1^{p\alpha} (1-t_2)^{p\beta} dt_2 dt_1 \right)^{1/p} \right. \\
& \quad \left. + \left(\int_0^1 \int_0^1 (1-t_1)^{p\alpha} (1-t_2)^{p\beta} dt_2 dt_1 \right)^{1/p} \right\} \\
& \quad \times \left(\int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} (t_1 a + (1-t_1)b, t_2 c + (1-t_2)d) \right|^q dt_2 dt_1 \right)^{1/q}.
\end{aligned} \tag{28}$$

Because $|\partial^2 f / \partial t_1 \partial t_2|$ is s -convex function on the coordinates on Δ , by (8), we have

$$\begin{aligned}
& \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} (t_1 a + (1-t_1)b, t_2 c + (1-t_2)d) \right|^q dt_2 dt_1 \\
& \leq \frac{\left| (\partial^2 f / \partial t_1 \partial t_2)(a, c) \right|^q + \left| (\partial^2 f / \partial t_1 \partial t_2)(b, c) \right|^q + \left| (\partial^2 f / \partial t_1 \partial t_2)(a, d) \right|^q + \left| (\partial^2 f / \partial t_1 \partial t_2)(b, d) \right|^q}{(s+1)^2}.
\end{aligned} \tag{29}$$

So

$$\begin{aligned}
& \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \right. \\
& \quad \times \left[J_{a^+, c^+}^{\alpha, \beta} f(b, d) + J_{a^+, d^-}^{\alpha, \beta} f(b, c) + J_{b^-, c^+}^{\alpha, \beta} f(a, d) + J_{b^-, d^-}^{\alpha, \beta} f(a, c) \right] - A \Bigg| \\
& \leq \frac{(b-a)(d-c)}{[(\alpha p + 1)(\beta p + 1)]^{1/p}} \\
& \quad \times \left(\frac{\left| (\partial^2 f / \partial t_1 \partial t_2)(a, c) \right|^q + \left| (\partial^2 f / \partial t_1 \partial t_2)(b, c) \right|^q + \left| (\partial^2 f / \partial t_1 \partial t_2)(a, d) \right|^q + \left| (\partial^2 f / \partial t_1 \partial t_2)(b, d) \right|^q}{(s+1)^2} \right)^{1/q},
\end{aligned} \tag{30}$$

which completes the proof. $\square$

Theorem 14. Let $f : \Delta = [a, b] \times [c, d] \subseteq [0, \infty)^2 \rightarrow [0, \infty)$ be a partial differentiable mapping with $0 \leq a < b, 0 \leq c < d$. If $|\partial^2 f / \partial u \partial v|^q$ is s -concave on the coordinates in the second sense on Δ for some fixed $s \in (0, 1]$ and $q > 1$, then one has the inequalities:

$$\begin{aligned}
& \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \right. \\
& \quad \times \left[J_{a^+, c^+}^{\alpha, \beta} f(b, d) + J_{a^+, d^-}^{\alpha, \beta} f(b, c) \right. \\
& \quad \left. + J_{b^-, c^+}^{\alpha, \beta} f(a, d) + J_{b^-, d^-}^{\alpha, \beta} f(a, c) \right] - A \Bigg| \\
& \leq \frac{(b-a)(d-c)}{[(\alpha p + 1)(\beta p + 1)]^{1/p}} \\
& \quad \times 4^{(s-1)/q} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right|,
\end{aligned} \tag{31}$$

where $(1/p) + (1/q) = 1$ and A is as given in Lemma 9.

Proof. Similarly as in Theorem 13, we obtain

$$\begin{aligned}
& \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right. \\
& \quad \left. + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \right. \\
& \quad \times \left[J_{a^+, c^+}^{\alpha, \beta} f(b, d) + J_{a^+, d^-}^{\alpha, \beta} f(b, c) \right. \\
& \quad \left. + J_{b^-, c^+}^{\alpha, \beta} f(a, d) + J_{b^-, d^-}^{\alpha, \beta} f(a, c) \right] - A \Bigg| \\
& \leq \frac{(b-a)(d-c)}{4} \\
& \quad \times \left\{ \left(\int_0^1 \int_0^1 t_1^{p\alpha} t_2^{p\beta} dt_2 dt_1 \right)^{1/p} \right. \\
& \quad \left. + \left(\int_0^1 \int_0^1 (1-t_1)^{p\alpha} t_2^{p\beta} dt_2 dt_1 \right)^{1/p} \right. \\
& \quad \left. + \left(\int_0^1 \int_0^1 t_1^{p\alpha} (1-t_2)^{p\beta} dt_2 dt_1 \right)^{1/p} \right. \\
& \quad \left. + \left(\int_0^1 \int_0^1 (1-t_1)^{p\alpha} (1-t_2)^{p\beta} dt_2 dt_1 \right)^{1/p} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left(\left(\int_0^1 t_1^{p\alpha} (1-t_2)^{p\beta} dt_2 dt_1 \right)^{1/p} \right. \\
& + \left. \left(\left(\int_0^1 (1-t)^{p\alpha} (1-s)^{p\beta} dt_2 dt_1 \right)^{1/p} \right) \right\} \\
& \times \left(\left(\int_0^1 \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} (t_1 a + (1-t_1)b, t_2 c \right. \right. \right. \\
& \quad \left. \left. \left. + (1-t_2)d \right) \right|^q dt_2 dt_1 \right)^{1/q}.
\end{aligned} \tag{32}$$

Because $|\partial^2 f / \partial t_1 \partial t_2|$ is s -concave function on the coordinates on Δ , by the reversed direction of (8), then one has

$$\begin{aligned}
& \left| \int_0^1 \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} (t_1 a + (1-t_1)b, t_2 c + (1-t_2)d) \right|^q dt_2 dt_1 \right. \\
& \quad \left. \leq 4^{s-1} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right|^q \right.
\end{aligned} \tag{33}$$

Then

$$\begin{aligned}
& \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \right. \\
& \quad \times [J_{a^+, c^+}^{\alpha, \beta} f(b, d) + J_{a^+, d^-}^{\alpha, \beta} f(b, c) \\
& \quad \left. + J_{b^-, c^+}^{\alpha, \beta} f(a, d) + J_{b^-, d^-}^{\alpha, \beta} f(a, c)] - A \right| \\
& \leq \frac{(b-a)(d-c)}{[(\alpha p + 1)(\beta p + 1)]^{1/p}} \\
& \quad \times 4^{(s-1)/q} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right|.
\end{aligned} \tag{34}$$

We get the desired results. $\square$

4. Conclusion

In this paper, we obtain some Hermite-Hadamard type inequalities for coordinated s -convex functions via Riemann-Liouville integrals. An interesting topic is whether we can use the methods in this paper to establish the Hermite-Hadamard inequalities for other kinds of convex functions on the coordinates via Riemann-Liouville integrals.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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