Some Generalized Gronwall-Bellman Type Impulsive Integral Inequalities and Their Applications

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This paper investigates some generalized Gronwall-Bellman type impulsive integral inequalities containing integration on infinite intervals. Some new results are obtained, which generalize some existing conclusions. Our result is also applied to study a boundary value problem of differential equations with impulsive terms.

1. Introduction

It is well known that Gronwall-Bellman type integral inequalities involving functions of one and more than one independent variables play important roles in the study of existence, uniqueness, boundedness, stability, invariant manifolds, and other qualitative properties of solutions of the theory of differential and integral equations. A lot of contributions to its generalization have been archived by many researchers (see [1–14]). Pachpatte [15] especially studied the following inequality:

\[
    u(x) \leq a(x) + \int_{x}^{\infty} b(s) u(s) \, ds \tag{1}
\]

containing integration on infinite integral and used it in the study of terminal value problems for Gronwall-Bellman type differential equations. Then, Cheung and Ma [16] generalized it into two independent variables with a nonlinear term:

\[
    u(x, y) \leq a(x, y) + c(x, y) \int_{x}^{\infty} \int_{y}^{\infty} d(s, t) \omega(u(s, t)) \, ds \, dt. \tag{2}
\]

Along the development of the theory of impulsive differential systems, more and more attention is paid to generalizations of Gronwall-Bellman’s results for discontinuous functions (that is, impulsive integral inequalities) and their applications (see [17–25]). Among them, one of the important things is that Samoilenko and Perestyuk [17] considered

\[
    u(x) \leq c + \int_{x_{0}}^{x} f(s) u(s) \, ds + \sum_{x_{0} < x_{j} < x} \beta_{j} u(x_{j} - 0) \tag{3}
\]

about the nonnegative piecewise continuous function \( u(x) \) where \( c, \beta_{j} \) are nonnegative constants, \( f(x) \) is a positive function, and \( x_{j} \) are the first kind discontinuity points of the function \( u(x) \). Then Borysenko [18] investigated integral inequalities with two independent variables:

\[
    u(x, y) \leq a(x, y) + \int_{x_{0}}^{x} \int_{y_{0}}^{y} \tau(s, t) u(s, t) \, ds \, dt + \sum_{(x_{0}, y_{0}) < (x, y_{0}) < (x, y) \leq (x_{i}, y_{i})} \beta_{i} u(x_{i} - 0, y_{i} - 0). \tag{4}
\]

Here \( u(x, y) \) is an unknown nonnegative continuous function with the exception of the points \( (x_{i}, y_{i}) \) where there is a finite jump \( u(x_{i} - 0, y_{i} - 0) \neq u(x_{i} + 0, y_{i} + 0) \) for \( i = 1, 2, \ldots \).
In 2013, Zheng [25] considered the following delay integral inequalities containing integration on infinite intervals:

\[
\begin{align*}
    u(x) &\leq c + \int_x^\infty f_1(x, s) u(\tau(s)) \, ds \\
    &+ \int_x^\infty f_2(x, s) \omega(u(\tau(s))) \, ds \\
    &+ \sum_{x<\xi, \xi<\infty} \beta_i u(x_i - 0),
\end{align*}
\]

\[
(5)
\]

\[
(\omega(0) = 0)
\]

In what follows, our main aim is to avoid such conditions. Our main aim here, motivated by the work above, is to discuss the following much more general integral inequality:

\[
\begin{align*}
    u(x) &\leq a(x) + \sum_{k=1}^m \int_x^\infty f_k(x, s) \omega_k(u(\sigma_k(s))) \, ds \\
    &+ \sum_{x<\xi, \xi<\infty} \beta_i u^m(x_i - 0), \quad m > 0,
\end{align*}
\]

\[
(7)
\]

\[
(\omega(0) = 0)
\]

Consider (7) and assume that

\[
\begin{align*}
    &H_1 \quad f_k(x, s) \quad (k = 1, 2) \quad \text{is a continuous and nonnegative function for } x, s \in \mathbb{R}_+ \quad \text{and is bounded in } x \in \mathbb{R}_+ \quad \text{for each fixed } s \in \mathbb{R}_+; \\
    &H_2 \quad \omega_1(u) \quad \text{and} \quad \omega_2(u) \quad \text{are continuous and nonnegative functions on } [0, \infty) \quad \text{and positive on } (0, \infty) \quad \text{such that} \quad \omega_-(u)/\omega_+(u) \quad \text{is nondecreasing}; \\
    &H_3 \quad u(x) \quad \text{is a nonnegative and continuous function defined on } \mathbb{R}_+ \quad \text{with the first kind of discontinuities at the points } x_i \quad \text{where } i = 1, 2, \ldots, n \quad \text{and } 0 < x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = \infty; \\
    &H_4 \quad a(x) \quad \text{is a continuous and bounded function for } x \in \mathbb{R}_+ \quad \text{and } a(\infty) \neq 0; \beta_i \quad \text{is a nonnegative constant for any positive integer } i;
\end{align*}
\]

Let \( W_j(u) = \int_{\bar{u}_j}^u (dz/\omega(z)) \) for \( u \geq \bar{u}_j \) and \( j = 1, 2 \) where \( \bar{u}_j \) is a given positive constant. Clearly, \( W_j \) is strictly increasing so its inverse \( W_j^{-1} \) is well defined, continuous, and increasing in its corresponding domain.

**Theorem 1.** Suppose that \( (H_1)-(H_3) \) hold and \( u(x) \) satisfies (7) for a positive constant \( m \). If one lets \( u_{j-1}(x) = u(x) \) for \( x \in [x_{j-1}, x_j] \), \( i = 1, 2, \ldots, n + 1 \), then the estimate of \( u(x) \) is recursively given by

\[
\begin{align*}
    u_{j-1}(x) &\leq W_j^{-1} \left[ W_j \circ W_{i-1}^{-1} \left( W_i \circ (r_{j-1}(x)) + \int_{x_{j-1}}^{x_j} f_1(x, s) \, ds \right) \right] \\
    &+ \int_{x_{j-1}}^{x_j} f_2(x, s) \, ds
\end{align*}
\]

\[
(9)
\]

for \( x \in [x_{j-1}, x_j] \) and \( i = 1, 2, \ldots, n + 1 \), where

\[
\begin{align*}
    &r_n(x) = \sup_{x \leq \tau < \infty} |a(\tau)|, \\
    &\bar{f}_k(x, s) = \sup_{x \leq \tau \leq \infty} f_k(\tau, s), \quad k = 1, 2, \\
    &r_{i-1}(x) = r_n(x) + \frac{2}{\beta_i} \frac{2}{m} \int_{x_{j-1}}^{x_j} f_k(x, s) \omega_k(u_j(\sigma(s))) \, ds \\
    &+ \sum_{j=k+1}^{\infty} \frac{2}{\beta_i} \frac{2}{m} \int_{x_{j-1}}^{x_j} f_k(x, s) \omega_k(u_j(\sigma(s))) \, ds
\end{align*}
\]

\[
(10)
\]

