Research Article

Higher Integrability for Very Weak Solutions of Inhomogeneous A -Harmonic Form Equations

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The higher integrability for very weak solutions of A -harmonic form equations $d^* A(x, u, du) = B(x, u, du)$ has been proved.

1. Introduction

The aim of the present paper is to prove the higher integrability for very weak solutions of A -harmonic form equation

$$d^* A(x, u, du) = B(x, u, du), \quad (1)$$

with the more general growth conditions than (4); that is, we assume that $A : \Omega \times \bigwedge^{l-1}(\mathbf{R}^n) \times \bigwedge^l(\mathbf{R}^n) \rightarrow \bigwedge^l(\mathbf{R}^n)$, $B : \Omega \times \bigwedge^{l-1}(\mathbf{R}^n) \times \bigwedge^l(\mathbf{R}^n) \rightarrow \bigwedge^{l-1}(\mathbf{R}^n)$ satisfy the following conditions on a bounded convex domain Ω :

$$\begin{aligned} |A(x, u, \xi)| &\leq \beta_1 |\xi|^{p-1} + \gamma_1 |u - u_\Omega|^{p-1} + f_1(x), \\ |B(x, u, \xi)| &\leq \beta_2 |\xi|^s + \gamma_2 |u - u_\Omega|^s + f_2(x), \\ \langle A(x, u, \xi), \xi \rangle &\geq \alpha |\xi|^p, \\ A(x, u, \lambda \xi) &= \lambda |\lambda|^{p-2} A(x, u, \xi), \end{aligned} \quad (2)$$

for almost every $x \in \Omega$, all $(l-1)$ -differential forms u , and l -differential forms ξ . Here, $\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2 > 0$ are positive constants, $0 < s < p-1$, and $\max\{1, p-1\} < r < p < n$ is a fixed exponent associated with (1), the nonnegative functions $f_1, f_2 \in L^{t/(p-1)}(\Omega)$ for $t > p$.

Definition 1. A differential form $u \in W_{\text{loc}}^{1,r}(\Omega, \bigwedge^{l-1})$ with $\max\{1, p-1\} < r < p$ is called a very weak solution to (1) if u satisfies

$$\int_{\Omega} \langle A(x, u, du), d\varphi \rangle dx = \int_{\Omega} B(x, u, du) \varphi dx \quad (3)$$

for all $\varphi \in W^{1,r/(r-p+1)}(\Omega, \bigwedge^{l-1})$ with compact support. The special type of (1) is

$$d^* A(x, du) = 0, \quad (4)$$

where $A : \Omega \times \bigwedge^l(\mathbf{R}^n) \rightarrow \bigwedge^l(\mathbf{R}^n)$ satisfies the conditions

$$|A(x, \xi)| \leq \beta |\xi|^{p-1}, \quad \langle A(x, \xi), \xi \rangle \geq \alpha |\xi|^p \quad (5)$$

for almost every $x \in \Omega$ and all $\xi \in \bigwedge^l(\mathbf{R}^n)$. Here, $\alpha, \beta > 0$ are constants and $1 < p < n$ is a fixed exponent associated with (4). $u \in W_{\text{loc}}^{1,p}(\Omega, \bigwedge^{l-1})$ is an A -harmonic tensor in Ω if u satisfies (4) in Ω .

When u is a 0-form, that is, u is a function, (1) is equivalent to

$$\text{div } A(x, u, \nabla u) = B(x, u, \nabla u). \quad (6)$$

Lots of results have been obtained in recent years about different versions of the A -harmonic equation; see [1–13].

In 1994, Iwaniec and Sbordone [3] first introduced weakly A -harmonic mapping. The word *weak* means that the integrable exponent r of u is smaller than the natural exponent p . In 1995, Stroffolini [14] gave the higher integrability result of weakly A -harmonic tensors. In 2010, Gao and Wang [15]

gave an alternative proof of the higher integrability result of weakly A -harmonic tensors by introducing the definition of weak WT_2 -class of differential forms.

In this paper, we continue to consider the higher integrability. To the generalized form of (1), under some general conditions (2) given above on the operator A , we obtain the higher integrability for very weak solutions to (1).

The following is our main results.

Theorem 2. *Let Ω be a bounded convex domain of $\mathbf{R}^n$. There exist exponents $1 < r_1 = r_1(n, p, \beta_1, \beta_2) < p < r_2 = r_2(n, p, \beta_1, \beta_2) < \infty$ such that if $u \in W_{\text{loc}}^{1, r_1}(\Omega, \wedge^{l-1})$ is a very weak solution of (1), then $u \in W_{\text{loc}}^{1, r_2}(\Omega, \wedge^{l-1})$. In particular, $u \in W_{\text{loc}}^{1, p}(\Omega, \wedge^{l-1})$ is a weak solution of (1) in the usual sense.*

Remark 3. To prove theorem, we have to estimate the integral of some power of $|u|$ and $|\varphi|$ by means of $|\nabla u|$ and $|\nabla \varphi|$, respectively. We deal with this difficulty by imbedding inequalities for differential forms. In addition, to reduce the integrable exponent of du , we use Lemma 7.

2. Notion and Lemmas

We keep using the traditional notation.

