Research Article

Laplacian Spectral Characterization of Some Unicyclic Graphs

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Received 17 June 2014; Accepted 25 August 2014; Published 3 September 2014

Academic Editor: Heping Zhang

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Let $W(n; q, m_1, m_2)$ be the unicyclic graph with n vertices obtained by attaching two paths of lengths m_1 and m_2 at two adjacent vertices of cycle C_q . Let $U(n; q, m_1, m_2, \dots, m_s)$ be the unicyclic graph with n vertices obtained by attaching s paths of lengths $m_1, m_2, \dots, m_s$ at the same vertex of cycle C_q . In this paper, we prove that $W(n; q, m_1, m_2)$ and $U(n; q, m_1, m_2, \dots, m_s)$ are determined by their Laplacian spectra when q is even.

1. Introduction

Let G be a simple, undirected graph with n vertices. Let A be the adjacency matrix of G and let D be the diagonal matrix of vertex degrees of G . The matrices $L = D - A$ and $Q = D + A$ are called the *Laplacian matrix* and *signless Laplacian matrix* of G , respectively. The multiset of eigenvalues of A and L are called the *A-spectrum* and *L-spectrum* of G , respectively. The eigenvalues of A and L are called the *A-eigenvalues* and *L-eigenvalues* of G , respectively. We use $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$ and $\mu_1(G) \geq \mu_2(G) \geq \dots \geq \mu_n(G) = 0$ to denote the *A-eigenvalues* and the *L-eigenvalues* of G , respectively. Two graphs are said to be *L-cospectral* (*A-cospectral*) if they have the same *L-spectrum* (*A-spectrum*). A graph G is said to be *determined by its L-spectrum* (*A-spectrum*) if there is no other nonisomorphic graph *L-cospectral* (*A-cospectral*) with G . Let $\phi_A(G, x)$, $\phi_L(G, x)$, and $\phi_Q(G, x)$ denote the characteristic polynomials of the adjacency matrix, the Laplacian matrix, and the signless Laplacian matrix of G , respectively. As usual, P_n , C_n , and K_n stand for the path, the cycle, and the complete graph with n vertices, respectively. Let $\ell(G)$ denote the line graph of G . A tree is called *starlike* if it has exactly one vertex of degree larger than 2. Let $T_{a,b,c}$ denote the starlike tree with a vertex v of degree 3 such that $T_{a,b,c} - v = P_a \cup P_b \cup P_c$.

For a connected graph G with n vertices, G is called a *unicyclic graph* if G has n edges. Which graphs are determined by their spectrum is a difficult problem in the theory of graph

spectra. Here, we introduce some results on spectral characterizations of unicyclic graphs. Let $U(n; q, m_1, m_2, \dots, m_s)$ be the unicyclic graph with n vertices obtained by attaching s paths of lengths $m_1, m_2, \dots, m_s$ ($m_i \geq 1$) at the same vertex of cycle C_q (see Figure 1). Haemers et al. [1] proved that $U(n; q, m_1)$ is determined by its *A-spectrum* when q is odd, and all $U(n; q, m_1)$ are determined by their *L-spectra*. It is also known that $U(n; q, m_1)$ is determined by its *A-spectrum* when q is even [2]. Liu et al. [3] proved that $U(n; q, m_1, m_2)$ is determined by its *L-spectrum*. It is known that $U(n; q, 1, 1, \dots, 1)$ is determined by its *L-spectrum*, and $U(n; q, 1, 1, \dots, 1)$ is determined by its *A-spectrum* if q is odd (see [4]). Boulet [5] proved that the sun graph is determined by its *L-spectrum*. Shen and Hou [6] gave a class of unicyclic graphs with even girth that are determined by their *L-spectra*.

Let $W(n; q, m_1, m_2)$ be the unicyclic graph with n vertices obtained by attaching two paths of lengths m_1 and m_2 ($m_1, m_2 \geq 1$) at two adjacent vertices of cycle C_q (see Figure 1). In this paper, we prove that $W(n; q, m_1, m_2)$ and $U(n; q, m_1, m_2, \dots, m_s)$ are determined by their *L-spectra* when q is even.