2. **Main Results**

In what follows, \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}_+ = [0, \infty) \), and \( D_1z(x, y) \) denotes the first-order partial derivative of \( z(x, y) \) with respect to \( x \).
provided that
\[
W_1 (r_{i-1} (x)) + \int_x^{x_i} f_1 (x, s) \, ds \leq \int_{\delta_1}^{\infty} \frac{dz}{\omega_1 (z)},
\]
\[
W_2 \circ W_1^{-1} \left( W_1 (r_{i-1} (x)) + \int_x^{x_i} f_1 (x, s) \, ds \right) + \int_x^{x_i} f_2 (x, s) \, ds \leq \int_{\delta_1}^{\infty} \frac{dz}{\omega_2 (z)}.
\]

Proof. From the assumptions, we know that \( r_n (x) \) and \( f_k (x, s) \) \( (k = 1, 2) \) are well defined. Moreover, \( r_n (x) \) is nonnegative and nonincreasing in \( x \) and \( f_k (x, s) \) is nonnegative and nonincreasing in \( x \) and satisfies \( a(x) \leq r_n (x), f_k (x, s) \leq \tilde{f}_k (x, s). \)

Case 1. If \( x \in [x_n, \infty) \) (in fact, \( x_{n+1} = \infty \)), from the definition of \( \sigma_k \), we have \( \sigma_k (x) \in [x_n, \infty) \) \( (k = 1, 2) \). According to (7) and (10) we get
\[
u (x) \leq r_n (x) + \sum_{k=1}^{2} \int_{x}^{\infty} \tilde{f}_k (x, s) \omega_k \left( u (\sigma_k (s)) \right) \, ds.
\]
Take any fixed \( T \in [x_n, \infty) \), and we investigate the following inequality:
\[
u (x) \leq r_n (T) + \sum_{k=1}^{2} \int_{x}^{\infty} \tilde{f}_k (T, s) \omega_k \left( u (\sigma_k (s)) \right) \, ds
\]
for \( x \in [T, \infty) \). Let
\[
z (x) = \sum_{k=1}^{2} \int_{x}^{\infty} \tilde{f}_k (T, s) \omega_k \left( u (\sigma_k (s)) \right) \, ds
\]
and let \( z(\infty) = 0 \). Hence, \( u(x) \leq r_n (T) + z(x) \). Clearly, \( z(x) \) is a nonnegative, nonincreasing, and differentiable function for \( x \in [T, \infty) \). The assumption \( a(\infty) \neq 0 \) yields that \( r_n (T) + z(x) > 0 \). Thus
\[
z' (x)
\]
\[
\frac{z'}{\omega_1 (r_n (T) + z (x))}
\]
Integrating both sides of the above inequality from \( x \) to \( \infty \), we obtain
\[
W_1 (r_n (T)) - W_1 (r_n (T) + z (x)) 
\]
\[
\geq - \int_{x}^{\infty} \tilde{f}_1 (T, s) \, ds
\]
(16)
for \( x \in [T, \infty) \), where \( \phi(x) = \omega_2 (x) / \omega_1 (x) \), so
\[
W_1 (r_n (T) + z (x)) \leq W_1 (r_n (T)) + \int_{x}^{\infty} \tilde{f}_1 (T, s) \, ds
\]
(17)
or, equivalently,
\[
\xi (x) \leq W_1 (r_n (T)) + \int_{x}^{\infty} \tilde{f}_1 (T, s) \, ds
\]
(18)
and satisfies \( a(x) \leq r_n (x), f_k (x, s) \leq \tilde{f}_k (x, s). \).

\[
\xi (x) = W_1 (r_n (T) + z (x))
\]
(19)
It is easy to check that \( \xi (x) \leq z_1 (x), z_1 (\infty) = W_1 (r_n (T)) \) and \( z_1 (x) \) is differentiable, positive, and nonincreasing on \([T, \infty)\). Since \( \phi(W_1^{-1} (u)) \) is nondecreasing, from the assumption \( (H_2) \), we have
\[
\frac{z_1' (x)}{\phi (W_1^{-1} (z_1 (x)))}
\]
\[
= \frac{\tilde{f}_1 (T, x) \omega_1 (u (\sigma_1 (x))) - \tilde{f}_2 (T, x) \omega_2 (u (\sigma_2 (x)))}{\omega_1 (r_n (T) + z (x))} 
\]
\[
\geq \frac{- \tilde{f}_1 (T, x) \omega_1 (r_n (T) + z (x))}{\omega_1 (r_n (T) + z (x))}
\]
\[
\geq \frac{- \tilde{f}_1 (T, x) \omega_1 (r_n (T) + z (x))}{\omega_1 (r_n (T) + z (x))} 
\]
\[
\geq - \frac{\tilde{f}_1 (T, x) \omega_1 (r_n (T) + z (x))}{\omega_1 (r_n (T) + z (x))}
\]
\[
\geq - \frac{\tilde{f}_1 (T, x) \omega_1 (r_n (T) + z (x))}{\omega_1 (r_n (T) + z (x))}
\]
\[
= W_2 \circ W_1^{-1} (z_1 (\infty)) - W_2 \circ W_1^{-1} (z_1 (x))
\]
(21)
Integrating both sides of (20) from \( x \) to \( \infty \), we obtain

\[
W_2\left(r_n(T)\right) - W_2 \circ W_1^{-1}\left(z_1(x)\right) \\
= \int_x^\infty \frac{z_1'(s)}{\phi(W_1^{-1}(z_1(s)))} ds \\
\geq - \int_x^\infty \frac{\tilde{f}_1(T, s)}{\phi\left(W_1^{-1}\left(W_1(r_n(T)) + \int_s^\infty \tilde{f}_1(T, \tau) d\tau\right)\right)} ds \\
- \int_x^\infty \tilde{f}_2(T, s) ds \\
\geq -W_2 \circ W_1^{-1}\left(W_1(r_n(T)) + \int_x^\infty \tilde{f}_1(T, s) ds\right) \\
+ W_2\left(r_n(T)\right) - \int_x^\infty \tilde{f}_2(T, s) ds.
\] (22)