Let Ω be a bounded convex domain of $\mathbf{R}^n$, let $e_1, e_2, \dots, e_n$ be the standard unit basis of $\mathbf{R}^n$, and let $\wedge^l = \wedge^l(\mathbf{R}^n)$ be the linear space of l -covectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_l}$, corresponding to all ordered l -tuples $I = (i_1, i_2, \dots, i_l)$, $1 \leq i_1 < i_2 < \dots < i_l \leq n$, $l = 0, 1, \dots, n$. Let $\mathbf{R} = \mathbf{R}^1$. The Grassmann algebra $\wedge = \oplus \wedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$, the inner product in $\wedge$ is given by $\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$ with summation over all l -tuples $I = (i_1, i_2, \dots, i_l)$ and all integers $l = 0, 1, \dots, n$. The Hodge star operator $*$: $\wedge \rightarrow \wedge$ is denoted by rules $*1 = e_1 \wedge e_2 \wedge \dots \wedge e_n$ and $\alpha \wedge * \beta = \beta \wedge * \alpha = \langle \alpha, \beta \rangle (*1)$ for all $\alpha, \beta \in \wedge$. The norm of $\alpha \in \wedge$ is given by formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = *(\alpha \wedge * \alpha) \in \wedge^0 = \mathbf{R}$. The Hodge star is an isometric isomorphism on $\wedge$ with $*$: $\wedge^l \rightarrow \wedge^{n-l}$ and $* * (-1)^{l(n-l)} : \wedge^l \rightarrow \wedge^l$. Balls are denoted by B and ρB is the ball with the same center as B and with $\text{diam}(\rho B) = \rho \text{diam}(B)$. We do not distinguish balls from cubes throughout this paper. The n -dimensional Lebesgue measure of a measurable set $E \subseteq \mathbf{R}^n$ is denoted by $|E|$.

Differential forms are important generalizations of real functions and distributions; note that a 0-form is the usual function in $\mathbf{R}^n$. A differential l -form ω on Ω is a Schwartz distribution on Ω with values in $\wedge^l(\mathbf{R}^n)$. We use $D'(\Omega, \wedge^l)$ to denote the space of all differential l -forms $\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \dots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_l}$. We write $L^p(\Omega, \wedge^l)$ for the l -forms with $\omega_I \in L^p(\Omega, \mathbf{R})$ for all ordered l -tuples I . Thus, $L^p(\Omega, \wedge^l)$ is a Banach space with norm

$$\begin{aligned} \|\omega\|_{p, \Omega} &= \left(\int_{\Omega} |\omega(x)|^p dx \right)^{1/p} \\ &= \left(\int_{\Omega} \left(\sum |\omega_I(x)|^2 \right)^{p/2} dx \right)^{1/p}. \end{aligned} \quad (7)$$

For $\omega \in D'(\Omega, \wedge^l)$, the vector-valued differential form $\nabla \omega = (\partial \omega / \partial x_1, \dots, \partial \omega / \partial x_n)$ consists of differential forms $\partial \omega / \partial x_i \in D'(\Omega, \wedge^l)$ where the partial differentiations are applied to the coefficients of ω . As usual, $W^{1, p}(\Omega, \wedge^l)$ is used to denote the Sobolev space of l -forms, which equals $L^p(\Omega, \wedge^l) \cap L^p_1(\Omega, \wedge^l)$ with norm

$$\begin{aligned} \|\omega\|_{W^{1, p}(\Omega, \wedge^l)} &= \|\omega\|_{W^{1, p}(\Omega, \wedge^l)} \\ &= \text{diam}(\Omega)^{-1} \|\omega\|_{p, \Omega} + \|\nabla \omega\|_{p, \Omega}. \end{aligned} \quad (8)$$

The notations $W^{1, p}_{\text{loc}}(\Omega, \mathbf{R})$ and $W^{1, p}_{\text{loc}}(\Omega, \wedge^l)$ are self-explanatory. We denote the exterior derivative by $d : D'(\Omega, \wedge^l) \rightarrow D'(\Omega, \wedge^{l+1})$ for $l = 0, 1, \dots, n$. Its formal adjoint operator $d^* : D'(\Omega, \wedge^{l+1}) \rightarrow D'(\Omega, \wedge^l)$ is given by $d^* = (-1)^{n-l+1} * d *$ on $D'(\Omega, \wedge^{l+1})$, $l = 0, 1, \dots, n$. A differential l -form $u \in D'(\Omega, \wedge^l)$ is called a closed form if $du = 0$ in Ω . It is called exact if there exists a differential form $\alpha \in D'(\Omega, \wedge^{l-1})$ such that $u = d\alpha$. Poincaré lemma implies that exact forms are closed.

From [1, 16], if $D \subset \mathbf{R}^n$ is a bounded convex domain, to each $y \in D$, there corresponds a linear operator $K_y : C^\infty(D, \wedge^l) \rightarrow C^\infty(D, \wedge^{l-1})$ defined by

$$\begin{aligned} (K_y \omega)(x; \xi_1, \dots, \xi_{l-1}) \\ = \int_0^1 t^{l-1} \omega(tx + y - ty; x - y, \xi_1, \dots, \xi_{l-1}) dt \end{aligned} \quad (9)$$

and a decomposition $\omega = d(K_y \omega) + K_y(d\omega)$. A homotopy operator $T : C^\infty(D, \wedge^l) \rightarrow C^\infty(D, \wedge^{l-1})$ is defined by averaging K_y over all points y in D ; that is,

$$T\omega = \int_D \varphi(y) K_y \omega dy, \quad (10)$$

where $\varphi \in C^\infty_0(D)$ is normalized by $\int_D \varphi(y) dy = 1$. Then, there is also a decomposition

$$\omega = d(T\omega) + T(d\omega). \quad (11)$$

The l -form $\omega_D \in D'(D, \wedge^l)$ is defined by

$$\omega_D = \begin{cases} |D|^{-1} \int_D \omega(y) dy & \text{if } l = 0 \\ d(T\omega) & \text{if } l = 1, 2, \dots, n \end{cases} \quad (12)$$

for all $\omega \in L^p(D, \wedge^l)$. Clearly, ω_D is a closed form and, for $l > 0$, ω_D is an exact form.

We need the following lemmas.

Lemma 4 (see [16, 17]). *Let $\omega \in D'(D, \wedge^l)$ be such that $d\omega \in L^p(D, \wedge^{l+1})$; then, $\omega - \omega_D$ is in $W^{1, p}(D, \wedge^l)$, and*

$$\|\omega - \omega_D\|_{p, D} \leq C(n, p) \text{diam}(D) \|d\omega\|_{p, D} \quad (13)$$

holds for a cube or a ball D in $\mathbf{R}^n$, $l = 0, 1, \dots, n$, and $1 < p < \infty$.