2. Preliminaries

In this section, we give some lemmas which play important roles throughout this paper.

FIGURE 1: Two classes of unicyclic graphs.

Lemma 1 (see [7]). Let G be a graph. For the adjacency matrix and the Laplacian matrix, the following can be obtained from the spectrum:

- (i) the number of vertices,
- (ii) the number of edges.

For the adjacency matrix, the following follows from the spectrum:

- (iii) the number of closed walks of any length.

For the Laplacian matrix, the following follows from the spectrum:

- (iv) the number of components,
- (v) the number of spanning trees.

Lemma 2 (see [8]). For a bipartite graph G , one has $\phi_L(G, x) = \phi_Q(G, x)$.

Lemma 3 (see [8]). Let G be a graph with n vertices and m edges. Then

$$\phi_A(\ell(G), x) = (x+2)^{m-n} \phi_Q(G, x+2). \quad (1)$$

For a graph G with n vertices, let $\phi_L(G, x) = l_0 x^n + l_1 x^{n-1} + \dots + l_n$. Oliveira et al. determined the first four coefficients of $\phi_L(G, x)$ as follows.

Lemma 4 (see [9]). Let G be a graph with n vertices and m edges, and let $d_1, d_2, \dots, d_n$ be the degree sequence of G . Then

$$\begin{aligned} l_0 &= 1, & l_1 &= -2m = -\sum_{i=1}^n d_i, \\ l_2 &= 2m^2 - m - \frac{1}{2} \sum_{i=1}^n d_i^2, \\ l_3 &= \frac{1}{3} \left[-4m^3 + 6m^2 + 3m^2 \sum_{i=1}^n d_i^2 \right. \\ &\quad \left. - \sum_{i=1}^n d_i^3 - 3 \sum_{i=1}^n d_i^2 + 6N_G(C_3) \right], \end{aligned} \quad (2)$$

where $N_G(C_3)$ is the number of triangles in G .

For a graph G , the subdivision graph of G , denoted by $S(G)$, is the graph obtained from G by inserting a new vertex in each edge of G .

Lemma 5 (see [8]). Let G be a graph with n vertices and m edges. Then

$$\phi_A(S(G), x) = x^{m-n} \phi_Q(G, x^2). \quad (3)$$

Lemma 6 (see [8]). Let u be a vertex of G , let $N(u)$ be the set of all vertices adjacent to u , and let $C(u)$ be the set of all cycles containing u . Then

$$\begin{aligned} \phi_A(G, x) &= x \phi_A(G-u, x) - \sum_{v \in N(u)} \phi_A(G-u-v, x) \\ &\quad - 2 \sum_{Z \in C(u)} \phi_A(G-V(Z), x), \end{aligned} \quad (4)$$

where $V(Z)$ is the vertex set of Z .

Lemma 7 (see [10]). Consider $\phi_A(P_n, 2) = n+1$.

Lemma 8 (see [1]). Let G be a graph with n vertices and let v be a vertex of G . Then $\lambda_1(G) \geq \lambda_1(G-v) \geq \lambda_2(G) \geq \lambda_2(G-v) \geq \dots \geq \lambda_{n-1}(G-v) \geq \lambda_n(G)$.

Lemma 9 (see [5]). Let G be a graph with edge set $E(G)$. Then

$$\mu_1(G) \leq \max \{d(u) + d(v) : uv \in E(G)\}, \quad (5)$$

where $d(u)$ stands for the degree of vertex u .

Lemma 10 (see [11]). For a connected graph G with at least two vertices, one has $\mu_1(G) \geq \Delta(G) + 1$, where $\Delta(G)$ denotes the maximum vertex degree of G ; equality holds if and only if $\Delta(G) = n-1$.

Lemma 11 (see [12]). Let G be a connected graph with $n \geq 3$ vertices and let d_2 be the second maximum degree of G . Then $d_2 \leq \mu_2(G)$.