Thus,

\[
W_2 \circ W_1^{-1}(z_1(x)) \\
\leq W_2 \circ W_1^{-1}\left(W_1(r_n(T)) + \int_x^\infty \tilde{f}_1(T, s) ds\right) \\
+ \int_x^\infty \tilde{f}_2(T, s) ds.
\] (23)

We have by (11)

\[
u(x) \leq z(x) + r_n(T) \\
\leq W_1^{-1}(\xi(x)) \leq W_1^{-1}(z_1(x)) \\
\leq W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1(r_n(T)) + \int_x^\infty \tilde{f}_1(T, s) ds\right) \\
+ \int_x^\infty \tilde{f}_2(T, s) ds\right].
\] (24)

Since the inequality above is true for any \( x \in [T, \infty) \), we obtain

\[
u(T) \leq W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1(r_n(T)) + \int_T^\infty \tilde{f}_1(T, s) ds\right) \\
+ \int_T^\infty \tilde{f}_2(T, s) ds\right].
\] (25)

Replacing \( T \) by \( x \) and \( \infty \) by \( x_{n+1} \) yields

\[
u(x) \leq W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1(r_n(x)) + \int_x^{x_{n+1}} \tilde{f}_1(x, s) ds\right) \\
+ \int_x^{x_{n+1}} \tilde{f}_2(x, s) ds\right].
\] (26)

This means that (9) is true for \( x \in [x_n, \infty) \) if we replace \( u(x) \) with \( u_n(x) \).

Case 2. If \( x \in [x_{n-1}, x_n) \), (7) becomes

\[
u(x) \leq r_n(x) \\
+ \sum_{k=1}^{n} \int_{x_k}^{x_{k+1}} f_k(x, s) \omega_k(u_n(\sigma_k(s))) ds \\
+ \beta_n u_n^m(x_n - 0) \\
+ \sum_{k=1}^{n} \int_x^{x_k} f_k(x, s) \omega_k(u(\sigma_k(s))) ds \\
\leq r_{n-1}(x) + \sum_{k=1}^{n} \int_x^{x_k} f_k(x, s) \omega_k(u(\sigma_k(s))) ds,
\] (27)

where the definition of \( r_{n-1}(x) \) is given in (10), which is similar to (12). Then, we obtain

\[
u(x) \leq W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1(r_{n-1}(x)) + \int_x^{x_n} f_1(x, s) ds\right) \\
+ \int_x^{x_n} \tilde{f}_2(x, s) ds\right].
\] (28)

This implies that (9) is true for \( x \in [x_{n-1}, x_n) \) if we replace \( u(x) \) by \( u_{n-1}(x) \).

Case 3. If (7) is true for \( x \in [x_i, x_{i+1}) \), that is,

\[
u_i(x) \leq W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1(r_i(x)) + \int_x^{x_{i+1}} \tilde{f}_1(x, s) ds\right) \\
+ \int_x^{x_{i+1}} \tilde{f}_2(x, s) ds\right],
\] (29)

then, for \( x \in [x_{i-1}, x_i) \), (7) becomes

\[
u(x) \leq r_n(x) \\
+ \sum_{j=1}^{n} \sum_{k=1}^{2} \int_{x_j}^{x_{j+1}} f_k(x, s) \omega_k(u_j(\sigma_k(s))) ds \\
+ \sum_{j=1}^{n} \beta_j u_j^m(x_j - 0) \\
+ \sum_{k=1}^{2} \int_x^{x_{j+1}} f_k(x, s) \omega_k(u(\sigma_k(s))) ds \\
\leq r_{i-1}(x) + \sum_{k=1}^{2} \int_x^{x_i} f_k(x, s) \omega_k(u(\sigma_k(s))) ds,
\] (30)

where we use the fact that the estimate of \( u(x) \) is already known for \( x \in [x_j, x_{j+1}) \), \( j = i, i + 1, \ldots, n \). By assumption
(29), again (30) is the same as (27) if we replace $r_{n-1}(x)$ by $r_{i-1}(x)$ and $x_n$ by $x_i$. Thus, by (28), we have

$$u(x) \leq W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1\left(r_{i-1}(x)\right) + \int_x^{x_i} \tilde{f}_1(x,s)ds\right)\right]$$

$$+ \int_x^{x_i} \tilde{f}_2(x,s)ds.]$$

(31)

This completes the proof of Theorem 1 by mathematical induction.

Remark 2. Zheng [25] investigated (5) which is the special case of (7). His results are under the assumptions that $a(x) = c$, $f_1(x,s)$, $f_2(x,s)$ are decreasing in $s$ for every fixed $s$ and $\omega \in \varphi$. In our result, these assumptions are avoided.

Consider the inequality

$$\varphi(u(x)) \leq a(x) + \sum_{k=1}^{n} \int_{x_k}^{\infty} f_k(x,s) \omega_k(u(\sigma_k(s)))ds$$

$$+ \sum_{x < x_k < \infty} \beta_k \psi(u(x_i - 0)),$$

(32)

which looks much more complicated than (7).

Corollary 3. In addition to the assumptions (H1)–(H5), suppose that $\varphi(u)$ is positive on $(0, \infty)$, $\varphi(u)$ is positive and strictly increasing on $(0, \infty)$, and $u(x)$ satisfies (32). If one lets $u_{i-1}(x) = u(x)$ for $x \in [x_{i-1}, x_i)$, $i = 1, 2, \ldots, n + 1$, then the estimate of $u(x)$ is recursively given by

$$u_{i-1}(x) \leq \varphi^{-1}\left[W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1\left(r_{i-1}(x)\right) + \int_x^{x_i} \tilde{f}_1(x,s)ds\right)\right]$$

$$\times \left(W_1\left(r_{i-1}(x)\right) + \int_x^{x_i} \tilde{f}_1(x,s)ds\right)$$

$$+ \int_x^{x_i} \tilde{f}_2(x,s)ds\right\},$$

(33)

where $W_j(u) = \int_{s_j}^{u} (dz/\omega_j(\varphi^{-1}(z)))$, $r_{n}(x)$, and $\tilde{f}_k(x,s)$ are given in Theorem 1 and $r_{i-1}(x)$ is defined as follows:

$$r_{i-1}(x) = r_{n}(x) + \sum_{j=1}^{n} \sum_{j=1}^{x_{i-1}} f_k(x,s) \omega_k(u_j(\sigma_k(s)))ds$$

$$+ \sum_{j=1}^{n} \beta_j \psi(u_j(x_i - 0)), \quad i = 1, 2, \ldots, n,$$

(34)

provided that

$$W_1\left(r_{i-1}(x)\right) + \int_x^{x_i} \tilde{f}_1(x,s)ds \leq \int_{s_i}^{\infty} \frac{dz}{\omega(x)},$$

$$W_2 \circ W_1^{-1}\left(W_1\left(r_{i-1}(x)\right) + \int_x^{x_i} \tilde{f}_1(x,s)ds\right)$$

$$+ \int_x^{x_i} \tilde{f}_2(x,s)ds \leq \int_{s_i}^{\infty} \frac{dz}{\omega(x)}.$$