Lemma 5 (see [18]). Suppose X and Y are vectors of an inner product space. Then,

$$| |X|^\varepsilon X - |Y|^\varepsilon Y | \leq \frac{1-\varepsilon}{1+\varepsilon} 2^{-\varepsilon} |X-Y|^{1+\varepsilon}, \quad (14)$$

for $-1 < \varepsilon \leq 0$, and

$$| |X|^\varepsilon X - |Y|^\varepsilon Y | \leq (1+\varepsilon) (|Y| + |X-Y|)^\varepsilon |X-Y|, \quad (15)$$

for $\varepsilon \geq 0$.

Lemma 6 (see [14]). Let D be a cube or a ball, and $\omega \in L^s(D, \wedge^l)$ with $d\omega \in L^s(D, \wedge^{l+1})$. Then,

$$\begin{aligned} & \frac{1}{\text{diam } D} \left(\int_D |\omega - \omega_D|^s \right)^{1/s} \\ & \leq C(n, s) \left(\int_D |d\omega|^{ns/(n+s-1)} \right)^{(n+s-1)/ns}. \end{aligned} \quad (16)$$

Here, we denote by $\int_D$ the integral mean over D .

Lemma 7 (see [19], page 122, and Proposition 1.1). Let Q be an n -cube. Suppose

$$\begin{aligned} \int_{Q_R(x_0)} |g|^q dx & \leq b \left(\int_{Q_{2R}(x_0)} |g| dx \right)^q \\ & + \int_{Q_{2R}(x_0)} |f|^q dx + \theta \int_{Q_{2R}(x_0)} |g|^q dx \end{aligned} \quad (17)$$

for each $x_0 \in Q$ and each $R < (1/2) \text{dist}(x_0, \partial Q) = R_0$, where R_0, b, θ are constants with $b > 1, R_0 > 0, 0 \leq \theta < 1$. Then, $g(x) \in L^p_{\text{loc}}(Q)$ for $p \in [q, q + \varepsilon)$ and

$$\begin{aligned} & \left(\int_{Q_R} |g|^p dx \right)^{1/p} \\ & \leq C \left\{ \left(\int_{Q_{2R}} |g|^q dx \right)^{1/q} + \left(\int_{Q_{2R}} |f|^p dx \right)^{1/p} \right\} \end{aligned} \quad (18)$$

for $Q_{2R} \subset Q, R < R_0$, where C and ε are positive constants depending only on b, θ, q, n .

3. Proof of Theorem 2

Let $u \in W^{1,r}_{\text{loc}}(\Omega, \wedge^{l-1})$ be a very weak solution of (1) and let $Q(2R) \subset \Omega$ be a cube. Fix a cutoff function $\eta(x) \in C^\infty_0(Q(2R))$ such that $0 \leq \eta \leq 1, |\nabla \eta| \leq C(n)/R$, and $\eta \equiv 1$ on $Q(R)$. Adopting a usual convention, C will denote a constant whose value may change in any two occurrences, and only the relevant dependences will be specified, as, for example, in $C(n)$.

Step 1. In order to take a suitable test form in the weak solutions of (1), we do a Hodge decomposition [16, 17] to distribution tensors fields $|d(\eta(u - u_{Q(2R)}))|^{r-p} d(\eta(u - u_{Q(2R)})) \in L^{r/(r-p+1)}(Q(2R), \wedge^l)$. With the aid of Hodge decomposition,

$$|d(\eta(u - u_{Q(2R)}))|^{r-p} d(\eta(u - u_{Q(2R)})) = d\varphi + h, \quad (19)$$

where $d\varphi, h \in L^{r/(r-p+1)}(Q(2R), \wedge^l)$, and

$$\|h\|_{r/(r-p+1)} \leq C(n)(p-r) \|d(\eta(u - u_{Q(2R)}))\|_r^{r-p+1}; \quad (20)$$

then, we have

$$\begin{aligned} & \|d\varphi\|_{r/(r-p+1)} \\ & \leq \| |d(\eta(u - u_{Q(2R)}))|^{r-p} d(\eta(u - u_{Q(2R)})) \|_{r/(r-p+1)} \\ & \quad + \|h\|_{r/(r-p+1)} \leq \|d(\eta(u - u_{Q(2R)}))\|_r^{r-p+1} \\ & \quad + C(n)(p-r) \|d(\eta(u - u_{Q(2R)}))\|_r^{r-p+1} \\ & \leq C(n) \|d(\eta(u - u_{Q(2R)}))\|_r^{r-p+1}. \end{aligned} \quad (21)$$

For $d\varphi \in L^{r/(r-p+1)}(Q(2R), \wedge^l)$, it is clear that $\varphi - \varphi_{Q(2R)} \in W^{1,r/(r-p+1)}(Q(2R), \wedge^{l-1})$ by Lemma 4. We can use $\varphi - \varphi_{Q(2R)} \in W^{1,r/(r-p+1)}(Q(2R), \wedge^{l-1})$ as a test form for (1). Let

$$\begin{aligned} E & = |d(\eta(u - u_{Q(2R)}))|^{r-p} d(\eta(u - u_{Q(2R)})) \\ & \quad - |\eta d(u - u_{Q(2R)})|^{r-p} \eta d(u - u_{Q(2R)}). \end{aligned} \quad (22)$$

Combining the above formula with (19), we get

$$\begin{aligned} & d(\varphi - \varphi_{Q(2R)}) \\ & = d\varphi \\ & = |d(\eta(u - u_{Q(2R)}))|^{r-p} d(\eta(u - u_{Q(2R)})) - h \\ & = E - h + |\eta d(u - u_{Q(2R)})|^{r-p} \eta d(u - u_{Q(2R)}). \end{aligned} \quad (23)$$