Lemma 12 (see [8]). Let G be a graph with n vertices and let e be an edge of G . Then $\mu_1(G) \geq \mu_1(G-e) \geq \mu_2(G) \geq \mu_2(G-e) \geq \dots \geq \mu_{n-1}(G-e) \geq \mu_n(G) = \mu_n(G-e) = 0$.

For a graph G , let $N_G(M)$ denote the number of subgraphs of G which are isomorphic to graph M .

Lemma 13 (see [13]). Let G be a graph and let $N_G(k)$ be the number of closed walks of length k in G . Then

$$\begin{aligned} N_G(3) &= 6N_G(C_3), \\ N_G(5) &= 30N_G(C_3) + 10N_G(C_5) + 10N_G(U(4; 3, 1)). \end{aligned} \quad (6)$$

3. Main Results

Lemma 14. Let G be a unicyclic graph with n vertices, and G contains an even cycle C_q . Let H be a graph L -cospectral with G . Then the following statements hold.

- (1) H is a unicyclic graph with n vertices, and the girth of H is q .
- (2) The line graphs $\ell(G)$ and $\ell(H)$ are A -cospectral.
- (3) The subdivision graphs $S(G)$ and $S(H)$ are A -cospectral, and $\sqrt{\mu_i(G)} = \lambda_i(S(G))$ ($i = 1, 2, \dots, n$).

Proof. By Lemma 1, H is a unicyclic graph with n vertices, and the girth of H is q . Since q is even, G and H are bipartite. By Lemma 2, one has $\phi_Q(G, x) = \phi_L(G, x) = \phi_L(H, x) = \phi_Q(H, x)$. Lemma 3 implies that line graphs $\ell(G)$ and $\ell(H)$ are A -cospectral. By Lemma 5, subdivision graphs $S(G)$ and $S(H)$ are A -cospectral, and $\sqrt{\mu_i(G)} = \lambda_i(S(G))$ ($i = 1, 2, \dots, n$). $\square$

Theorem 15. The unicyclic graph $G = W(n; q, m_1, m_2)$ is determined by its L -spectrum when q is even.

Proof. Let H be any graph L -cospectral with G . By Lemma 14, we know that H is a unicyclic graph with n vertices, the girth of H is q , and $\ell(G)$ and $\ell(H)$ are A -cospectral. By Lemmas 1 and 13, we have $N_{\ell(H)}(C_3) = N_{\ell(G)}(C_3) = 2$. So the maximum degree of H does not exceed 3. Suppose that there are a_i vertices of degree i ($i = 1, 2, 3$) in H . From Lemma 4, we have

$$\begin{aligned} \sum_{i=1}^3 a_i &= n, \\ \sum_{i=1}^3 i a_i &= 2n, \end{aligned} \quad (7)$$

$$\sum_{i=1}^3 i^2 a_i = 2 \times 3^2 + 4(n-4) + 2 = 4n + 4.$$

Solving the above equations, we get $a_1 = 2$, $a_2 = n-4$, $a_3 = 2$. So H and G have the same degree sequence. Then, one of the following holds.

- (1) H is the unicyclic graph obtained by attaching two paths of lengths l_1 and l_2 at two nonadjacent vertices of cycle C_q .
- (2) $H = W(n; q, l_1, l_2)$; that is, H is the unicyclic graph obtained by attaching two paths of lengths l_1 and l_2 at two adjacent vertices of cycle C_q .
- (3) H is the graph shown in Figure 2.

Next, we discuss each of these three cases listed above.

Case 1 (H is the unicyclic graph obtained by attaching two paths of lengths l_1 and l_2 at two nonadjacent vertices of cycle C_q). Since $\ell(G)$ and $\ell(H)$ are A -cospectral, by Lemma 1, $\ell(G)$ and $\ell(H)$ have the same number of closed walks of any length. It is not difficult to see that $N_{\ell(G)}(C_5) = N_{\ell(H)}(C_5)$. By Lemma 13, we have $N_{\ell(H)}(U(4; 3, 1)) = N_{\ell(G)}(U(4; 3, 1))$.

FIGURE 2: Graph H .