(35)

Proof. Let $\varphi(u(x)) = h(x)$. Since the function $\varphi$ is strictly increasing on $[0, \infty)$, its inverse $\varphi^{-1}$ is well defined. And (32) becomes

$$h(x) \leq a(x) + \sum_{k=1}^{n} \int_{x_k}^{\infty} f_k(x,s) \omega_k(h(\sigma_k(s)))ds$$

$$+ \sum_{x < x_k < \infty} \beta_k \psi(h(x_i - 0)).$$

(36)

Let $\tilde{\omega}_k = \omega_k \circ \varphi^{-1}$ and $\tilde{\psi} = \psi \circ \varphi^{-1}$; (36) becomes

$$h(x) \leq a(x) + \sum_{k=1}^{n} \int_{x_k}^{\infty} f_k(x,s) \tilde{\omega}_k(h(\sigma_k(s)))ds$$

$$+ \sum_{x < x_k < \infty} \beta_k \tilde{\psi}(h(x_i - 0)).$$

(37)

It is easy to see that $\tilde{\psi}(u) > 0$, $\tilde{\omega}_1(u)$ and $\tilde{\omega}_2(u)$ are continuous and nonnegative functions on $[0, \infty)$, and $\tilde{\omega}_1(u)/\tilde{\omega}_1(u)$ is nondecreasing on $[0, \infty)$. Even though $\tilde{\psi}(u)$ is much more general, using the same way in Theorem 1, for $x \in [x_{i-1}, x_i)$, $i = 1, 2, \ldots, n + 1$, we can obtain the estimate of $u(x)$:

$$u_{i-1}(x) \leq \varphi^{-1}\left[W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1\left(r_{i-1}(x)\right) + \int_x^{x_i} \tilde{f}_1(x,s)ds\right)\right]$$

$$\times \left(W_1\left(r_{i-1}(x)\right) + \int_x^{x_i} \tilde{f}_1(x,s)ds\right)$$

$$+ \int_x^{x_i} \tilde{f}_2(x,s)ds\right\},$$

(38)

This completes the proof of Corollary 3.

If $\varphi(u) = u^\lambda$ where $\lambda > 0$ is a constant, we can study the inequality

$$u^\lambda(x) \leq a(x) + \sum_{k=1}^{n} \int_{x_k}^{\infty} f_k(x,s) \omega_k(u(\sigma_k(s)))ds$$

$$+ \sum_{x < x_k < \infty} \beta_k \psi(u(x_i - 0)).$$

(39)

According to Corollary 3, we have the following result.

Corollary 4. In addition to the assumptions (H1)–(H5), suppose that $\varphi(u)$ is positive on $(0, \infty)$ and $u(x)$ satisfies (39).
If one lets $u_{i-1}(x) = u(x)$ for $x \in [x_{i-1}, x_i)$, $i = 1, 2, \ldots, n + 1$, then the estimate of $u(x)$ is recursively given by

$$u_{i-1}(x) \leq \left\{ W_2^{-1} \left[ W_2 \circ W_1^{-1} \left( W_1 \left( r_{i-1}(x) \right) + \int_{x}^{x_i} \int_{x}^{y} f_1(x, s, t) \, ds \, dt \right) \right] \right\}^{1/\lambda}, \quad (40)$$

where $W_j(u) = \int_{a_j}^{u} \left( dz/\omega(z^{1/\lambda}) \right)$, $r_n(x)$, $r_{i-1}(x)$, and $f_k(x, s)$ are given in Corollary 3.

Let

$$\Omega = \bigcup_{i,j \geq 1} \Omega_{ij},$$

$$\Omega_{ij} = \{(x, y) : x_{i-1} \leq x < x_i, y_{j-1} \leq y < y_j\}, \quad (41)$$

for $i, j = 1, 2, \ldots, n + 1$, $0 < x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = \infty$, and $0 < y_0 < y_1 < y_2 < \cdots < y_n < y_{n+1} = \infty$.

Consider (8) and assume that

- (C1) $f_k(x, y, s, t)$ $(k = 1, 2)$ is continuous and nonnegative on $\Omega \times \Omega$ and bounded in $(x, y) \in \Omega$ for each fixed $(s, t) \in \Omega$ and satisfies $f_k(x, y, s, t) = 0$ $(k = 1, 2)$ if $(s, t) \in \Omega_{ij}, i \neq j$ for arbitrary $i, j = 1, 2, \ldots, n + 1$;
- (C2) $\omega_1(u)$ and $\omega_2(u)$ are continuous and nonnegative functions on $[0, \infty)$ and are positive on $(0, \infty)$ such that $\omega_2(u)/\omega_1(u)$ is nondecreasing;
- (C3) $u(x, y)$ is nonnegative and continuous on $\Omega$ with the exception of the points $(x_i, y_j)$ where there is a finite jump: $u(x_i, 0, y_j) = u(x_i, 0, y_j + 0)$, $i = 1, 2, \ldots, n$;
- (C4) $\omega_1(u)$ is continuous and bounded for $(x, y) \in \Omega$ and $\omega_1(\infty, \infty) \neq 0$; $\beta_0$ is a constant nonnegative for any positive integer $i$;
- (C5) $\sigma_k(x)$ and $\tau_k(y)$ $(k = 1, 2)$ are continuous and nonnegative such that $\sigma_k(x) \geq x$ and $\sigma_k(x) \leq x_j$ for $x \in [x_{i-1}, x_i)$, $i = 1, 2, \ldots, n + 1$, and $\tau_k(y) \geq y$ and $\tau_k(y) \leq y_j$ for $y \in [y_{j-1}, y_j)$, $i = 1, 2, \ldots, n + 1$.