Then, by Definition 1,

$$\begin{aligned} & \int_{Q(2R)} \langle A(x, u, du), \\ & \quad E - h + |\eta d(u - u_{Q(2R)})|^{r-p} \eta d(u - u_{Q(2R)}) \rangle dx \\ & = \int_{Q(2R)} B(x, u, du) (\varphi - \varphi_{Q(2R)}) dx. \end{aligned} \quad (24)$$

That is,

$$\begin{aligned} & \int_{Q(2R)} \langle A(x, u, du), |\eta d(u - u_{Q(2R)})|^{r-p} \\ & \quad \times \eta d(u - u_{Q(2R)}) \rangle dx \\ & = \int_{Q(2R)} \langle A(x, u, du), h \rangle dx \\ & \quad - \int_{Q(2R)} \langle A(x, u, du), E \rangle dx \\ & \quad + \int_{Q(2R)} B(x, u, du) (\varphi - \varphi_{Q(2R)}) dx \\ & = I_1 + I_2 + I_3. \end{aligned} \quad (25)$$

Step 2. In this part, we are devoted to estimate every integration in (25), respectively. In the following, we will especially be concerned about the coefficient of $\int_{Q(2R)} |du|^r dx$. In our case, r is sufficiently close to p . We can estimate $C(n, p, r, |\Omega|)$ independently of r ; then, we will write constants $C(n, p, r, s, |\Omega|) = C(n, p, s, |\Omega|) = C$.

Noticing that $u_{Q(2R)}$ satisfies $du_{Q(2R)} = 0$, then by condition (2), the left integration in (25) becomes

$$\begin{aligned} & \int_{Q(2R)} \langle A(x, u, du), \\ & \quad |\eta d(u - u_{Q(2R)})|^{r-p} \eta d(u - u_{Q(2R)}) \rangle dx \\ &= \int_{Q(2R)} \langle A(x, u, du), |\eta du|^{r-p} \eta du \rangle dx \\ &= \int_{Q(2R)} |\eta du|^{r-p} \eta \langle A(x, u, du), du \rangle dx \\ &\geq \int_{Q(R)} |du|^{r-p} \langle A(x, u, du), du \rangle dx \\ &\geq \alpha \int_{Q(R)} |du|^r dx. \end{aligned} \quad (26)$$

In the following, we will estimate I_1 , I_2 , and I_3 , respectively.

Estimate of I_1 . By (2), Hölder's inequality, and (20),

$$\begin{aligned} |I_1| &\leq \int_{Q(2R)} |A(x, u, du)| |h| dx \\ &\leq \int_{Q(2R)} (\beta_1 |du|^{p-1} + \gamma_1 |u - u_{Q(2R)}|^{p-1} + f_1(x)) |h| dx \\ &\leq \beta_1 \left(\int_{Q(2R)} |du|^r \right)^{(p-1)/r} \\ &\quad \times \left(\int_{Q(2R)} |h|^{r/(r-p+1)} dx \right)^{(r-p+1)/r} \\ &\quad + \gamma_1 \left(\int_{Q(2R)} |u - u_{Q(2R)}|^r \right)^{(p-1)/r} \\ &\quad \times \left(\int_{Q(2R)} |h|^{r/(r-p+1)} dx \right)^{(r-p+1)/r} \\ &\quad + \left(\int_{Q(2R)} |f_1|^{r/(p-1)} \right)^{(p-1)/r} \\ &\quad \times \left(\int_{Q(2R)} |h|^{r/(r-p+1)} dx \right)^{(r-p+1)/r} \\ &\leq \beta_1 C(n) (p-r) \left(\int_{Q(2R)} |du|^r \right)^{(p-1)/r} \\ &\quad \times \left(\int_{Q(2R)} |d(\eta(u - u_{Q(2R)}))|^r dx \right)^{(r-p+1)/r} \\ &\quad + \gamma_1 C(n) (p-r) \left(\int_{Q(2R)} |u - u_{Q(2R)}|^r \right)^{(p-1)/r} \end{aligned}$$

$$\begin{aligned} & \times \left(\int_{Q(2R)} |d(\eta(u - u_{Q(2R)}))|^r dx \right)^{(r-p+1)/r} \\ &+ C(n) (p-r) \left(\int_{Q(2R)} |f_1|^{r/(p-1)} \right)^{(p-1)/r} \\ & \times \left(\int_{Q(2R)} |d(\eta(u - u_{Q(2R)}))|^r dx \right)^{(r-p+1)/r} \\ &= I_{11} + I_{12} + I_{13}. \end{aligned} \quad (27)$$

Estimate of I_{11} . By Lemma 6 and by noticing that Ω is a bounded convex domain, we have

$$\begin{aligned} & \left(\int_{Q(2R)} |u - u_{Q(2R)}|^r dx \right)^{1/r} \\ & \leq C(n, r) R^{1/r} \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/nr} \\ & \leq C(n, r, |\Omega|) \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/nr}; \end{aligned} \quad (28)$$

then, by

$$\begin{aligned} & \left(\int_{Q(2R)} |d(\eta(u - u_{Q(2R)}))|^r dx \right)^{(r-p+1)/r} \\ &= \left(\int_{Q(2R)} |\eta du + (u - u_{Q(2R)}) d\eta|^r dx \right)^{(r-p+1)/r} \\ &\leq C(p, r) \left(\int_{Q(2R)} |\eta du|^r dx \right)^{(r-p+1)/r} \\ &\quad + C(p, r) \left(\int_{Q(2R)} |(u - u_{Q(2R)}) d\eta|^r dx \right)^{(r-p+1)/r} \\ &\leq C(p, r) \left(\int_{Q(2R)} |du|^r dx \right)^{(r-p+1)/r} \\ &\quad + \frac{C(n, p, r, |\Omega|)}{R^{r-p+1}} \\ &\quad \times \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)(r-p+1)/nr} \end{aligned} \quad (29)$$