Note that $m_1 + m_2 + q = l_1 + l_2 + q = n$. If $m_1 \geq 2$ or $m_2 \geq 2$, then $N_{\ell(G)}(U(4; 3, 1)) \geq 7$ and $N_{\ell(H)}(U(4; 3, 1)) \leq 6$. If $m_1 = m_2 = 1$, then $N_{\ell(G)}(U(4; 3, 1)) = 6$ and $N_{\ell(H)}(U(4; 3, 1)) = 4$. Hence $N_{\ell(H)}(U(4; 3, 1)) \neq N_{\ell(G)}(U(4; 3, 1))$, a contradiction.

Case 2 (H is the unicyclic graph $W(n; q, l_1, l_2)$). From Lemma 14, we know that the subdivision graphs $S(G)$ and $S(H)$ (shown in Figure 3) are A -cospectral. Let $p_f = \phi_A(P_f, x)$; from Lemmas 6 and 7, we have

$$\begin{aligned} \phi_A(S(G), x) &= x p_{2m_1+2m_2+2q-1} \\ &\quad - (p_{2m_1} p_{2q-2+2m_2} + p_{2m_2} p_{2q-2+2m_1}) \\ &\quad - 2 p_{2m_1} p_{2m_2}, \\ \phi_A(S(G), 2) &= 2(2m_1 + 2m_2 + 2q) \\ &\quad - (2m_1 + 1)(2q + 2m_2 - 1) \\ &\quad - (2m_2 + 1)(2q + 2m_1 - 1) \\ &\quad - 2(2m_1 + 1)(2m_2 + 1) \\ &= -4(m_1 q + m_2 q + 4m_1 m_2), \end{aligned} \quad (8)$$

$$\begin{aligned} \phi_A(S(H), x) &= x p_{2l_1+2l_2+2q-1} \\ &\quad - (p_{2l_1} p_{2q-2+2l_2} + p_{2l_2} p_{2q-2+2l_1}) \\ &\quad - 2 p_{2l_1} p_{2l_2}, \\ \phi_A(S(H), 2) &= 2(2l_1 + 2l_2 + 2q) \\ &\quad - (2l_1 + 1)(2q + 2l_2 - 1) \\ &\quad - (2l_2 + 1)(2q + 2l_1 - 1) \\ &\quad - 2(2l_1 + 1)(2l_2 + 1) \\ &= -4(l_1 q + l_2 q + 4l_1 l_2). \end{aligned}$$

By $\phi_A(S(G), 2) = \phi_A(S(H), 2)$, we get $-4(m_1 q + m_2 q + 4m_1 m_2) = -4(l_1 q + l_2 q + 4l_1 l_2)$. By $m_1 + m_2 + q = l_1 + l_2 + q = n$, we get $m_1 m_2 = l_1 l_2$. Hence, $m_1 = l_1, m_2 = l_2$ or $m_1 = l_2, m_2 = l_1$, G and H are isomorphic.

Case 3 (H is the graph shown in Figure 2). It is well known that the largest L -eigenvalue of a path is less than 4, and the largest L -eigenvalue of an even cycle is 4. Lemma 12 implies that $\mu_2(G) < 4$. Let u and v be the two vertices of degree 3 in H (see Figure 2). If u and v are nonadjacent, there exists an edge e of H such that $H - e = C_q \cup T_{l_1, l_2, n-l_1-l_2-q-1}$. By Lemmas

FIGURE 3: Two subdivision graphs.

FIGURE 4: Two subdivision graphs.

10 and 12, we get $\mu_2(H) \geq 4$, a contradiction to $\mu_2(G) < 4$. So u and v are adjacent.