**Theorem 5.** Suppose that (C1)–(C5) hold and $u(x, y)$ satisfies (8) for a positive constant $m$. If one lets $u_i(x, y) = u(x, y)$ for $(x, y) \in \Omega_{ii}, i = 1, 2, \ldots, n$, then the estimate of $u(x, y)$ is recursively given by

$$u_{i-1}(x, y) \leq W_2^{-1} \left[ W_2 \circ W_1^{-1} \left( W_1 \left( r_{i-1}(x, y) \right) + \int_{x}^{x_i} \int_{y}^{y_i} f_1(x, y, s, t) \, ds \, dt \right) \right] \times \left( W_1 \left( r_{i-1}(x, y) \right) + \int_{x}^{x_i} \int_{y}^{y_i} f_1(x, y, s, t) \, ds \, dt \right), \quad (42)$$

for $(x, y) \in \Omega_{ii}, i = 1, 2, \ldots, n + 1$, where

$$r_n(x, y) = \sup_{x \leq \xi \leq \infty} \sup_{x \leq \eta \leq \infty} |a(\xi, \eta)|,$$

$$f_k(x, y, s, t) = \sup_{x \leq \xi \leq \infty} f_k(\xi, \eta, s, t),$$

$$r_{i-1}(x, y) = r_n(x, y)$$

$$+ \sum_{j=1}^{n} \int_{x}^{x_i} \int_{y}^{y_i} f_k(x, y, s, t) \omega_k \left( u_j(s, t) \right) \, ds \, dt$$

$$+ \sum_{j=1}^{n} \beta_j u_j(x, y, s, t) \omega_k \left( x_j - 0, y_j - 0 \right), \quad i = 1, 2, \ldots, n,$$

provided that

$$W_1 \left( r_{i-1}(x, y) \right) + \int_{x}^{x_i} \int_{y}^{y_i} f_1(x, y, s, t) \, ds \, dt \leq \int_{a_i}^{\infty} \frac{dz}{\omega_1(z)},$$

$$W_2 \circ W_1^{-1} \left( W_1 \left( r_{i-1}(x, y) \right) + \int_{x}^{x_i} \int_{y}^{y_i} f_1(x, y, s, t) \, ds \, dt \right) \geq \int_{a_i}^{\infty} \frac{dz}{\omega_2(z)}.$$

**Proof.** Obviously, for any $(x, y) \in \Omega$, $r_n(x, y)$ is positive and nonincreasing with respect to $x$ and $y$; $f_k(x, y, s, t)$ $(k = 1, 2)$ is nonnegative and nonincreasing with respect to $x$ and $y$ for each fixed $s$ and $t$. They satisfy $a(x, y) \leq r_n(x, y)$ and $f_k(x, y, s, t) \leq f_k(x, y, s, t)$.

**Case I.** If $(x, y) \in \Omega_{n+1,n+1} = \{(x, y) : x_n \leq x < x_{n+1}, y_n \leq y < y_{n+1}\}$, we have from (8)

$$u(x, y) \leq r_n(x, y)$$

$$+ \sum_{k=1}^{2} \int_{x}^{x_i} \int_{y}^{y_i} \bar{f}_k(x, y, s, t) \omega_k \left( u(\sigma_k(s), \tau_k(t)) \right) \, ds \, dt.$$

Take any fixed $\bar{x} \in [x_n, \infty)$, $\bar{y} \in [y_n, \infty)$, and for arbitrary $x \in [\bar{x}, \infty)$, $y \in [\bar{y}, \infty)$, we get

$$u(x, y) \leq r_n(x, y)$$

$$+ \sum_{k=1}^{2} \int_{x}^{x_i} \int_{y}^{y_i} \bar{f}_k(x, \bar{y}, s, t) \omega_k \left( u(\sigma_k(s), \tau_k(t)) \right) \, ds \, dt.$$
Let 
\[ z(x, y) = r_n(\tilde{x}, \tilde{y}) \]
\[ + \sum_{k=1}^{n} \int_{x}^{\infty} \int_{y}^{\infty} \tilde{f}_k(\tilde{x}, \tilde{y}, s, t) \omega_k \times (u(\sigma_2(s), \tau_1(t))) \, ds \, dt \]
and let \( z(\infty, y) = r_n(\tilde{x}, \tilde{y}) \). Hence, \( u(x, y) \leq z(x, y) \). Clearly, \( z(x, y) \) is a nonnegative, nonincreasing, and differentiable function for \( x \in [\tilde{x}, \infty) \) and \( y \in [\tilde{y}, \infty) \). Since \( a(\infty, \infty) \neq 0 \) and \( \omega_1(z(x, y)) > 0 \), we have
\[ D_1 z(x, y) = \frac{\int_{x}^{\infty} \int_{y}^{\infty} \tilde{f}_1(\tilde{x}, \tilde{y}, x, t) \omega_1(u(\sigma_1(x), \tau_1(t))) \, dt}{\omega_1(z(x, y))} \]
\[ - \frac{\int_{x}^{\infty} \int_{y}^{\infty} \tilde{f}_2(\tilde{x}, \tilde{y}, x, t) \omega_2(u(\sigma_2(x), \tau_1(t))) \, dt}{\omega_1(z(x, y))} \]
\[ \geq - \frac{\int_{x}^{\infty} \int_{y}^{\infty} \tilde{f}_1(\tilde{x}, \tilde{y}, x, t) \omega_1(z(x, t)) \, dt}{\omega_1(z(x, y))} \]
\[ - \frac{\int_{x}^{\infty} \int_{y}^{\infty} \tilde{f}_2(\tilde{x}, \tilde{y}, x, t) \omega_2(z(x, t)) \, dt}{\omega_1(z(x, y))} \]
\[ \geq - \int_{y}^{\infty} \int_{x}^{\infty} \tilde{f}_1(\tilde{x}, \tilde{y}, x, t) \, dt \]
\[ - \int_{y}^{\infty} \int_{x}^{\infty} \tilde{f}_2(\tilde{x}, \tilde{y}, x, t) \frac{\omega_2(z(x, t))}{\omega_1(z(x, t))} \, dt. \]

Integrating both sides of the above inequality from \( x \) to \( \infty \), we obtain
\[ W_1(z(\infty, y)) - W_1(z(x, y)) \]
\[ \geq - \int_{x}^{\infty} \int_{y}^{\infty} \tilde{f}_1(\tilde{x}, \tilde{y}, s, t) \, ds \, dt \]
\[ - \int_{x}^{\infty} \int_{y}^{\infty} \tilde{f}_2(\tilde{x}, \tilde{y}, s, t) \frac{\omega_2(z(s, t))}{\omega_1(z(s, t))} \, ds \, dt. \]

Thus,
\[ W_1(z(x, y)) \leq W_1(r_n(\tilde{x}, \tilde{y})) \]
\[ + \int_{x}^{\infty} \int_{y}^{\infty} \tilde{f}_1(\tilde{x}, \tilde{y}, s, t) \, ds \, dt \]
\[ + \int_{x}^{\infty} \int_{y}^{\infty} \tilde{f}_2(\tilde{x}, \tilde{y}, s, t) \phi(z(s, t)) \, ds \, dt. \]