with Young's inequality

$$\begin{aligned} ab &\leq \frac{\tau a^p}{p} + \frac{\tau^{-p'/p} b^{p'}}{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \\ &\quad \forall a, b > 0, \quad \tau > 0, \quad p > 1, \end{aligned} \quad (30)$$

we have

$$\begin{aligned}
 |I_{11}| &= \beta_1 C(n) (p-r) \\
 &\quad \times \left(\int_{Q(2R)} |du|^r \right)^{(p-1)/r} \\
 &\quad \times \left(\int_{Q(2R)} |d(\eta(u - u_{Q(2R)}))|^r dx \right)^{(r-p+1)/r} \\
 &\leq \beta_1 C(n, p, r) (p-r) \int_{Q(2R)} |du|^r dx \\
 &\quad + \beta_1 C(n, p, r, |\Omega|) (p-r) \tau \int_{Q(2R)} |du|^r dx \\
 &\quad + \frac{\beta_1 C(n, p, r, |\Omega|) (p-r) \tau^{-(p-1)/(r-p+1)}}{R^r} \\
 &\quad \times \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/n}.
 \end{aligned} \tag{31}$$

Noticing that r is sufficiently close to p , there exists a constant $\varepsilon' = 1/2$ such that $0 < p-r \leq \varepsilon' < 1$. Then, we have $p-1 < (p-1)/(r-p+1) \leq (p-1)/(1-\varepsilon')$, and $\tau^{-(p-1)/(r-p+1)} \leq C$. (31) becomes

$$\begin{aligned}
 |I_{11}| &\leq \beta_1 C(p-r) (1+\tau) \int_{Q(2R)} |du|^r dx \\
 &\quad + \frac{\beta_1 C(p-r)}{R^r} \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/n}.
 \end{aligned} \tag{32}$$

Estimate of I_{12} . By Lemma 4 and noticing that Ω is a bounded convex domain, we have

$$\begin{aligned}
 &\left(\int_{Q(2R)} |u - u_{Q(2R)}|^r dx \right)^{1/r} \\
 &\leq C(n, r) R \left(\int_{Q(2R)} |du|^r dx \right)^{1/r} \\
 &\leq C(n, r, |\Omega|) \left(\int_{Q(2R)} |du|^r dx \right)^{1/r};
 \end{aligned} \tag{33}$$

then, by the above inequality, (29), and Young's inequality,

$$\begin{aligned}
 |I_{12}| &= \gamma_1 C(n) (p-r) \\
 &\quad \times \left(\int_{Q(2R)} |u - u_{Q(2R)}|^r \right)^{(p-1)/r} \\
 &\quad \times \left(\int_{Q(2R)} |d(\eta(u - u_{Q(2R)}))|^r dx \right)^{(r-p+1)/r} \\
 &\leq \gamma_1 C(n, p, r, |\Omega|) (p-r) \left(\int_{Q(2R)} |du|^r dx \right)^{(p-1)/r} \\
 &\quad \times \left(\int_{Q(2R)} |d(\eta(u - u_{Q(2R)}))|^r dx \right)^{(r-p+1)/r} \\
 &\leq \gamma_1 C(n, p, r, |\Omega|) (p-r) \int_{Q(2R)} |du|^r dx \\
 &\quad + \gamma_1 C(n, p, r, |\Omega|) (p-r) \tau \int_{Q(2R)} |du|^r dx \\
 &\quad + \frac{\gamma_1 C(n, p, r, |\Omega|) (p-r) \tau^{-(p-1)/(r-p+1)}}{R^r} \\
 &\quad \times \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/n} \\
 &\leq \gamma_1 C(p-r) (1+\tau) \int_{Q(2R)} |du|^r dx + \frac{\gamma_1 C(p-r)}{R^r} \\
 &\quad \times \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/n}.
 \end{aligned} \tag{34}$$

Estimate of I_{13} . By (31) and Young's inequality, we have

$$\begin{aligned}
 |I_{13}| &= C(n) (p-r) \\
 &\quad \times \left(\int_{Q(2R)} |f_1|^{r/(p-1)} dx \right)^{(p-1)/r} \\
 &\quad \times \left(\int_{Q(2R)} |d(\eta(u - u_{Q(2R)}))|^r dx \right)^{(r-p+1)/r} \\
 &\leq 2C(n, p, r, |\Omega|) (p-r) \tau \int_{Q(2R)} |f_1|^{r/(p-1)} dx \\
 &\quad + C(n, p, r, |\Omega|) (p-r) \tau^{-(p-1)/(r-p+1)} \int_{Q(2R)} |du|^r dx \\
 &\quad + \frac{C(n, p, r, |\Omega|) (p-r) \tau^{-(p-1)/(r-p+1)}}{R^r} \\
 &\quad \times \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/n}
 \end{aligned}$$

$$\begin{aligned}
&\leq 2C(p-r)\tau \int_{Q(2R)} |f_1|^{r/(p-1)} dx + C(p-r) \\
&\quad \times \int_{Q(2R)} |du|^r dx \\
&\quad + \frac{C(p-r)}{R^r} \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/n}.
\end{aligned} \tag{35}$$

Combining (27), (31), (34), and (35) yields

$$\begin{aligned}
|I_1| &\leq C(p-r)(\beta_1 + \beta_1\tau + \gamma_1 + \gamma_1\tau + 1) \\
&\quad \times \int_{Q(2R)} |du|^r dx \\
&\quad + \frac{C(p-r)}{R^r} (\beta_1 + \gamma_1 + 1) \\
&\quad \times \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/n} \\
&\quad + 2C(p-r)\tau \left(\int_{Q(2R)} |f_1|^{r/(p-1)} dx \right).
\end{aligned} \tag{36}$$