From Lemma 14, we know that the subdivision graphs $S(G)$ and $S(H)$ (shown in Figure 4) are A -cospectral. Let $p_f = \phi_A(P_f, x)$; from Lemmas 6 and 7, we have

$$\begin{aligned}
 & \phi_A(S(G), x) \\
 &= xp_{2m_1+2m_2+2q-1} \\
 &\quad - (p_{2m_1}p_{2q-2+2m_2} + p_{2m_2}p_{2q-2+2m_1}) \\
 &\quad - 2p_{2m_1}p_{2m_2}, \\
 & \phi_A(S(G), 2) \\
 &= 2(2m_1 + 2m_2 + 2q) \\
 &\quad - (2m_1 + 1)(2q + 2m_2 - 1) \\
 &\quad - (2m_2 + 1)(2q + 2m_1 - 1) \\
 &\quad - 2(2m_1 + 1)(2m_2 + 1) \\
 &= -4(m_1q + m_2q + 4m_1m_2), \\
 & \phi_A(S(H), x) \\
 &= xp_{2q-1}\phi_A(T_{1,2l_1,2l_2}, x) \\
 &\quad - (p_{2q-1}p_{2l_1+2l_2+1} + 2p_{2q-2}\phi_A(T_{1,2l_1,2l_2}, x)) \\
 &\quad - 2\phi_A(T_{1,2l_1,2l_2}, x), \\
 & \phi_A(S(H), 2) \\
 &= 2 \times 2q\phi_A(T_{1,2l_1,2l_2}, 2)
 \end{aligned}$$

$$\begin{aligned}
 & - [2q(2l_1 + 2l_2 + 2) + 2(2q - 1)\phi_A(T_{1,2l_1,2l_2}, 2)] \\
 & - 2\phi_A(T_{1,2l_1,2l_2}, 2) \\
 &= -4q(l_1 + l_2 + 1). \tag{9}
 \end{aligned}$$

Since $\phi_A(S(G), 2) = \phi_A(S(H), 2)$, we have $-4q(l_1 + l_2 + 1) = -4(m_1q + m_2q + 4m_1m_2)$. By $l_1 + l_2 + 1 = m_1 + m_2$, we get $m_1m_2 = 0$, a contradiction to $m_1, m_2 > 0$. $\square$

Here, we describe a classic method to count the number of closed walks of a given length in a graph (see [2, 13, 14]). For a graph G , $N_G(k)$ stands for the number of closed walks of length k in G and $N_G(M)$ stands for the number of subgraphs of G which are isomorphic to graph M . Let $\omega_k(M)$ be the number of closed walks of length k of graph M which contains all edges of M , and $M_k(G)$ denotes the set of all connected subgraphs M of G such that $\omega_k(M) \neq 0$. Then

$$N_G(k) = \sum_{M \in M_k(G)} N_G(M) \omega_k(M). \tag{10}$$

Lemma 16. Let $G = U(n; q, m_1, m_2, \dots, m_s)$ and $G' = U(n; q, l_1, l_2, \dots, l_s)$ be L -cospectral graphs. If q is even, then G and G' are isomorphic.

Proof. If q is even, by Lemma 14, $\ell(G)$ and $\ell(G')$ are A -cospectral. From Lemma 1, we get $N_{\ell(G)}(k) = N_{\ell(G')}(k)$ for any positive integer k . Suppose $m_1 \leq m_2 \leq \dots \leq m_s$, $l_1 \leq l_2 \leq \dots \leq l_s$. Let $r_i = \min\{m_i, l_i\}$ ($i = 1, 2, \dots, s$). If $m_1 \neq l_1$, by $m_1 + m_2 + \dots + m_s = l_1 + l_2 + \dots + l_s$, we know that $M_{2r_1+3}(\ell(G)) = M_{2r_1+3}(\ell(G'))$. For any $M \in M_{2r_1+3}(\ell(G))$ and $M \neq U(3 + r_1; 3, r_1)$, we have $N_{\ell(G)}(M) = N_{\ell(G')}(M)$. Since $N_{\ell(G)}(U(3 + r_1; 3, r_1)) \neq N_{\ell(G')}(U(3 + r_1; 3, r_1))$, by (10), we get $N_{\ell(G)}(2r_1 + 3) \neq N_{\ell(G')}(2r_1 + 3)$, a contradiction. So we have $m_1 = l_1$. Similar to the above arguments, by counting the number of closed walks of length $2r_i + 3$ ($i = 2, 3, \dots, s$),

we can get $m_i = l_i$ ($i = 2, 3, \dots, s$). Hence G and G' are isomorphic. $\square$