Note that
\[ \int_{x}^{\infty} \frac{D_1 z_1(s, y)}{\phi(W_1^{-1}(z_1(s, y)))} \, ds \]
\[ = \int_{x}^{\infty} D_1 z_1(s, y) \frac{\omega_1(W_1^{-1}(z_1(s, y)))}{\omega_2(W_1^{-1}(z_1(s, y)))} \, ds 
\]
\[ = \int_{W_1^{-1}(z_1(s, y))}^{W_1^{-1}(r_n(\tilde{x}, \tilde{y}))} \frac{du}{\omega_2(u)}. \]
Since the above inequality is true for any \( x \in [\bar{x}, \infty) \), \( y \in [\bar{y}, \infty) \), we obtain

\[
\begin{align*}
u(x, y) \leq W_2^{-1}\left(W_2 \circ W_1^{-1}\left(W_1(r_n(\bar{x}, \bar{y})) \right) + \int_{\bar{x}}^{\infty} \int_{\bar{y}}^{\infty} f_1(\bar{x}, \bar{y}, s, t) \, ds \, dt \right) \\
\times \left(W_1(r_n(x, y)) + \int_{x}^{x_{n+1}} \int_{y}^{y_{n+1}} f_1(x, y, s, t) \, ds \, dt \right) \\
+ \int_{x}^{x_{n+1}} \int_{y}^{y_{n+1}} f_2(x, y, s, t) \, ds \, dt.
\end{align*}
\]  

(58)

Replacing \( \bar{x}, \bar{y}, \) and \( \infty \) by \( x, y, \) and \( x_{n+1}, \) respectively, yields

\[
u(x, y) \leq W_2^{-1}\left(W_2 \circ W_1^{-1}\left(W_1(r_n(x, y)) \right) + \int_{x}^{x_{n+1}} \int_{y}^{y_{n+1}} f_1(x, y, s, t) \, ds \, dt \right) \\
\times \left(W_1(r_n(x, y)) + \int_{x}^{x_{n+1}} \int_{y}^{y_{n+1}} f_1(x, y, s, t) \, ds \, dt \right) \\
+ \int_{x}^{x_{n+1}} \int_{y}^{y_{n+1}} f_2(x, y, s, t) \, ds \, dt.
\]

(59)

This means that (42) is true for \((x, y) \in \Omega_{n+1,n+1}\) and \(i = n\) if we replace \(u(x, y)\) with \(u_n(x, y)\).

Case 2. If \((x, y) \in \Omega_{n,n} = \{(x, y) : x_{n-1} \leq x < x_n, \ y_{n-1} \leq y < y_n\}\), (8) becomes

\[
u(x, y) \leq r_n(x, y)
\]

\[
+ \sum_{k=1}^{n-1} \int_{x_k}^{x_{n+1}} \int_{y_k}^{y_{n+1}} f_k(x, y, s, t) \omega_k \times (u_n(\sigma_k(s), \tau_k(t))) \, ds \, dt
\]

\[
+ (\beta_n u_n(x_n-0, y_{n-0})
\]

\[
+ \sum_{k=1}^{n-1} \int_{x_k}^{x_{n+1}} \int_{y_k}^{y_{n+1}} f_k(x, y, s, t) \omega_k \times (u(\sigma_k(s), \tau_k(t))) \, ds \, dt
\]

\[
\leq r_{n-1}(x, y)
\]

\[
+ \sum_{k=1}^{n-1} \int_{x_k}^{x_{n+1}} \int_{y_k}^{y_{n+1}} f_k(x, y, s, t) \omega_k \times (u(\sigma_k(s), \tau_k(t))) \, ds \, dt,
\]

(60)

where the definition of \(r_{n-1}(x, y)\) is given in (43). Note that the estimate of \(u_n(x, y)\) is known. Clearly, (60) is the same as (45)
if we replace \( r_n(x, y) \) and \((\infty, \infty)\) by \( r_{n-1}(x, y) \) and \((x_n, y_n)\). Thus, by (59), we have

\[
u(x, y) \leq W_2^{-1} \left[ W_2 \circ W_1^{-1} \right. \times \left( W_1(r_{n-1}(x, y)) + \int_x^y \int_y^{y_n} f_1(x, y, s, t) \, ds \, dt \right) + \int_x^y \int_y^{y_n} f_2(x, y, s, t) \, ds \, dt \bigg].
\]

(61)

This implies that (42) is true for \((x, y) \in \Omega_{n,n} \) and \(i = n - 1\) if we replace \( u(x, y) \) by \( u_{n-1}(x, y) \).

**Case 3.** Assume that (42) is true for \((x, y) \in \Omega_{i+1,i+1} = \{(x, y) : x_i \leq x < x_{i+1}, y_i \leq y < y_{i+1}\}. Then for \((x, y) \in \Omega_{i,i} = \{(x, y) : x_i \leq x < x_i, y_{i-1} \leq y < y_i\}, (8) becomes

\[
u(x, y) \leq r_n(x, y) + \sum_{j=i}^{n-1} \sum_{k=1}^{n-j} \int_{x_j}^{x_{j+1}} \int_{y_j}^{y_{j+1}} f_k(x, y, s, t, \omega_k) \bigg( u(x, y, s, t) \bigg) \, ds \, dt
\]

(62)

which looks much more complicated than (8).

**Corollary 7.** In addition to the assumptions \((C_1)-(C_3)\), suppose that \( \psi(u) \) is positive on \((0, \infty)\), \( \varphi(u) \) is positive and strictly increasing on \((0, \infty)\), and \( u(x, y) \) satisfies (64) for a positive constant \( m \). If one lets \( u_i(x, y) = u(x, y) \), then the estimate of \( u(x, y) \) is recursively given by

\[
u(x, y) \leq \varphi^{-1} \left[ W_2^{-1} \left[ W_2 \circ W_1^{-1} \right. \times \left( W_1(r_{i-1}(x, y)) + \int_x^y \int_y^{y_i} f_1(x, y, s, t) \, ds \, dt \right) + \int_x^y \int_y^{y_i} f_2(x, y, s, t) \, ds \, dt \bigg].
\]

(65)

where \( W_i(u) = \int_{z_i}^{z_i} (dz/\omega_i(\varphi^{-1}(z))) \), \( r_n(x, y) \), and \( f_i(x, y, s, t) \) are given in Theorem 5. \( r_{i-1}(x, y) \) is defined as follows:

\[
r_{i-1}(x, y) = r_n(x, y) + \sum_{j=i}^{n-1} \sum_{k=1}^{n-j} \int_{x_j}^{x_{j+1}} \int_{y_j}^{y_{j+1}} f_k(x, y, s, t, \omega_k) \bigg( u(x, y, s, t) \bigg) \, ds \, dt
\]

(66)
provided that
\[ W_1(r_{i-1}(x, y)) + \int_x^{x_i} \int_y^{y_i} \tilde{f}_1(x, y, s, t) \, ds \, dt \leq \int_{\Omega_i} \frac{dz}{\omega_1(z)}, \]
\[ W_2 \circ W_1^{-1} \left[ W_1(r_{i-1}(x, y)) + \int_x^{x_i} \int_y^{y_i} \tilde{f}_1(x, y, s, t) \, ds \, dt \right] + \int_x^{x_i} \int_y^{y_i} \tilde{f}_2(x, y, s, t) \, ds \, dt \leq \int_{\Omega_i} \frac{dz}{\omega_2(z)} \]  
(67)

The proof is similar to Corollary 3.