Estimate of I_2 . Consider (22), and let

$$\begin{aligned}
X &= d(\eta(u - u_{Q(2R)})), \\
Y &= \eta d(u - u_{Q(2R)}), \quad \varepsilon = r - p
\end{aligned} \tag{37}$$

in Lemma 5; then, by Lemma 5, we have

$$\begin{aligned}
|E| &\leq 2^{p-r} \frac{p-r+1}{r-p+1} |(u - u_{Q(2R)}) d\eta|^{r-p+1} \\
&\leq C |(u - u_{Q(2R)}) d\eta|^{r-p+1}.
\end{aligned} \tag{38}$$

By (2), (38), Hölder's inequality, and Young's inequality,

$$\begin{aligned}
|I_2| &\leq \int_{Q(2R)} |A(x, u, du)| |E| dx \\
&\leq C \int_{Q(2R)} (\beta_1 |du|^{p-1} + \gamma_1 |u - u_{Q(2R)}|^{p-1} + f_1(x)) \\
&\quad \times |(u - u_{Q(2R)}) d\eta|^{r-p+1} dx \\
&\leq \beta_1 C \left(\int_{Q(2R)} |du|^r dx \right)^{(p-1)/r} \\
&\quad \times \left(\int_{Q(2R)} |(u - u_{Q(2R)}) d\eta|^r dx \right)^{(r-p+1)/r} \\
&\quad + \gamma_1 C \left(\int_{Q(2R)} |u - u_{Q(2R)}|^r dx \right)^{(p-1)/r} \\
&\quad \times \left(\int_{Q(2R)} |(u - u_{Q(2R)}) d\eta|^r dx \right)^{(r-p+1)/r}
\end{aligned}$$

$$\begin{aligned}
&+ C \left(\int_{Q(2R)} |f_1|^{r/(p-1)} dx \right)^{(p-1)/r} \\
&\quad \times \left(\int_{Q(2R)} |(u - u_{Q(2R)}) d\eta|^r dx \right)^{(r-p+1)/r} \\
&\leq \beta_1 C \tau \int_{Q(2R)} |du|^r dx + \frac{\beta_1 C \tau^{-(p-1)/(r-p+1)}}{R^r} \\
&\quad \times \int_{Q(2R)} |u - u_{Q(2R)}|^r dx \\
&\quad + \gamma_1 C \tau \int_{Q(2R)} |u - u_{Q(2R)}|^r dx \\
&\quad + \frac{\gamma_1 C \tau^{-(p-1)/(r-p+1)}}{R^r} \int_{Q(2R)} |u - u_{Q(2R)}|^r dx \\
&\quad + C \tau \int_{Q(2R)} |f_1|^{r/(p-1)} dx \\
&\quad + \frac{C \tau^{-(p-1)/(r-p+1)}}{R^r} \int_{Q(2R)} |u - u_{Q(2R)}|^r dx.
\end{aligned} \tag{39}$$

Combined with (28), the above inequality becomes

$$\begin{aligned}
|I_2| &\leq \beta_1 C \tau \int_{Q(2R)} |du|^r dx \\
&\quad + C \tau \int_{Q(2R)} |f_1|^{r/(p-1)} dx \\
&\quad + \frac{C}{R^r} (\beta_1 + \gamma_1\tau + \gamma_1 + 1) \\
&\quad \times \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/n}.
\end{aligned} \tag{40}$$

Estimate of I_3 . By (2),

$$\begin{aligned}
|I_3| &\leq \int_{Q(2R)} |B(x, u, du)| |\varphi - \varphi_{Q(2R)}| dx \\
&\leq \int_{Q(2R)} (\beta_2 |du|^s + \gamma_2 |u - u_{Q(2R)}|^s + f_2(x)) \\
&\quad \times |\varphi - \varphi_{Q(2R)}| dx \\
&= \beta_2 \int_{Q(2R)} |du|^s |\varphi - \varphi_{Q(2R)}| dx \\
&\quad + \gamma_2 \int_{Q(2R)} |u - u_{Q(2R)}|^s |\varphi - \varphi_{Q(2R)}| dx \\
&\quad + \int_{Q(2R)} f_2(x) |\varphi - \varphi_{Q(2R)}| dx \\
&= I_{31} + I_{32} + I_{33}.
\end{aligned} \tag{41}$$

Estimate of I_{31} . By Hölder's inequality, Lemma 4, and Young's inequality with $sr/(p-1) < r$, it yields

$$\begin{aligned}
 |I_{31}| &\leq \beta_2 \left(\int_{Q(2R)} |du|^{sr/(p-1)} dx \right)^{(p-1)/r} \\
 &\quad \times \left(\int_{Q(2R)} |\varphi - \varphi_{Q(2R)}|^{r/(r-p+1)} dx \right)^{(r-p+1)/r} \\
 &\leq \beta_2 C(n, p, r) \text{diam}(Q(2R)) \\
 &\quad \times \left(\int_{Q(2R)} |du|^{sr/(p-1)} dx \right)^{(p-1)/r} \\
 &\quad \times \left(\int_{Q(2R)} |d\varphi|^{r/(r-p+1)} dx \right)^{(r-p+1)/r} \\
 &\leq \beta_2 C(n, p, r, |\Omega|) \tau_1 \int_{Q(2R)} |d\varphi|^{r/(r-p+1)} dx \\
 &\quad + \beta_2 C(n, p, r, |\Omega|) \tau_1^{-(r-p+1)/(p-1)} \int_{Q(2R)} |du|^{sr/(p-1)} dx \\
 &\leq \beta_2 C(n, p, r, |\Omega|) \tau_1 \int_{Q(2R)} |d\varphi|^{r/(r-p+1)} dx \\
 &\quad + \beta_2 C(n, p, r, |\Omega|) \tau_1^{-(r-p+1)/(p-1)} \tau_2 \int_{Q(2R)} |du|^r dx \\
 &\quad + \beta_2 C(n, p, r, |\Omega|) \tau_1^{-(p-1)/(r-p+1)} \tau_2^{-s/(p-1-s)} \\
 &\leq \beta_2 C \tau_1 \int_{Q(2R)} |d\varphi|^{r/(r-p+1)} dx \\
 &\quad + \beta_2 C \tau_1^{-(r-p+1)/(p-1)} \tau_2 \int_{Q(2R)} |du|^r dx \\
 &\quad + \beta_2 C(n, p, r, |\Omega|) \tau_1^{-(p-1)/(r-p+1)} \tau_2^{-s/(p-1-s)} \\
 &\leq \beta_2 C \tau_1 \int_{Q(2R)} |d\varphi|^{r/(r-p+1)} dx \\
 &\quad + \beta_2 C \tau_2 \int_{Q(2R)} |du|^r dx + \beta_2 C.
 \end{aligned} \tag{42}$$