Theorem 17. *The unicyclic graph $G = U(n; q, m_1, m_2, \dots, m_s)$ is determined by its L -spectrum when q is even.*

Proof. Let G' be any graph L -cospectral with G . By Lemma 14, G' is a unicyclic graph with n vertices, and the girth of G' is q . Let v be the vertex of degree $s + 2$ in the subdivision graph $S(G) = U(2n; 2q, 2m_1, 2m_2, \dots, 2m_s)$; then $S(G) - v = P_{2q-1} \cup P_{2m_1} \cup P_{2m_2} \cup \dots \cup P_{2m_s}$. Since the largest A -eigenvalue of a path is less than 2, by Lemmas 8 and 14, we get $\sqrt{\mu_2(G)} = \lambda_2(S(G)) < 2$, $\mu_2(G) < 4$. Suppose $d_1 \geq d_2 \geq \dots \geq d_n$ is the degree sequence of G' . By Lemma 11, we have $d_2 \leq 3$. From Lemmas 9 and 10, we get $s + 3 < \mu_1(G) \leq s + 4$, $d_1 + d_2 \geq \mu_1(G) > s + 3$, and $d_1 + 1 < \mu_1(G) \leq s + 4$. By $d_2 \leq 3$, we have $s < d_1 < s + 3$.

If $d_1 = s + 2$, applying Lemma 4, we have

$$\begin{aligned} \sum_{i=2}^n d_i &= \underbrace{2 + 2 + \dots + 2}_{n-s-1} + \underbrace{1 + 1 + \dots + 1}_s, \\ \sum_{i=2}^n d_i^2 &= \underbrace{2^2 + 2^2 + \dots + 2^2}_{n-s-1} + \underbrace{1^2 + 1^2 + \dots + 1^2}_s. \end{aligned} \quad (11)$$

Since $\sum_{i=2}^n d_i^2$ is minimal if and only if $|d_i - d_j| \leq 1$ for any $i, j \in \{2, 3, \dots, n\}$, the degree sequences of G and G' are both $s + 2, \underbrace{2, 2, \dots, 2}_{n-s-1}, \underbrace{1, 1, \dots, 1}_s$. Lemma 16 implies that G and G' are isomorphic.

If $d_1 = s + 1$, by $d_1 + d_2 > s + 3$ and $d_2 < 4$, we get $d_2 = 3$. Suppose that there are a_3 three, a_2 two, and a_1 one in $d_2, d_3, \dots, d_n$. By Lemma 4, we have

$$\begin{aligned} \sum_{i=1}^3 a_i + 1 &= n, \\ \sum_{i=1}^3 i a_i + (s + 1) &= s + 2(n - s - 1) + (s + 2), \\ \sum_{i=1}^3 i^2 a_i + (s + 1)^2 &= s + 4(n - s - 1) + (s + 2)^2. \end{aligned} \quad (12)$$

Solving the above equations, we get $a_1 = 2s - 1$, $a_2 = n - 3s$, $a_3 = s$. From Lemma 4, we have

$$\sum_{i=1}^3 i^3 a_i + (s + 1)^3 = s + 8(n - s - 1) + (s + 2)^3. \quad (13)$$

$s = 0$ or $s = 1$ is the solution of the above equation. Then $d_1 = 1$ or $d_1 = 2$, a contradiction to $d_2 = 3$. $\square$

The join of two graphs G and H , denoted by $G \times H$, is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H . Some results on spectral characterizations of graphs obtained by join operation can be found in [15–20]. For a unicyclic graph G , if G is determined by its L -spectrum and $G \neq C_6$, then $G \times K_r$ is determined by its L -spectrum (cf.

[18, Theorem 4.4]). Hence, we can obtain the following two results from Theorems 15 and 17.

Corollary 18. *Let $G = W(n; q, m_1, m_2)$. Then $G \times K_r$ is determined by its L -spectrum when q is even.*

Corollary 19. *Let $G = U(n; q, m_1, m_2, \dots, m_s)$. Then $G \times K_r$ is determined by its L -spectrum when q is even.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by the National Natural Science Foundation of China under Grant 11371109 and the Fundamental Research Funds for the Central Universities.

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