If \( \psi(u) = u^\lambda \), where \( \lambda > 0 \) is a constant, we can study the inequality
\[ u^\lambda(x, y) \leq a(x, y) + \frac{2}{\lambda} \int_x^{x_i} \int_y^{y_i} f_k(x, y, s, t) \times \omega_k \left( u(\sigma_k(s), r_k(t)) \right) \, ds \, dt 
+ \sum_{x < x_i, y < y_i < \infty} \beta_i \psi \left( u(x_i - 0, y_i - 0) \right). \]  
(68)

According to Corollary 7, we have the following result.

**Corollary 8.** In addition to the assumptions (C1)–(C5), suppose that \( \psi(u) \) is positive on \((0, \infty)\) and \( u(x, y) \) satisfies (68) for a positive constant \( \lambda \). If one lets \( u(x, y) = u(x, y) \) for \( (x, y) \in \Omega_i \), then the estimate of \( u(x, y) \) is recursively given by
\[ u_{i-1}(x, y) \leq \left\{ W_2^{-1} \left[ W_2 \circ W_1^{-1} \right] \times \left( W_1(r_{i-1}(x, y)) + \int_x^{x_i} \int_y^{y_i} \tilde{f}_1(x, y, s, t) \, ds \, dt \right) + \int_x^{x_i} \int_y^{y_i} \tilde{f}_2(x, y, s, t) \, ds \, dt \right\} \frac{1}{\lambda}, \]  
(69)

where \( W_j(u) = \int_x^{x_i} (dz / \omega_j(z^{1/\lambda})) \), \( r_n(x, y), r_{i-1}(x, y) \), and \( \tilde{f}_j(x, y, s, t) \) are given in Corollary 7.

### 3. Applications

**Example 9.** Consider the following impulsive differential equation:
\[ \frac{dg}{dx} = F(x, g), \quad x \neq x_i, \]  
(70)
\[ \Delta g |_{x=x_i} = l_i(x), \quad g(\infty) = \theta \neq 0, \]  
(71)

where \( g : \mathbb{R} \rightarrow \mathbb{R}, F : \mathbb{R}^2 \rightarrow \mathbb{R}, l_i : \mathbb{R} \rightarrow \mathbb{R} \) and \( i = 1, 2, \ldots, n \), \( 0 < x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = \infty \). Here, \( \theta \) is a constant.

Assume that
\[ (A_1) \ |F(x, g)| \leq h_1(x)e^{\lambda g} + h_2(x)e^{2\lambda g} \text{ where } h_1 \text{ and } h_2 \text{ are nonnegative, bounded, and continuous on } \mathbb{R}^2; \]
\[ (A_2) \ |l_i(g)| \leq \beta_i |g|^m \text{ where } \beta_i \text{ and } m \text{ are nonnegative constants}. \]

**Theorem 10.** Suppose that \((A_1)\) and \((A_2)\) hold. If one lets \( g_{n-1}(x) = g(x) \) for \( x \in [x_{i-1}, x_i], i = 1, 2, \ldots, n + 1 \), then the solution of (70) has an estimate for \( x \in [x_{i-1}, x_i] \):
\[ |g_{n-1}(x)| \leq -\frac{1}{2} \ln \left( \left( e^{-r_{i-1}(x)} - \int_x^{x_i} h_1(s) \, ds \right)^2 - 2 \int_x^{x_i} h_2(s) \, ds \right), \]  
(72)

where \( r_n(x) = |\theta| \) and
\[ r_{i-1}(x) = r_n(x) + \sum_{j=1}^n \int_{x_j}^{x_{i-1}} h_1(s) e^{\lambda g(s)} \, ds \]  
\[ + \sum_{j=1}^n \int_{x_j}^{x_i} h_2(s) e^{2\lambda g(s)} \, ds \]  
(73)
\[ + \sum_{j=1}^n \beta_j |g_j(x_j - 0)|^m, \quad i = 1, 2, \ldots, n, \]
\[ \left( e^{-r_{i-1}(x)} - 2 \int_x^{x_i} h_1(s) \, ds \right)^2 - 2 \int_x^{x_i} h_2(s) \, ds > 0. \]

**Proof.** Integrating (70) from \( x \) to \( \infty \) and using the initial conditions (71), we get
\[ g(x) = \theta - \int_x^{\infty} F(s, g) \, ds - \sum_{x < x_i, y_i < \infty} l_i(g(x_i - 0)), \]  
(74)

which implies that
\[ |g(x)| \leq |\theta| + \int_x^{\infty} h_1(s) e^{\lambda g(s)} \, ds + \int_x^{\infty} h_2(s) e^{2\lambda g(s)} \, ds \]  
(75)
\[ + \sum_{x < x_i, y_i < \infty} \beta_j |g(x_j - 0)|^m. \]

Let
\[ u(x) = |g(x)|, \quad a(x) = |\theta|, \quad \sigma_1(x) = \sigma_2(x) = x, \]
\[ f_1(x, s) = h_1(s), \quad f_2(x, s) = h_2(s), \quad \omega_1(u) = e^u, \]
\[ \omega_2(u) = e^{2u}. \]  
(76)
Thus, (75) is the same as (7). It is easy to obtain that for any positive constants $\bar{u}_1$ and $\bar{u}_2$

$$r_n(x) = |\emptyset|, \quad f_1(x, s) = h_1(s), \quad f_2(x, s) = h_2(s),$$

$$W_1(u) = \int_{\bar{u}_1}^{u} \frac{dz}{\omega_1(z)} = \int_{\bar{u}_1}^{u} e^{-z}dz = e^{-\bar{u}_1} - e^{-z},$$

$$W_1^{-1}(u) = -\ln(e^{-\bar{u}_1} - u),$$

$$W_2(u) = \int_{\bar{u}_2}^{u} \frac{dz}{\omega_2(z)} = \int_{\bar{u}_2}^{u} e^{-z}dz = \frac{1}{2}(e^{-2\bar{u}_2} - e^{-2u}),$$

$$W_2^{-1}(u) = -\frac{1}{2}\ln(e^{-2\bar{u}_2} - 2u),$$

$$r_{i-1}(x) = r_n(x) + \sum_{j=i}^{n} \int_{x_j}^{x_{i+1}} h_1(s) e^{g_j(s)}ds$$

$$+ \sum_{j=i}^{n} \int_{x_j}^{x_{i+1}} h_2(s) e^{2g_j(s)}ds$$

$$+ \sum_{j=i}^{n} \beta_j g_j(x_j - 0)^m.$$

(77)