By (21) and (28),

$$\begin{aligned}
 &\|d\varphi\|_{r/(r-p+1)}^{r/(r-p+1)} \\
 &\leq C(n) \|d(\eta(u - u_{Q(2R)}))\|_r^r \\
 &\leq C(n) \|\eta du + (u - u_{Q(2R)}) d\eta\|_r^r \\
 &\leq C(n) (\|\eta du\|_r^r + \|(u - u_{Q(2R)}) d\eta\|_r^r) \\
 &\leq C(n, p, r) \left(\|du\|_r^r + \frac{C(n)}{R^r} \|u - u_{Q(2R)}\|_r^r \right) \\
 &\leq C \|du\|_r^r + \frac{C}{R^r} \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/n}.
 \end{aligned} \tag{43}$$

Then, combining (42) and (43),

$$\begin{aligned}
 |I_{31}| &\leq \beta_2 C(\tau_1 + \tau_2) \int_{Q(2R)} |du|^r dx \\
 &\quad + \frac{\beta_2 C \tau_1}{R^r} \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/n} + \beta_2 C.
 \end{aligned} \tag{44}$$

Estimate of I_{32} . Similarly, by Hölder's inequality, Lemma 4, Young's inequality, (43), and (33), it yields

$$\begin{aligned}
 |I_{32}| &\leq \gamma_2 \left(\int_{Q(2R)} |u - u_{Q(2R)}|^{sr/(p-1)} dx \right)^{(p-1)/r} \\
 &\quad \times \left(\int_{Q(2R)} |\varphi - \varphi_{Q(2R)}|^{r/(r-p+1)} dx \right)^{(r-p+1)/r} \\
 &\leq \gamma_2 C(n, p, r) \text{diam}(Q(2R)) \\
 &\quad \times \left(\int_{Q(2R)} |u - u_{Q(2R)}|^{sr/(p-1)} dx \right)^{(p-1)/r} \\
 &\quad \times \left(\int_{Q(2R)} |d\varphi|^{r/(r-p+1)} dx \right)^{(r-p+1)/r} \\
 &\leq \gamma_2 C(n, p, r, |\Omega|) \tau_1 \int_{Q(2R)} |d\varphi|^{r/(r-p+1)} dx \\
 &\quad + \gamma_2 C(n, p, r, |\Omega|) \tau_1^{-(r-p+1)/(p-1)} \\
 &\quad \times \int_{Q(2R)} |u - u_{Q(2R)}|^{sr/(p-1)} dx \\
 &\leq \gamma_2 C(n, p, r, |\Omega|) \tau_1 \int_{Q(2R)} |d\varphi|^{r/(r-p+1)} dx \\
 &\quad + \gamma_2 C(n, p, r, |\Omega|) \tau_1^{-(r-p+1)/(p-1)} \tau_2 \\
 &\quad \times \int_{Q(2R)} |u - u_{Q(2R)}|^r dx \\
 &\quad + \gamma_2 C(n, p, r, |\Omega|) \tau_1^{-(r-p+1)/(p-1)} \tau_2^{-s/(p-1-s)} \\
 &\leq \gamma_2 C(n, p, r, |\Omega|) (\tau_1 + \tau_1^{-(r-p+1)/(p-1)} \tau_2) \\
 &\quad \times \int_{Q(2R)} |du|^r dx \\
 &\quad + \frac{\gamma_2 C(n, p, r, |\Omega|) \tau_1}{R^r} \\
 &\quad \times \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/n} \\
 &\quad + \gamma_2 C(n, p, r, |\Omega|) \tau_1^{-(r-p+1)/(p-1)} \tau_2^{-s/(p-1-s)}
 \end{aligned}$$

$$\begin{aligned}
&\leq \gamma_2 C (\tau_1 + \tau_2) \int_{Q(2R)} |du|^r dx \\
&\quad + \frac{\gamma_2 C \tau_1}{R^r} \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/n} + \gamma_2 C.
\end{aligned} \tag{45}$$