Therefore, for any nonnegative $i$ and $x \in [x_{i-1}, x_i)$

$$|g_{i-1}(x)|$$

$\leq -\frac{1}{2} \ln \left( e^{-r_{i-1}(x)} - \int_{x}^{x_{i+1}} h_1(s) ds \right)^2 - 2 \int_{x}^{x_{i+1}} h_2(s) ds > 0.$

(78)

provided that

$$\left( e^{-r_{i-1}(x)} - 2 \int_{x}^{x_{i+1}} h_1(s) ds \right)^2 - 2 \int_{x}^{x_{i+1}} h_2(s) ds > 0.$$  

(79)

Example II. Consider the following partial differential equation with an impulsive term:

$$\frac{\partial^2 v(x, y)}{\partial x \partial y} = H(x, y, v(x, y)),

(x, y) \in \Omega_{2i}, \quad x \neq x_i, \quad y \neq y_j,$$

$$\Delta v|_{x=x_i, y=y_j} = I_i(v),$$

$$v(x, \infty) = \phi_1(x), \quad v(\infty, y) = \phi_2(y),$$

$$\phi_1(\infty) = \phi_2(\infty) \neq 0,$$

where $v : \mathbb{R}^2 \to \mathbb{R}$, $H : \mathbb{R}^2 \to \mathbb{R}$, $I_i : \mathbb{R} \to \mathbb{R}$, and $i = 1, 2, \ldots, n + 1$.

Assume that

$$(B_1) \ |H(x, y, v(x, y))| \leq h_1(x, y)e^{\beta_1 v(x, y)} + h_2(x, y)e^{\beta_2 v(x, y)}$$

where $h_1, h_2$ are nonnegative, bounded, and continuous on $\Omega$, $h_1(x, y) = 0$, $h_2(x, y) = 0$ for $(x, y) \in \Omega_{2i}, i \neq j$, $i, j = 1, 2, \ldots, n + 1$;

$$(B_2) \ |I_i(v)| \leq \beta_i |v|^m$$

where $\beta_i$ and $m$ are nonnegative constants.

Theorem 12. Suppose that $(B_1)$ and $(B_2)$ hold. If one lets $v_i(x, y) = v(x, y)$ for $(x, y) \in \Omega_{2i}$, then the solution of system (80) has an estimate for $(x, y) \in \Omega_{2i}$:

$$|v_i(x, y)| \leq \frac{1}{2} \ln \left( e^{-r_{i-1}(x, y)} - \int_{x}^{x_i} \int_{y}^{y_j} h_1(s, t) ds dt \right)^2$$

$$- 2 \int_{x}^{x_i} \int_{y}^{y_j} h_2(s, t) ds dt,$$

(81)

where

$$r_n(x, y) = \sup_{x \leq \xi < \infty, y \leq \eta < \infty} \left| \phi_1(\xi) + \phi_2(\eta) - \phi_1(\infty) \right| > 0,$$

$$r_{i-1}(x, y) = r_n(x, y)$$

$$+ \sum_{j=i}^{n} \int_{x_j}^{x_{i+1}} \int_{y_j}^{y_{i+1}} h_1(s, t) e^{\beta_j g_j(s, t)} ds dt$$

$$+ \sum_{j=i}^{n} \int_{x_j}^{x_{i+1}} \int_{y_j}^{y_{i+1}} h_2(s, t) e^{2\beta_j g_j(s, t)} ds dt$$

$$+ \sum_{j=i}^{n} \beta_j |v_j(x_j - 0, y_j - 0)|^m,$$

$$\left( e^{-r_{i-1}(x, y)} - \int_{x}^{x_i} \int_{y}^{y_j} h_1(s, t) ds dt \right)^2$$

$$- 2 \int_{x}^{x_i} \int_{y}^{y_j} h_2(s, t) ds dt > 0.$$

(82)

Proof. The solution of (80) with an initial value is given by

$$v(x, y) = v(x, \infty) + v(\infty, y) - v(\infty, \infty)$$

$$+ \int_{x}^{\infty} \int_{y}^{\infty} H(s, t, v(s, t)) ds dt$$

$$+ \sum_{x < \xi < \infty, y < \eta < \infty} I_i(v(x_i - 0, y_i - 0))$$

$$= \phi_1(x) + \phi_2(y) - \phi_1(\infty)$$

$$+ \int_{x}^{\infty} \int_{y}^{\infty} H(s, t, v(s, t)) ds dt$$

$$+ \sum_{x < \xi < \infty, y < \eta < \infty} I_i(v(x_i - 0, y_i - 0)),$$

(83)
which implies that
\[
\begin{align*}
|v(x, y)| & \leq |\phi_1(x) + \phi_2(y) - \phi_1(\infty)| \\
& + \int_x^\infty \int_y^\infty h_1(s, t) e^{\int_s^t (\sigma(s,t)) \, ds \, dt} \, ds \, dt \\
& + \int_x^\infty \int_y^\infty h_2(s, t) e^{\int_s^t (\sigma(s,t)) \, ds \, dt} \, ds \, dt \\
& + \sum_{x < x_j < y_j < y} \beta_j |v(x_j - 0, y_j - 0)|^m.
\end{align*}
\]  

Similar to Theorem 10, we can obtain, for \((x, y) \in \Omega_{ij},\)
\[
|v_i(x, y)| \leq -\frac{1}{2} \ln \left( e^{-2\sigma(x,y)} \left( 1 - \int_x^{x_i} \int_y^{y_i} h_1(s, t) \, ds \, dt \right) \right)^2 \\
- 2 \int_x^{x_i} \int_y^{y_i} h_2(s, t) \, ds \, dt.
\]

**Remark 13.** From Examples 9 and 11, we know that \(\omega_1(u) = e^u.\) Clearly, \(\omega_1(2u) = e^{2u} \leq \omega_1(2)\omega_1(u) = e^2 e^u\) does not hold for large \(u > 0.\) Thus, \(\omega_1(u) = e^u\) does not belong to class \(\varphi\) in [25]. Hence, the results in [25] cannot be applied to inequality (75).

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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**References**