Estimate of I_{33} . Consider

$$\begin{aligned}
|I_{33}| &= \int_{Q(2R)} f_2(x) |\varphi - \varphi_{Q(2R)}| dx \\
&\leq \left(\int_{Q(2R)} f_2^{r/(p-1)} dx \right)^{(p-1)/r} \\
&\quad \times \left(\int_{Q(2R)} |\varphi - \varphi_{Q(2R)}|^{r/(r-p+1)} dx \right)^{(r-p+1)/r} \\
&\leq C(n, r) \text{diam}(Q(2R)) \left(\int_{Q(2R)} f_2^{r/(p-1)} dx \right)^{(p-1)/r} \\
&\quad \times \left(\int_{Q(2R)} |d\varphi|^{r/(r-p+1)} dx \right)^{(r-p+1)/r} \\
&\leq C(n, r, |\Omega|) \left(\int_{Q(2R)} f_2^{r/(p-1)} dx \right)^{(p-1)/r} \\
&\quad \times \left(\int_{Q(2R)} |d(\eta(u - u_{Q(2R)}))|^r dx \right)^{(r-p+1)/r} \\
&\leq C(n, p, r, |\Omega|) \left(\int_{Q(2R)} f_2^{r/(p-1)} dx \right)^{(p-1)/r} \\
&\quad \times \left(\int_{Q(2R)} |du|^r dx \right)^{(r-p+1)/r} \\
&\quad + \frac{C(n, p, r, |\Omega|)}{R^{r-p+1}} \left(\int_{Q(2R)} f_2^{r/(p-1)} dx \right)^{(p-1)/r} \\
&\quad \times \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)(r-p+1)/nr} \\
&\leq C(n, p, r, |\Omega|) \tau \int_{Q(2R)} |du|^r dx \\
&\quad + \frac{C(n, p, r, |\Omega|) \tau}{R^r} \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/n} \\
&\quad + C(n, p, r, |\Omega|) 2\tau^{-(r-p+1)/(p-1)} \int_{Q(2R)} |f_2|^{r/(p-1)} dx \\
&\leq C\tau \int_{Q(2R)} |du|^r dx \\
&\quad + \frac{C\tau}{R^r} \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/n} \\
&\quad + C \int_{Q(2R)} |f_2|^{r/(p-1)} dx.
\end{aligned} \tag{46}$$

Then, combining (41), (44), (45), and (46),

$$\begin{aligned}
|I_3| &\leq C(\beta_2 \tau_1 + \beta_2 \tau_2 + \gamma_2 \tau_1 + \gamma_2 \tau_2 + \tau) \\
&\quad \times \int_{Q(2R)} |du|^r dx \\
&\quad + \frac{C(\beta_2 \tau_1 + \gamma_2 \tau_1 + \tau)}{R^r} \\
&\quad \times \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/n} \\
&\quad + C \int_{Q(2R)} |f_2|^{r/(p-1)} dx + C(\beta_2 + \gamma_2).
\end{aligned} \tag{47}$$

Combining with (27), (36), (40), and (47), we get

$$\begin{aligned}
&\alpha \int_{Q(R)} |du|^r dx \\
&\leq C((p-r)(\beta_1 + \beta_1 \tau + \gamma_1 + \gamma_1 \tau + 1) \\
&\quad + (\beta_1 \tau + \beta_2 \tau_1 + \beta_2 \tau_2 + \gamma_2 \tau_1 + \gamma_2 \tau_2 + \tau)) \\
&\quad \times \int_{Q(2R)} |du|^r dx \\
&\quad + \frac{C}{R^r} ((p-r)(\beta_1 + \gamma_1 + 1) + (\beta_1 + \gamma_1 \tau + \gamma_1 + 1) \\
&\quad + (\beta_2 \tau_1 + \gamma_2 \tau_1 + \tau)) \\
&\quad \times \left(\int_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/n} \\
&\quad + (2C(p-r)\tau + C\tau) \\
&\quad \times \int_{Q(2R)} |f_1|^{r/(p-1)} dx \\
&\quad + C \int_{Q(2R)} |f_2|^{r/(p-1)} dx + C(\beta_2 + \gamma_2).
\end{aligned} \tag{48}$$

Step 3. A higher integrability is obtained by a weak reverse Hölder inequality. Now, we are in a position to take r sufficiently close to p , such that $(p-r)(\beta_1 + \beta_1 \tau + \gamma_1 + \gamma_1 \tau + 1) < \min\{\alpha/8C, 1/2\}$, and take τ , τ_1 , and τ_2 small enough such that $\beta_1 \tau + \beta_2 \tau_1 + \beta_2 \tau_2 + \gamma_2 \tau_1 + \gamma_2 \tau_2 + \tau < \alpha/8C$. Then, the summation of the coefficients of the first term in the right-hand side of (48) is smaller than α . This implies

$$\begin{aligned}
\oint_{Q(R)} |du|^r dx &\leq \theta \oint_{Q(2R)} |du|^r dx \\
&\quad + C \left(\oint_{Q(2R)} |du|^{nr/(n+r-1)} dx \right)^{(n+r-1)/n} \\
&\quad + C \oint_{Q(2R)} (|f_1|^{r/(p-1)} + |f_2|^{r/(p-1)} + 1) dx.
\end{aligned} \tag{49}$$

Setting $g = |du|^{nr/(n+r-1)}$, $q = (n+r-1)/n$, and $f = C(|f_1|^{r/(p-1)} + |f_2|^{r/(p-1)} + 1)^{n/(n+r-1)}$, we obtain from Lemma 7 that

$$\begin{aligned} & \left(\int_{Q(R)} |du|^{r'} dx \right)^{1/r'} \\ & \leq C \left(\int_{Q(2R)} |du|^r dx \right)^{1/r} \\ & \quad + C \left(\int_{Q(2R)} \left(|f_1|^{r/(p-1)} + |f_2|^{r/(p-1)} + 1 \right)^{r'/r} dx \right)^{1/r'} \end{aligned} \quad (50)$$

for some $r' = r'(p, s, n, \alpha/\beta) > r$. The above inequality implies that du satisfies a weak reverse Hölder inequality. The integrability exponent of du has improved from $r = p - \varepsilon_0$ to $r' = p - \varepsilon_0 + \varepsilon_1$.

We are now in a position to repeatedly use Lemma 7 to improve the degree of integrability of du that allows us to increase the exponent r' even further. This idea can trace from a series of works of Iwaniec and his coworkers [3, 18, 20]. Reasoning as before $u \in W_{\text{loc}}^{1, r''}(\Omega, \wedge^{l-1})$ with some $r'' > r'$ and the reverse Hölder inequality (50) remains valid with the exponent r'' in place of r' . We improve the degree of integrability of ∇u again and again. It is clear that one can reach any number $q \in (r_1, r_2)$ to conclude with $u \in W_{\text{loc}}^{1, q}(\Omega, \wedge^{l-1})$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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